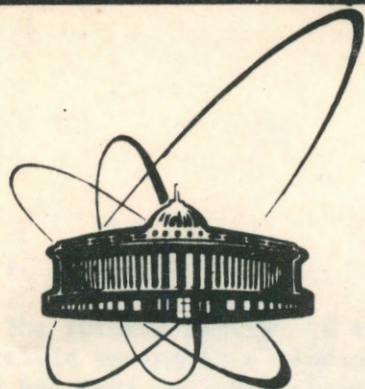


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ОБЪЕДИНЕННЫЙ  
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AN INVERSE PROBLEM  
FOR CLASSICAL RELATIVISTIC MECHANICS

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## 1. INTRODUCTION

The inverse problem of a nonrelativistic classical point particle moving in a confining potential is well known<sup>1,2/</sup> and has been solved by inversion of an Abelian integral equation. The data for the reconstruction of the (symmetric) potential are the values of the full period  $T(E)$ .

The same problem is solved in this letter for a relativistic classical particle trapped in a symmetric confining potential  $\phi(x)$  which is the fourth component of a 4-potential. The Newtonian equation of motion and the relation between the full energy and 3-momentum are used. The resulting integral eqn. for  $T(E)$  is inverted by a Laplace-transformation technique, and finally we get the explicit formula for  $x(\phi)$  without any approximations. The nonrelativistic limit is considered too. This new example of an exactly solvable inverse problem manifests the classical limit of a (future) relativistic quantum inverse problem in WKB-approximation.

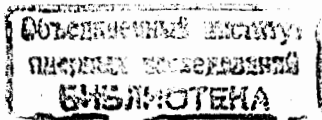
## 2. FORMULATION OF THE DIRECT PROBLEM

The Lorentz-invariant action for a point particle moving in a 4-potential  $A^\mu$  with the components  $(A_{\mu i}, V)^*$ , which only depends on  $x$ , is:

$$S = \int (-m_0 c^2 + \frac{e}{c} A_\mu u^\mu) dr, \quad (1)$$

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\* The conventions are:  $-c^2 dr^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$  with  $\alpha, \beta = 1, \dots, 4$ ,  $i, k = 1, 2, 3$ ,  $\eta_{\alpha\beta} = \text{diag}(1, 1, 1, -1)$ ,  $x^\alpha \sim (x^i, ct)$ ,  $dt/dr = \gamma$ ,  $v^i = dx^i/dt$ . Then the components of the four-velocity are:  $u^\alpha \sim \gamma(v^i, c)$ ; and of the four-force,  $K^\alpha \sim \gamma(F_i v_i/c)$ , where  $F_i$  is the Newtonian force. The rest mass is  $m_0$  and  $m = \gamma m_0$ .



and leads at  $A_1 = 0$  and  $\phi = eV$  in one dimension to the equation of motion:

$$\frac{dp}{dt} = F(x) = -\frac{d\phi}{dx}, \quad (p = m \frac{dx}{dt}). \quad (2)$$

The connection between the total relativistic energy  $H$  and the 3-momentum in this case is

$$(H - \phi)^2 = p^2 c^2 + m_0^2 c^4. \quad (3)$$

Note, that a Lorentz-invariant scalar potential  $\phi(x^\mu)$  can, as is well known, be coupled only to the mass-term. This would lead to a velocity-dependent force, but in this paper we restrict ourselves to investigations of pure  $x$ -dependence of the force. The problem is to define the (symmetric) potential  $\phi(x)$  if, for a fixed  $E$ , the time for a full period  $T(E)$  is known. It is assumed, that the inverse function  $x(\phi)$  exists for  $x > 0$ .

From (2) it immediately follows the full period

$$T(E) = 4 \int_0^E \frac{d\phi}{\frac{d\phi}{dp} F(\phi)}, \quad (4)$$

and fixing the zero of  $\phi(x)$  in its minimum, with  $H = m_0 c^2 + E = \text{const}$ , one obtains from (3):

$$\frac{d\phi}{dp} = \frac{c \sqrt{\left[ \frac{E - \phi}{m_0 c^2} \right]^2 + 2 \frac{E - \phi}{m_0 c^2}}}{1 + \frac{E - \phi}{m_0 c^2}}. \quad (5)$$

with the dimensionless abbreviations

$$y = \frac{E}{m_0 c^2} > 0, \quad z = \frac{\phi}{m_0 c^2} > 0, \quad r(z) = -\frac{4m_0 c}{F(z)} \quad (6)$$

the period will be

$$T(y) = \int_0^y r(z) \frac{1 + y - z}{\sqrt{(y-z)^2 + 2(y-z)}} dz. \quad (7)$$

### 3. THE INVERSION FORMULA

If we assume that the period  $T(y)$  is known, eqn.(7) is an integral eqn. determining the force  $r(z)$ . It is solved by the Laplace transformation

$$\hat{f}(s) \equiv \hat{L} f(x) = \int_0^\infty dx e^{-sx} f(x) \quad (8)$$

which results (see Appendix) in the factorized formula

$$\hat{T}(s) \equiv \hat{r}(s) e^s K_1(s), \quad (9)$$

where  $K_1(s)$  denotes the modified Bessel- (or McDonald-) function of the third kind<sup>3/</sup>. Using the identity

$$\frac{d}{dz} \int \frac{e^{sz}}{s} ds = \int e^{sz} ds, \quad (10)$$

one obtains the inversion-formula for the Laplace transformed function  $x(z)$ :

$$x(z) = \frac{c}{8\pi i} \frac{d}{dz} \int_{\epsilon - i\infty}^{\epsilon + i\infty} ds \frac{e^{sz}}{s^2} \frac{e^{-s}}{K_1(s)} \hat{T}(s). \quad (11)$$

With the convolution theorem the result finally shows

$$x(z) = \frac{c}{4} \frac{d}{dz} \int_0^z dy G(z-y) T(y) \quad (12)$$

with

$$G(z) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} ds \frac{e^{sz}}{s^2} \frac{e^{-s}}{K_1(s)}. \quad (13)$$

Note that for the derivation of the explicit formulas (12) and (13) one does not need any approximations.

#### 4. THE NONRELATIVISTIC LIMIT

For the large  $|s|$  case, being important for small values of the potential  $z$ , there exists the asymptotic representation of the function  $K_1(s)^{1/3}$ :

$$K_1(s) = \sqrt{\frac{\pi}{2}} \frac{e^{-s}}{\sqrt{s}} \sum_{k=0}^{\infty} \frac{a_k}{z^k}, \quad (14)$$

with

$$a_k = \frac{1}{2^k k!} \frac{\Gamma(3/2 + k)}{\Gamma(3/2 - k)}. \quad (15)$$

Using the relation ( $n = 1, 2, \dots$ )

$$\hat{L} \left[ \frac{2^n z^{n-1/2}}{1 \cdot 3 \cdot \dots \cdot (2n-1) \sqrt{\pi}} \right] = s^{-(n+1/2)}, \quad (16)$$

we get for the kernel  $G(z)$  the following asymptotic series:

$$G(z) = \frac{2}{\pi} \sum_{j=1}^{\infty} \frac{\beta_{j-1} (2z)^{j-1/2}}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2j-1)}, \quad (17)$$

and the coefficients  $\beta_n$  are defined by the recursion formula ( $\beta_0 = \alpha_0 = 1$ ):

$$\beta_n = - \sum_{l=1}^{l=n} \alpha_l \beta_{n-l}. \quad (18)$$

The first term in the series (17) results in the known (e.g. <sup>1/2</sup>) inversion formula for nonrelativistic mechanics, expressed by the Abelian integral transformation

$$x(z)_{\text{nonrel.}} = \frac{\sqrt{2} c}{4\pi} \int_0^z \frac{dy T(y)}{\sqrt{z-y}}. \quad (19)$$

Inserting the constant value of the period  $T = 2\pi/\omega$ , eqn.(19) gives the oscillator-potential. Taking the next terms in (17) into account we get the potential with the constant period:

$$\phi(x) = \frac{m_0 \omega^2 x^2}{2} \left[ 1 + \frac{\omega^2 x^2}{16c^2} \right] + O \left[ \frac{1}{c^4} \right]. \quad (20)$$

Defining the oscillator as a potential admitting constant values of the period for all energies, we thus find this potential in the relativistic (kinematic) case from the inversion formula (11).

#### APPENDIX

The Laplace transform of  $T(y)$  is

$$\hat{T}(s) = \int_0^{\infty} dy e^{-sy} \int_0^y r(z) \frac{(1+y-z) dz}{\sqrt{2(y-z) + (y-z)^2}}.$$

Changing the succession of integration, it can be written as

$$\int_0^{\infty} dz r(z) \int_0^{\infty} dy e^{-sy} \frac{1+y-z}{\sqrt{2(y-z) + (y-z)^2}}.$$

In order to factorize these integrals we set  $\xi = y-z$  in the second one and get

$$\int_0^{\infty} dz r(z) e^{-sz} \int_0^{\infty} d\xi e^{-s\xi} \frac{1+\xi}{\sqrt{2\xi + \xi^2}}.$$

The first integral by definition is  $\hat{T}(s)$ , and the second can be expressed analytically and gives  $e^s K_1(s)$  so that formula (9) is proved.

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