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AN INVERSE PROBLEM FOR CLASSICAL RELATIVISTIC MECHANICS

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1. INTRODUCTION

The inverse problem of a nonrelativistic classical point particle moving in a confining potential is well known[1,2] and has been solved by inversion of an Abelian integral equation. The data for the reconstruction of the (symmetric) potential are the values of the full period T(E).

The same problem is solved in this letter for a relativistic classical particle trapped in a symmetric confining potential $\phi(x)$ which is the fourth component of a 4-potential. The Newtonian equation of motion and the relation between the full energy and 3-momentum are used. The resulting integral eqn. for T(E) is inverted by a Laplace-transformation technique, and finally we get the explicit formula for $x(\phi)$ without any approximations. The nonrelativistic limit is considered too. This new example of an exactly solvable inverse problem manifests the classical limit of a (future) relativistic quantum inverse problem in WKB-approximation.

2. FORMULATION OF THE DIRECT PROBLEM

The Lorentz-invariant action for a point particle moving in a 4-potential A^{μ} with the components (A_i, V)*, which only depends on x, is:

$$S = \int \left(-m_0 c^2 + \frac{e}{c} A_{\mu} u^{\mu}\right) d\tau, \qquad (1)$$

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* *The conventions are:* $-c^2 d\tau^2 = \eta_{\alpha\beta}\, dx^{\alpha} dx^{\beta}$ *with* $\alpha, \beta = 1,\dots 4$, $i, k = 1,2,3$, $\eta_{\alpha\beta} = \mathrm{diag}(1,1,1,-1)$, $x^{\alpha} \sim (x^i, ct)$, $dt/d\tau = \gamma$, $v^i = dx^i/dt$. *Then the components of the four-velocity are:* $u^a \sim \gamma(v^i, c)$; *and of the four-force,* $K^{\alpha} \sim \gamma(F_i v_i/c)$, *where* F_i *is the Newtonian force. The rest mass is* m_0 *and* $m = \gamma m_0$.

and leads at $A_i = 0$ and $\phi = eV$ in one dimension to the equation of motion:

$$\frac{dp}{dt} = F(x) = -\frac{d\phi}{dx}, \quad (p = m\frac{dx}{dt}). \tag{2}$$

The connection between the total relativistic energy H and the 3-momentum in this case is

$$(H - \phi)^2 = p^2c^2 + m_o^2c^4. \tag{3}$$

Note, that a Lorentz-invariant scalar potential $\phi(x^\mu)$ can, as is well known, be coupled only to the mass-term. This would lead to a velocity-dependent force, but in this paper we restrict ourselves to investigations of pure x-dependence of the force. The problem is to define the (symmetric) potential $\phi(x)$ if, for a fixed E, the time for a full period T(E) is known. It is assumed, that the inverse function $x(\phi)$ exists for $x > 0$.

From (2) it immediately follows the full period

$$T(E) = 4 \int_0^E \frac{d\phi}{\frac{d\phi}{dp} F(\phi)}, \tag{4}$$

and fixing the zero of $\phi(x)$ in its minimum, with $H = m_oc^2 + E = \text{const}$, one obtains from (3):

$$\frac{d\phi}{dp} = \frac{c\sqrt{[\frac{E-\phi}{m_oc^2}]^2 + 2\frac{E-\phi}{m_oc^2}}}{1 + \frac{E-\phi}{m_oc^2}}. \tag{5}$$

with the dimensionless abbreviations

$$y = \frac{E}{m_oc^2} > 0, \quad z = \frac{\phi}{m_oc^2} > 0, \quad r(z) = -\frac{4m_oc}{F(z)} \tag{6}$$

the period will be

$$T(y) = \int_0^y r(z) \frac{1 + y - z}{\sqrt{(y-z)^2 + 2(y-z)}}. \tag{7}$$

3. THE INVERSION FORMULA

If we assume that the period T(y) is known, eqn.(7) is an integral eqn. determining the force $r(z)$. It is solved by the Laplace transformation

$$\hat{f}(s) \equiv \hat{L} f(x) = \int_0^\infty dx\, e^{-sx} f(x) \tag{8}$$

which results (see Appendix) in the factorized formula

$$\hat{T}(s) \equiv \hat{r}(s) e^s K_1(s), \tag{9}$$

where $K_1(s)$ denotes the modified Bessel- (or McDonald-) function of the third kind /3/. Using the identity

$$\frac{d}{dz} \int \frac{e^{sz}}{s} ds = \int e^{sz} ds, \tag{10}$$

one obtains the inversion-formula for the Laplace transformed function x(z):

$$x(z) = \frac{c}{8\pi i} \frac{d}{dz} \int_{\epsilon - i\infty}^{\epsilon + i\infty} ds \frac{e^{sz}}{s^2} \frac{e^{-s}}{K_1(s)} \hat{T}(s). \tag{11}$$

With the convolution theorem the result finally shows

$$x(z) = \frac{c}{4} \frac{d}{dz} \int_0^z dy\, G(z-y) T(y) \tag{12}$$

with

$$G(z) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} ds \frac{e^{sz}}{s^2} \frac{e^{-s}}{K_1(s)}. \tag{13}$$

Note that for the derivation of the explicit formulas (12) and (13) one does not need any approximations.

4. THE NONRELATIVISTIC LIMIT

For the large $|s|$ case, being important for small values of the potential z, there exists the asymptotic representation of the function $K_1(s)$ /3/ :

$$K_1(s) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-s}}{\sqrt{s}}\sum_{k=0}^{\infty}\frac{a_k}{z^k}, \tag{14}$$

with

$$a_k = \frac{1}{2^k k!}\,\frac{\Gamma(3/2+k)}{\Gamma(3/2-k)}. \tag{15}$$

Using the relation (n = 1,2,...,)

$$\hat{L}\left[\frac{2^n z^{(n-1/2)}}{1.3.\ldots(2n-1)\sqrt{\pi}}\right] = s^{-(n+1/2)}, \tag{16}$$

we get for the kernel G(z) the following asymptotic series:

$$G(z) = \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{\beta_{j-1}(2z)^{j-1/2}}{1.3.5.\ldots(2j-1)}, \tag{17}$$

and the coefficients β_n are defined by the recursion formula ($\beta_o = a_o = 1$):

$$\beta_n = -\sum_{i=1}^{i=n} a_i\,\beta_{n-i}. \tag{18}$$

The first term in the series (17) results in the known (e.g. /2/) inversion formula for nonrelativistic mechanics, expressed by the Abelian integral transformation

$$x(z)_{nonrel.} = \frac{\sqrt{2}\,c}{4\pi}\int_0^z \frac{dy\,T(y)}{\sqrt{z-y}}. \tag{19}$$

Inserting the constant value of the period $T = 2\pi/\omega$, eqn.(19) gives the oscillator-potential. Taking the next terms in (17) into account we get the potential with the constant period:

$$\phi(x) = \frac{m_o\omega^2x^2}{2}\left[1+\frac{\omega^2x^2}{16c^2}\right] + \mathcal{O}\left[\frac{1}{c^4}\right]. \tag{20}$$

Defining the oscillator as a potential admitting constant values of the period for all energies, we thus find this potential in the relativistic (kinematic) case from the inversion formula (11).

APPENDIX

The Laplace transform of T(y) is

$$\hat{T}(s) = \int_0^{\infty} dy\, e^{-sy}\int_0^y r(z)\,\frac{(1+y-z)\,dz}{\sqrt{2(y-z)+(y-z)^2}}.$$

Changing the succession of integration, it can be written as

$$\int_0^{\infty} dz\, r(z)\int_0^{\infty} dy\, e^{-sy}\,\frac{1+y-z}{\sqrt{2(y-z)+(y-z)^2}}.$$

In order to factorize these integrals we set $\xi = y-z$ in the second one and get

$$\int_0^{\infty} dz\, r(z)\,e^{-sz}\int_0^{\infty} d\xi\, e^{-s\xi}\,\frac{1+\xi}{\sqrt{2\xi+\xi^2}}.$$

The first integral by definition is $\hat{r}(s)$, and the second can be expressed analytically and gives $e^sK_1(s)$ so that formula (9) is proved.

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