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институт
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исследований
дубна

E4-89-187

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MICROSCOPIC BOSON APPROACH TO NUCLEAR COLLECTIVE MOTION

Submitted to "Nuclear Physics A"

1989

1. Introduction

One of the most challenging problems in the microscopic theory of nuclear structure is to develop a suitable approximation scheme for describing collective motion[1]. The basic difficulty consists in the separation of the microscopic many-body Hilbert space into a collective and noncollective subspace. Such a separation requires that the hamiltonian of the system provides no serious coupling between the collective subspace and its orthogonal complement. Namely, the collective subspace must be an (approximate) <u>invariant subspace</u> of the hamiltonian. In group-theoretical models[2-5], the hamiltonian is artificially devised from the outset so as to automatically fulfill this "maximal-decoupling" condition. In actual nuclei, however, the choice of the proper collective subspace satisfying the maximal-decoupling condition is quite a formidable problem because of characteristic effects of the shell structure as well as the large number of nucleon states contributing to the collective mode. Usually, the most important collective degrees of freedom can only be "guessed" on the basis of physical intuition. A typical example is the interacting boson approximation (IBA)[6-8], which has achieved considerable success in nuclear applications[9] and whose microscopic justification is currently one of the most exciting areas of nuclear structure physics[10-14]. Once the fundamental collective degrees of freedom are specified, the actual structure of the collective subspace can be determined by several methods[15-20], based predominantly on the variational principle.

Besides the above "heuristic" approach to the collective subspace, much effort has also been devoted to formulating a <u>theory</u> of an optimally decoupled collective motion[21-31], which aims at specifying the proper collective subspace from the dynamics of the system (rather than assuming it a priori). The attempts in this direction have been undertaken within the framework of various "semiclassical" methods[21-28], but recently, there have also appeared some works dealing with the fully quantum mechanical formulation[29-31].

In this paper we try to combine both the "heuristic" and the "theoretical" approach. Our basic idea is as follows. We start from the nuclear shell-model hamiltonian with appropriate single-particle energies and an effective two-body nucleon-nucleon interaction. Instead of extracting the collective degrees of freedom in the fermion space we first transcribe the original shell-model hamiltonian in terms of many interacting bosons and study the whole problem in the boson space. There are several reasons that make the use of a boson representation advantageous:

1) it is known from experiment that the low-energy collective excitations of nuclei are approximately bosonic in nature ;
2) the bosons, if suitably chosen, may represent rather complicated fermion configurations, thereby incorporating a good deal of the fermion dynamics in a simple manner[20,32,33] ;
3) the boson formulation enables us to make contact with phenomenological models[34,35] in terms of which we have most of our physical insight into the nature of collective states ;
4) the use of boson operators acting on the boson space makes the calculations simpler.

Among various bosonization schemes[36-43] we have chosen the Dyson representation[42,43] because it yields a hamiltonian with at most quartic terms in the boson operators and it has proved to offer a powerful method for the description of nuclear collective motion[44-49]. Once the boson hamiltonian is constructed, we diagonalize its one-body part. This diagonalization enables us to identify (in the first approximation) the lowest-energy eigenvectors with the collective degrees of freedom. The two-body part of the boson hamiltonian then naturally contains terms which couple these collective bosons with the noncollective (higher-energy) ones. As a second step in our way to determining the "true" collective subspace (i.e. an approximate invariant subspace of the hamiltonian) we introduce a canonical transformation which eliminates the above coupling terms in the boson hamiltonian. This means that the collective bosons determined in the first step generate an approximate invariant subspace of the transformed boson hamiltonian. By carrying out an inverse transformation of the collective part of this transformed hamiltonian, we obtain the original boson hamiltonian in terms of new bosons which contain both collective and noncollective components. These new (renormalized) bosons then span an approximate invariant subspace of the original hamiltonian, and therefore, they can be regarded as the true collective degrees of freedom. In this sense our approach is "theoretical" because it utilizes nothing but the dynamical decoupling condition. On the other hand, we assume that the low-energy eigenvectors of the one-body part of the boson hamiltonian already constitute the main components of the true collective bosons. In this sense our approach is "heuristic".

The overall ideology of the present paper is rather close to that of refs. [29-31]. The principal aim is to formulate and test a quantum mechanical method for determining an optimally decoupled subspace which would be able to describe collective motion in nuclei. However, there are some differences in the way of achieving the goal as well as in details of the formulation. In ref. [29], the guiding dynamical principle to determine the collective subspace is the invariance of the equations of motion under the transformation into the collective representation. In refs. [30,31] the dynamical collective subspace is determined so as to satisfy a stationary condition with respect to variations toward the noncollective degrees of freedom. In our approach the basic dynamical condition is the explicit elimination of coupling terms in the hamiltonian expressed in terms of both the collective and noncollective operators. The essential point of all the three approaches is the nonlinearity of the transformation to the collective operators. In practice, the corresponding expansions must be truncated at a rather low order [29-31], including only the first nonlinear terms. In our approach we use a specific kind of the mean-field approximation (MFA)[19,20,49] which enables us to take into account approximately all the higher-order terms neglected in refs.[29-31]. The MFA acts coherently with the decoupling condition because it gives rise to some averaging of collective motion with respect to the noncollective variables. The last point in which the present approach differs from that of refs.[29-31] concerns the unphysical (spurious) boson states violating the Pauli principle. The authors of refs.[30,31] do not worry about this problem at all and in ref.[29] a claim is made that the low-lying collective states are not expected to contain essential contributions from spurious components. We discuss this problem in more detail and we particularly point out its relation to the quality of decoupling.

The paper is organized as follows. In sect. 2 we present the relevant formalism along the lines given above. In sect. 3 we apply the proposed method to the multi-level pairing hamiltonian and give explicit results for the even isotopes of Sn, Ni and Pb.
Sect. 4 contains a summary and perspectives.

2. Outline of the method

2.1. BOSONIZATION OF THE FERMION HAMILTONIAN

We consider a system of an even number of identical nucleons moving in several nondegenerate shell-model orbits. The nuclear hamiltonian within the model space is assumed to have the form

$$\hat{H} = \sum_i \varepsilon_i a_i^+ a_i + \sum_{ijkl} V_{ijkl}\, a_i^+ a_j^+ a_l a_k \quad , \qquad (1)$$

where a_i^+ , a_i are the creation and annihilation operators of the valence nucleons in the single-particle states $i \equiv (n_i, l_i, j_i, m_i)$,

ε_i stand for the corresponding single-particle energies and the coefficients V_{ijkl} represent the antisymmetrized matrix elements of an effective two-body nucleon-nucleon interaction. The indices i,j,k,l in (1) run over a complete set of single-particle states within the selected model space. The hamiltonian (1) is assumed to have the usual properties with respect to the space rotation, time-reversal and hermitean conjugation. In particular, the coefficients V_{ijkl} are required to satisfy the relation $V_{ijkl} = V_{klij}$. As explained in the introduction, it is convenient to analyze the original fermion problem in the boson space. For this purpose we introduce the boson creation and annihilation operators b^+_{ij} and b_{ij} , which satisfy the following antisymmetry and commutation relations

$$b^+_{ij} = - b^+_{ji} \quad , \tag{2}$$

$$\left.\begin{array}{l} [b_{ij}, b_{kl}] = [b^+_{ij}, b^+_{kl}] = 0 \\ [b_{ij}, b^+_{kl}] = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \end{array}\right\} . \tag{3}$$

These boson operators, together with the boson vacuum $|0)_B$, $b_{ij}|0)_B = 0$, define the relevant boson space within which we intend to work in the following. The fermion pair operators $a^+_i a^+_j$, $a_j a_i$ and $a^+_i a_j$ can be mapped onto the corresponding boson images according to the famous generalized Dyson prescription[42,43])

$$a^+_i a^+_j \rightarrow (a^+_i a^+_j)_D = P^+_{ij} = b^+_{ij} - \sum_{mn} b^+_{im} b^+_{jn} b_{mn} \quad , \tag{4a}$$

$$a_j a_i \rightarrow (a_j a_i)_D = b_{ij} \quad , \tag{4b}$$

$$a^+_i a_j \rightarrow (a^+_i a_j)_D = \sum_k b^+_{ik} b_{jk} \quad . \tag{4c}$$

The superiority of the Dyson mapping (DM) over other bosonization procedures[36-41]) lies in the finiteness of the boson expansion. On the other hand, the DM has an outward demerit that the transformation from the fermion to the boson space is nonunitary$\left[(a_j a_i)^+_D \neq (a^+_i a^+_j)_D\right]$, so that the mapped boson hamiltonian is nonhermitean. This fact has long been the major impediment for the practical use of the DM. Today, however, the difficulty associated with the nonhermiticity of the Dyson boson hamiltonian is completely solved[50-52]), and thus, one can fully exploit the advantage of the DM as to its finiteness. Moreover, it has been shown [53-56] that the Dyson representation provides a very fruitful and promising framework for the description of nuclear collective motion.

Using the expressions (4a) - (4c) , we can easily write down the Dyson boson image $\hat{H}_D$ of the shell-model hamiltonian (1) :

$$\hat{H}_D = \sum_{ijkl} (\epsilon_i \delta_{ik} \delta_{lj} + V_{ijkl}) b^+_{ij} b_{kl} - \sum_{ijklmn} V_{ijkl}\, b^+_{im}\, b^+_{jn}\, b_{mn} b_{kl} \quad . \tag{5}$$

In accordance with the structure of $\hat{H}_D$ we can define a set of new operators B^+_α , B_α such that the following relations hold

$$B^+_\alpha = \frac{1}{\sqrt{2}} \sum_{ij} \phi^\alpha_{ij}\, b^+_{ij} \quad , \qquad \phi^\alpha_{ij} = -\phi^\alpha_{ji} \, , \qquad \phi^{\alpha *}_{ij} = \phi^\alpha_{ij} \quad , \tag{6a}$$

$$[B_\alpha , B^+_{\alpha'}] = \delta_{\alpha\alpha'} \quad , \tag{6b}$$

$$\hat{H}_D\, B^+_\alpha |0)_B = E_\alpha\, B^+_\alpha |0)_B \quad . \tag{6c}$$

That is, the operators B^+_α create bosons which are linear combinations of b^+_{ij} , and the one-boson state $B^+_\alpha |0)_B$ is an eigenstate of $\hat{H}_D$. Substituting (5) and (6a) into (6c) , we obtain

$$(\epsilon_i + \epsilon_j)\, \phi^\alpha_{ij} + 2 \sum_{kl} V_{ijkl}\, \phi^\alpha_{kl} = E_\alpha\, \phi^\alpha_{ij} \quad . \tag{7}$$

Together with the normalization relation $\sum_{ij} \phi^\alpha_{ij}\, \phi^\alpha_{ij} = 1$, this is a standard linear eigenvalue problem. The symbol α with values 1,2,...,M (M=number of independent pairs of indices i,j) is used to label the bosons B^+_α according to their energies E_α as follows: $E_1 \leq E_2 \leq \ldots$. The boson B^+_1 is thus the most collective one, being characterized by almost equal contributions from a large number of b^+_{ij}-bosons. The higher is the index α , the lower is the degree of collectivity of B^+_α , i.e. there occurs a definite pair of indices $(i,j) = (k,l)$, for which $\phi^\alpha_{kl} \approx 1$, while for $(i,j) \neq (k,l)$ the coefficients ϕ^α_{ij} tend to zero. Of course, details of this behaviour depend on the hamiltonian as well as on the single-particle levels considered. In order to be able to express $\hat{H}_D$ in terms of the boson operators B^+_α , B_α , we demand that the transformation (6a) be unitary, i.e.

$$\sum_{ij} \phi^{\alpha}_{ij} \phi^{\alpha'}_{ij} = \delta_{\alpha\alpha'} , \tag{8a}$$

$$\sum_{\alpha} \phi^{\alpha}_{ij} \phi^{\alpha}_{kl} = \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) . \tag{8b}$$

The condition (8a) is equivalent to the requirement (6b) . The most important consequence of the relation (8b) is that it enables one to invert (6a) ,

$$b^{+}_{ij} = \sqrt{2} \sum_{\alpha} \phi^{\alpha}_{ij} B^{+}_{\alpha} . \tag{9}$$

Using (7) , (8) and (9) , we obtain the boson hamiltonian (5) in the form

$$\hat{H}_{D} = \sum_{\alpha} E_{\alpha} B^{+}_{\alpha} B_{\alpha} + \sum_{\alpha\beta\gamma\delta} W_{\alpha\beta\gamma\delta} B^{+}_{\alpha} B^{+}_{\beta} B_{\gamma} B_{\delta} , \tag{10}$$

where

$$W_{\alpha\beta\gamma\delta} = -4 \sum_{ijklmn} V_{ijkl} \phi^{\alpha}_{im} \phi^{\beta}_{jn} \phi^{\gamma}_{mn} \phi^{\delta}_{kl} \tag{11}$$

and ϕ^{α}_{ij} , E_{α} are all possible solutions of eq. (7) . As is seen from (10) , the effect of the transformation (6a) under the condition (6c) is to diagonalize the one-body part of the boson hamiltonian (5) . Since the new bosons B^{+}_{α} already incorporate some of the nucleon correlations [see eqs. (6a) , (7)] , the interaction between the B^{+}_{α} bosons is expected to be relatively weak when compared to their unperturbed energies E_{α} . To be more precise, let us denote the typical value of the coupling strength in (10) by W. Let further $\Delta E = E_{n} - E_{c}$ be the typical value of the energy difference between the noncollective and collective bosons. In the concrete numerical examples discussed in this paper we then find

$$\varepsilon = \frac{W}{\Delta E} \sim 0.1 - 0.2 . \tag{12}$$

2.2. EXTRACTION OF THE COLLECTIVE DEGREES OF FREEDOM

The Dyson hamiltonian (10) acts on the boson space $\mathcal{B} : \{|N;\{\alpha\})_{B}\}$ spanned by a set of states

$$|N;\{\alpha\})_{B} = B^{+}_{\alpha_1} B^{+}_{\alpha_2} \ldots B^{+}_{\alpha_N} |0)_{B} , \quad \alpha_i \in \mathcal{M} = \{1,2,\ldots,M\} , \tag{13}$$

where $N = \frac{n}{2}$ is half the number of valence nucleons considered in the original fermion problem and the symbol $\{\alpha\}$ stands for the set $\{\alpha_1, \alpha_2, \dots, \alpha_N\}$. Let M_c be the minimum kinds of boson operators necessary for describing the collective states. Let further $\mathcal{M}_c$, $\mathcal{M}_n$ be two subsets of $\mathcal{M}$ such that

$$\mathcal{M}_c = \{1, 2, \dots, M_c\} ,$$
$$\mathcal{M}_n = \{M_c+1, \dots, M\} . \tag{14}$$

Accordingly, we can separate the boson operators $\mathcal{B}^+_\alpha$, $\mathcal{B}_\alpha$ into the collective $(\alpha \in \mathcal{M}_c)$ and noncollective $(\alpha \in \mathcal{M}_n)$ ones. In the following the collective indices α, β, ... $\in \mathcal{M}_c$ will be denoted by c_1, c_2, ..., while those belonging to the set $\mathcal{M}_n$ (the noncollective ones) will be denoted by n_1, n_2, ... Now, we introduce two orthogonal subspaces of the boson space $\mathcal{G}$,

$$\mathcal{G}_c : \{|N; \{c\})_B\} , \qquad \mathcal{G}_n : \{|N; \{n\})_B\} , \qquad \mathcal{G}_c \oplus \mathcal{G}_n = \mathcal{G} ,$$

whose basis states are given by

$$|N; \{c\})_B = \mathcal{B}^+_{c_1} \mathcal{B}^+_{c_2} \dots \mathcal{B}^+_{c_N} |0)_B , \qquad c_i \in \mathcal{M}_c \tag{15}$$

$$|N; \{n\})_B = \mathcal{B}^+_{n_1} \dots \mathcal{B}^+_{n_k} \mathcal{B}^+_{c_{k+1}} \dots \mathcal{B}^+_{c_N} |0)_B , \quad n_i \in \mathcal{M}_n, c_j \in \mathcal{M}_c, k \geq 1 . \tag{16}$$

Hereafter, $\mathcal{G}_c$ will be called the <u>collective subspace</u>, while its orthogonal complement $\mathcal{G}_n$ will be called the <u>noncollective subspace</u>. Note that the number of collective bosons (M_c) remains unspecified at the present stage. To determine it, we are guided by the properties of the solutions of eqs. (7) in the concrete physical situation. This is the "heuristic" feature of our approach.

Let us now consider the matrix elements of the Dyson hamiltonian (10) between the collective states (15). Since the hamiltonian $\hat{H}_D$ is at most two-body, all that one has to know in order to evaluate its matrix elements between any two states of the form (15) are the commutators

$$[\hat{H}_D, \mathcal{B}^+_c] |0\rangle_B = E_c \mathcal{B}^+_c |0)_B , \tag{17a}$$

$$_B(0| [\mathcal{B}_c, \hat{H}_D] = E_c\, {}_B(0| \mathcal{B}_c , \tag{17b}$$

$$[[\hat{H}_D, B^+_{c_1}], B^+_{c_2}] = 2\sum_{c_3c_4} W_{c_3c_4c_1c_2} B^+_{c_3} B^+_{c_4} + 2\sum_{cn} W_{cnc_1c_2} B^+_c B^+_n + 2\sum_{cn} W_{ncc_1c_2} B^+_n B^+_c + 2\sum_{n_1n_2} W_{n_1n_2c_1c_2} B^+_{n_1} B^+_{n_2} , \tag{18a}$$

$$[B_{c_1}, [B_{c_2}, \hat{H}_D]] = 2\sum_{c_3c_4} W_{c_1c_2c_3c_4} B_{c_3} B_{c_4} + 2\sum_{cn} W_{c_1c_2cn} B_c B_n + 2\sum_{cn} W_{c_1c_2nc} B_n B_c + 2\sum_{n_1n_2} W_{c_1c_2n_1n_2} B_{n_1} B_{n_2} . \tag{18b}$$

It is seen from the double commutators (18) that the hamiltonian $\hat{H}_D$ by acting on the pure collective states (15) produces new states which have non-zero components both in the collective subspace (15) and in its orthogonal complement (16). In other words, the collective subspace spanned by the states (15) is <u>not</u> an invariant subspace of the hamiltonian $\hat{H}_D$. The coupling between the collective and noncollective subspaces is caused by the terms

$$\sum_{cn} W_{cnc_1c_2} B^+_c B^+_n , \sum_{cn} W_{ncc_1c_2} B^+_n B^+_c , \sum_{n_1n_2} W_{n_1n_2c_1c_2} B^+_{n_1} B^+_{n_2} ,$$
$$\sum_{cn} W_{c_1c_2cn} B_c B_n , \sum_{cn} W_{c_1c_2nc} B_n B_c , \sum_{n_1n_2} W_{c_1c_2n_1n_2} B_{n_1} B_{n_2} , \tag{19}$$

appearing in the double commutators (18a) and (18b).

In order to find the true collective subspace with the property of invariance under the action of $\hat{H}_D$, we have to specify new operators in terms of which the hamiltonian $\hat{H}_D$ does not contain the coupling terms. To achieve this, we first introduce a canonical transformation

$$\hat{H}_D \rightarrow \hat{\mathcal{H}}_D = e^{\hat{G}} \hat{H}_D e^{-\hat{G}} \tag{20}$$

with the generator

$$\hat{G} = \sum_{\alpha\beta\gamma\delta} g_{\alpha\beta\gamma\delta} B^+_\alpha B^+_\beta B_\gamma B_\delta \tag{21}$$

determined so that the transformed hamiltonian $\hat{\mathcal{H}}_D$ conserves (up to a certain order) the subspace spanned by the states (15). Using the formula

$$e^{\hat{G}}\,\hat{H}_D\,e^{-\hat{G}} = \hat{H}_D - [\hat{H}_D,\hat{G}] + \tfrac{1}{2}[[\hat{H}_D,\hat{G}],\hat{G}] - \dots \,, \tag{22}$$

we get

$$\hat{\mathcal{H}}_D = \sum_{\alpha} E_\alpha B^+_\alpha B_\alpha + \sum_{\alpha\beta\gamma\delta} G_{\alpha\beta\gamma\delta}\, B^+_\alpha B^+_\beta B_\gamma B_\delta + \dots \,, \tag{23a}$$

where

$$G_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta} - g_{\alpha\beta\gamma\delta}\,(E_\alpha + E_\beta - E_\gamma - E_\delta) \tag{23b}$$

and the dots represent the three- and more-body boson interactions which will be neglected in this paper. We now require that the one- plus two-body part of $\hat{\mathcal{H}}_D$, given explicitly in (23a), does not couple the collective subspace (15) to the rest of the boson space, i.e. that $\hat{\mathcal{H}}_D$ satisfies the following commutation relations

$$[[\hat{\mathcal{H}}_D, B^+_{c_1}], B^+_{c_2}] = 2\sum_{c_3 c_4} G_{c_3 c_4 c_1 c_2}\, B^+_{c_3} B^+_{c_4} \,, \tag{24a}$$

$$[B_{c_1},[B_{c_2},\hat{\mathcal{H}}_D]] = 2\sum_{c_3 c_4} G_{c_1 c_2 c_3 c_4}\, B_{c_3} B_{c_4} \,. \tag{24b}$$

It is clear from (18a,b) that this requirement can be achieved by putting

$$G_{cnc_1c_2} = G_{ncc_1c_2} = G_{n_1n_2c_1c_2} = G_{c_1c_2cn} = G_{c_1c_2nc} = G_{c_1c_2n_1n_2} = 0 \,, \tag{25}$$

i.e. by eliminating the coupling terms of the form (19) . Substituting (23b) into (25) , we obtain for the coefficients $g_{\alpha\beta\gamma\delta}$, determining the generator (21) , the following expression

$$g_{\alpha\beta\gamma\delta} = \frac{W_{\alpha\beta\gamma\delta}}{E_\alpha + E_\beta - E_\gamma - E_\delta} \,, \tag{26}$$

where the quadruplet of indices $(\alpha\beta\gamma\delta)$ belongs to the set

$$\mathcal{R} = \{(cnc_1c_2),(ncc_1c_2),(n_1n_2c_1c_2),(c_1c_2cn),(c_1c_2nc),(c_1c_2n_1n_2)\} \,. \tag{27}$$

Note that due to the nonhermiticity of the Dyson hamiltonian $\hat{H}_D$ [$W_{\alpha\beta\gamma\delta} \neq W_{\gamma\delta\alpha\beta}$ - see eq. (11)] , the transformation induced by the operator $e^{\hat{G}}$ is not unitary.

With $\hat{G}$ specified by (21), (26) and (27) , the transformed hamiltonian $\hat{\mathcal{H}}_D = e^{\hat{G}}\,\hat{H}_D\,e^{-\hat{G}}$ contains no two-body terms that couple the collective and noncollective subspaces, spanned by the states (15) and (16) , respectively. In other words, the one- plus two-body part $\hat{\mathcal{H}}_D^{\mathrm{I}+\mathrm{II}}$ of the hamiltonian $\hat{\mathcal{H}}_D$ satisfies the commutation relations (24a,b) , which means that the collective subspace (15) is left invariant under the action of $\hat{\mathcal{H}}_D^{\mathrm{I}+\mathrm{II}}$. Of course, $\hat{\mathcal{H}}_D$ contains the three- and more-body coupling terms as well (they are marked by the dots in eq. (23a)) . However, the contribution of these terms is expected to be rather small, because they appear proportional to εW , $\varepsilon^2 W$, ... , so that their magnitude falls down quite rapidly [see (12)] . Nevertheless, at least the three-body terms might be important in certain cases and some results on this point will be reported soon.

Now, with the use of the generator $\hat{G}$, we can construct the transform $\hat{\mathcal{X}}$ of any operator $\hat{X}$,

$$\begin{aligned} \hat{\mathcal{X}} = e^{\hat{G}} \hat{X} e^{-\hat{G}} &= \hat{X} - [\hat{X}, \hat{G}] + \tfrac{1}{2!}[[\hat{X}, \hat{G}], \hat{G}] - \dots \\ &= \sum_{k=0}^{\infty} \frac{(-)^k}{k!} [\dots[\hat{X}, \underbrace{\hat{G}], \dots, \hat{G}]}_{k} \quad . \end{aligned} \tag{28}$$

In particular, for $\hat{X} = B_\alpha^+$ we have

$$[B_\alpha^+, \hat{G}] = -\sum_{\alpha_1\alpha_2\alpha_3}{}' (g_{\alpha_1\alpha_2\alpha_3\alpha} + g_{\alpha_1\alpha_2\alpha\alpha_3}) B_{\alpha_1}^+ B_{\alpha_2}^+ B_{\alpha_3} , \tag{29a}$$

$$[[B_\alpha^+, \hat{G}], \hat{G}] = \sum_{\alpha_1\alpha_2\alpha_3} u_{\alpha\alpha_1\alpha_2\alpha_3} B_{\alpha_1}^+ B_{\alpha_2}^+ B_{\alpha_3} + \sum_{\alpha_1\dots\alpha_5} v_{\alpha\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} B_{\alpha_1}^+ B_{\alpha_2}^+ B_{\alpha_3}^+ B_{\alpha_4} B_{\alpha_5} , \tag{29b}$$

$$\vdots$$

where

$$u_{\alpha\alpha_1\alpha_2\alpha_3} = \sum_{\beta\gamma}{}' (g_{\beta\gamma\alpha_3\alpha} + g_{\beta\gamma\alpha\alpha_3})(g_{\alpha_1\alpha_2\gamma\beta} + g_{\alpha_1\alpha_2\beta\gamma}) , \tag{30a}$$

$$\begin{aligned} v_{\alpha\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = 2\sum_{\sigma}{}' \{ &(g_{\sigma\alpha_3\alpha_5\alpha} + g_{\sigma\alpha_3\alpha\alpha_5})(g_{\alpha_1\alpha_2\alpha_4\sigma} + g_{\alpha_1\alpha_2\sigma\alpha_4}) \\ &- (g_{\alpha_1\alpha_2\sigma\alpha} + g_{\alpha_1\alpha_2\alpha\sigma}) g_{\alpha_3\sigma\alpha_4\alpha_5} \} , \end{aligned} \tag{30b}$$

the quantities $g_{\alpha\beta\gamma\delta}$ are given by (26) and all summations $\sum'$ are restricted in such a way that, for a given α , all indices attached to the g 's belong to the set $\mathcal{R}$ given by (27) . It is seen from (28) and (29) that the transformed bosons

$$\mathcal{B}_\alpha^+ = e^{\hat{G}} B_\alpha^+ e^{-\hat{G}} \tag{31}$$

are complicated many-body operators involving infinite expansions in terms of $\mathcal{B}^+_\lambda$, $\mathcal{B}_\lambda$. Hence approximations are necessary in practical applications. In this paper we follow the approximation which has been suggested in ref. [57] and applied to some realistic problem in ref. [14] . It consists in replacing the operators of type $\mathcal{B}^+\mathcal{B}$ and $\mathcal{B}^+\mathcal{B}^+\mathcal{B}\mathcal{B}$ in (29a,b) by their expectation values in an appropriate state. More precisely, we make the replacement

$$\mathcal{B}^+_{\lambda_1}\mathcal{B}^+_{\lambda_2}\mathcal{B}_{\lambda_3} \rightarrow \frac{1}{2}\left\{\langle\mathcal{B}^+_{\lambda_2}\mathcal{B}_{\lambda_3}\rangle\mathcal{B}^+_{\lambda_1} + \langle\mathcal{B}^+_{\lambda_1}\mathcal{B}_{\lambda_3}\rangle\mathcal{B}^+_{\lambda_2}\right\}, \tag{32a}$$

$$\begin{aligned}\mathcal{B}^+_{\lambda_1}\mathcal{B}^+_{\lambda_2}\mathcal{B}^+_{\lambda_3}\mathcal{B}_{\lambda_4}\mathcal{B}_{\lambda_5} \rightarrow \frac{1}{3}\Big\{&\langle\mathcal{B}^+_{\lambda_2}\mathcal{B}^+_{\lambda_3}\mathcal{B}_{\lambda_4}\mathcal{B}_{\lambda_5}\rangle\mathcal{B}^+_{\lambda_1} + \\ &+\langle\mathcal{B}^+_{\lambda_1}\mathcal{B}^+_{\lambda_3}\mathcal{B}_{\lambda_4}\mathcal{B}_{\lambda_5}\rangle\mathcal{B}^+_{\lambda_2} + \langle\mathcal{B}^+_{\lambda_1}\mathcal{B}^+_{\lambda_2}\mathcal{B}_{\lambda_4}\mathcal{B}_{\lambda_5}\rangle\mathcal{B}^+_{\lambda_3}\Big\},\end{aligned} \tag{32b}$$

where

$$\langle\hat{X}\rangle = \frac{1}{N!}\,{}_B(0|(\beta)^N\;\hat{X}\,(\beta^+)^N|0)_B \tag{33}$$

with

$$\beta^+ = \sum_\alpha \xi_\alpha\,\mathcal{B}^+_\alpha\;,\qquad \sum_\alpha|\xi_\alpha|^2 = 1 \tag{34}$$

and $N = \frac{n}{2}$ is the number of bosons (equal to half the number of valence nucleons) . The coefficients ξ_α are determined variationally by minimizing the expectation value of the boson hamiltonian (10) in the state $\frac{1}{\sqrt{N!}}(\beta^+)^N|0)_B$. This boson condensate is an example of the (number-projected) coherent state [58,59] which has proved to be very useful in treating the many-boson systems [19,20,49,60-65].
Under the approximation (32) , we obtain from (29b) the following approximate expression for the double commutator

$$[[\mathcal{B}^+_\alpha,\hat{\mathcal{C}}],\hat{\mathcal{C}}] \cong \eta_\alpha\,\mathcal{B}^+_\alpha\;, \tag{35a}$$

where

$$\begin{aligned}\eta_\alpha = \;&\frac{1}{2}\sum_{\lambda_1\lambda_2}{}'\,(u_{\alpha\lambda_1\lambda_2\alpha} + u_{\alpha\lambda_2\alpha\lambda_1})\langle\mathcal{B}^+_{\lambda_1}\mathcal{B}_{\lambda_2}\rangle + \\ &+\frac{1}{3}\sum_{\lambda_1\lambda_2\lambda_3\lambda_4}(v_{\alpha\lambda_1\lambda_2\lambda_3\lambda_4\alpha} + v_{\alpha\lambda_1\lambda_2\lambda_3\alpha\lambda_4} + v_{\alpha\lambda_1\lambda_2\alpha\lambda_3\lambda_4}) \\ &\times\langle\mathcal{B}^+_{\lambda_1}\mathcal{B}^+_{\lambda_2}\mathcal{B}_{\lambda_3}\mathcal{B}_{\lambda_4}\rangle\end{aligned} \tag{35b}$$

with $u_{\dots}$ and $v_{\dots\dots}$ given by (30a) and (30b) , respectively.

Within the approximation (35) we have

$$[\cdots[\underbrace{\mathcal{B}_\alpha^+,\hat{G}],\hat{G}],\ldots,\hat{G}}_{2m}] = \eta_\alpha^m\, B_\alpha^+ \quad, \tag{36a}$$

$$[\ldots[\underbrace{B_\alpha^+,\hat{G}],\hat{G}],\ldots,\hat{G}}_{2m+1}] = \eta_\alpha^m[B_\alpha^+,\hat{G}] \quad, \tag{36b}$$

so that [see (28)]

$$\mathcal{B}_\alpha^+ = e^{\hat{G}}\, B_\alpha^+\, e^{-\hat{G}} \cong \left\{\sum_{m=0}^{\infty}\frac{1}{(2m)!}\,\eta_\alpha^m\right\} B_\alpha^+ - \left\{\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\,\eta_\alpha^m\right\}[B_\alpha^+,\hat{G}] \;. \tag{37}$$

By evaluating the sums in (37) we get

$$\mathcal{B}_\alpha^+ \cong \begin{cases} \cosh(\sqrt{\eta_\alpha})B_\alpha^+ - \dfrac{1}{\sqrt{\eta_\alpha}}\sinh(\sqrt{\eta_\alpha})[B_\alpha^+,\hat{G}]\;; & \eta_\alpha \geq 0 \\ \cos(\sqrt{-\eta_\alpha})B_\alpha^+ - \dfrac{1}{\sqrt{-\eta_\alpha}}\sin(\sqrt{-\eta_\alpha})[B_\alpha^+,\hat{G}]\;; & \eta_\alpha < 0\,. \end{cases} \tag{38}$$

Using the explicit expression (29a) for $[B_\alpha^+,\hat{G}]$, we finally obtain

$$\mathcal{B}_\alpha^+ = r_\alpha B_\alpha^+ - \sum_{\alpha_1\alpha_2\alpha_3} s_{\alpha\alpha_1\alpha_2\alpha_3}\, B_{\alpha_1}^+ B_{\alpha_2}^+ B_{\alpha_3} \;, \tag{39a}$$

where

$$r_\alpha = \begin{cases} \cosh(\sqrt{\eta_\alpha})\;; & \eta_\alpha \geq 0 \\ \cos(\sqrt{-\eta_\alpha})\;; & \eta_\alpha < 0\;, \end{cases} \tag{39b}$$

$$s_{\alpha\alpha_1\alpha_2\alpha_3} = -\left(g_{\alpha_1\alpha_2\alpha_3\alpha} + g_{\alpha_1\alpha_2\alpha\alpha_3}\right)\begin{cases} \dfrac{1}{\sqrt{\eta_\alpha}}\sinh(\sqrt{\eta_\alpha})\;; & \eta_\alpha \geq 0 \\ \dfrac{1}{\sqrt{-\eta_\alpha}}\sin(\sqrt{-\eta_\alpha})\;; & \eta_\alpha < 0\,. \end{cases} \tag{39c}$$

In a completely analogous way we get

$$\mathcal{B}_\alpha = \bar{r}_\alpha B_\alpha - \sum_{\alpha_1\alpha_2\alpha_3} \bar{s}_{\alpha\alpha_1\alpha_2\alpha_3}\, B_{\alpha_1}^+ B_{\alpha_2} B_{\alpha_3} \;. \tag{40}$$

Since the transformation $e^{\hat{G}}$ is not unitary, one has in general $(\mathcal{B}_\alpha^+)^+ \neq \mathcal{B}_\alpha$, i.e. the coefficients $\bar{r}_\alpha$, $\bar{s}_{\alpha\alpha_1\alpha_2\alpha_3}$ in (40) are not the complex conjugates of r_α , $s_{\alpha\alpha_1\alpha_2\alpha_3}$ appearing in (39a).

With the aid of (31) and (10), the transformed hamiltonian $\hat{\mathcal{H}}_B$ can be written as

$$\hat{\mathcal{H}}_D = e^{\hat{\mathcal{C}}} H_D e^{-\hat{\mathcal{C}}} = \sum_\alpha E_\alpha \mathcal{B}^+_\alpha \mathcal{B}_\alpha + \sum_{\alpha\beta\gamma\delta} W_{\alpha\beta\gamma\delta} \mathcal{B}^+_\alpha \mathcal{B}^+_\beta \mathcal{B}_\gamma \mathcal{B}_\delta \ . \tag{41}$$

Substituting into (41) the expressions (39) and (40), we obtain the transformed hamiltonian in terms of the original bosons $\mathcal{B}^+_\alpha$, $\mathcal{B}_\alpha$. Since the transformation $\hat{H}_D \to \hat{\mathcal{H}}_D = e^{\hat{\mathcal{C}}} \hat{H}_D e^{-\hat{\mathcal{C}}}$ has been designed so as to eliminate the two-body coupling between the collective and noncollective subspaces spanned by the states (15) and (16), respectively, we have

$$\hat{\mathcal{H}}_D = \hat{\mathcal{H}}_D^{coll} + \hat{\mathcal{H}}_D^{noncoll} + \Delta \hat{\mathcal{H}}_D \ , \tag{42}$$

where

$$\hat{\mathcal{H}}_D^{coll} = \sum_c \widetilde{E}_c \mathcal{B}^+_c \mathcal{B}_c + \sum_{c_1c_2c_3c_4} \widetilde{W}_{c_1c_2c_3c_4} \mathcal{B}^+_{c_1} \mathcal{B}^+_{c_2} \mathcal{B}_{c_3} \mathcal{B}_{c_4} \tag{43}$$

is a one- plus two-body hamiltonian acting on the collective subspace (15), $\hat{\mathcal{H}}_D^{noncoll}$ is a one- plus two-body hamiltonian acting on the noncollective subspace (16) and $\Delta \hat{\mathcal{H}}_D$ involves all the three- and more- body boson terms. As already explained, $\Delta \hat{\mathcal{H}}_D$ is expected to have a small effect on the collective quantities and its discussion will be postponed to a forthcoming paper. The parameters $\widetilde{E}_c$, $\widetilde{W}_{c_1c_2c_3c_4}$ appearing in the collective hamiltonian (43) are given by

$$\widetilde{E}_c = E_c \bar{r}_c r_c \ , \tag{44a}$$

$$\begin{aligned}\widetilde{W}_{c_1c_2c_3c_4} = \ & W_{c_1c_2c_3c_4} r_{c_1} r_{c_2} \bar{r}_{c_3} \bar{r}_{c_4} - E_{c_4} S_{c_4c_1c_2c_3} \bar{r}_{c_4} - E_{c_1} r_{c_1} \bar{S}_{c_1c_2c_3c_4} \\ & - \sum_{d_1d_2} W_{c_1c_2d_1d_2} r_{c_1} r_{c_2} \bar{r}_{d_1} \bar{S}_{d_2d_1c_3c_4} + \sum_{d_1d_2} E_{d_1} S_{d_1c_1c_2d_2} \bar{S}_{d_1d_2c_3c_4} \\ & - \sum_{d_1d_2} W_{d_1d_2c_3c_4} r_{d_2} S_{d_1c_1c_2d_2} \bar{r}_{c_3} \bar{r}_{c_4} + \sum_{d_1\ldots d_4} W_{d_1d_2d_3d_4} r_{d_2} S_{d_1c_1c_2d_2} \bar{r}_{d_3} \bar{S}_{d_4d_3c_3c_4} \ .\end{aligned} \tag{44b}$$

The collective hamiltonian $\hat{\mathcal{H}}_D^{coll}$ thus has a form of $\hat{H}_D$ truncated to collective bosons [cf. eq. (10)], but the associated collective parameters E_c, $W_{c_1c_2c_3c_4}$ are replaced by some renormalized values $\widetilde{E}_c$, $\widetilde{W}_{c_1c_2c_3c_4}$ according to (44a,b). From the physical point of view, the source of the renormalization is the coupling between the collective and noncollective degrees of freedom. Technically, however, there are two different mechanisms giving rise to the renormalized collective parameters. The first one is the contraction of some many-body terms arising from the multiple commutators in (28), (29) . It is closely related to the presence of nonlinear terms in (39) , (40) , and therefore, it is essentially governed by the quantities $s_{\alpha\beta\gamma\delta}$, $\bar{s}_{\alpha\beta\gamma\delta}$. This feature of our approach is completely shared with the approaches presented in refs.[29-31]. The second mechanism of the renormalization has its origin in the mean-field approximation which enables one to achieve the double commutator $[[\mathcal{B}^+_\alpha, \hat{\mathcal{C}}], \hat{\mathcal{C}}]$ to be proportional to $\mathcal{B}^+_\alpha$, thereby making it possible to evaluate approximately an averaged contribution of many terms in the transformed operators. This mechanism manifests itself mainly through the coefficients r_α , $\bar{r}_\alpha$ in (39), (40) and it leads to the renormalization of the one-boson energies [see (44a)]. This is a specific feature of our approach and it is not encountered in refs.[29-31], where $r_c = \bar{r}_c = 1$.

Now, we perform an inverse transformation of eq. (42),

$$\hat{\mathcal{H}}_D \longrightarrow e^{-\hat{\mathcal{C}}}\, \hat{\mathcal{H}}_D\, e^{\hat{\mathcal{C}}} = \hat{H}_D \quad . \tag{45}$$

Neglecting $\Delta\hat{\mathcal{H}}_D$ and denoting

$$\mathbf{B}^+_\alpha = e^{-\hat{\mathcal{C}}}\, \mathcal{B}^+_\alpha\, e^{\hat{\mathcal{C}}} \quad , \tag{46}$$

$$\hat{H}^{noncoll}_D(\mathbf{B}^+_n, \mathbf{B}_n) = e^{-\hat{\mathcal{C}}}\, \hat{\mathcal{H}}^{noncoll}_D\, e^{\hat{\mathcal{C}}} \quad , \tag{47}$$

we obtain

$$\hat{H}_D = \sum_c \widetilde{E}_c \mathbf{B}^+_c \mathbf{B}_c + \sum_{c_1 c_2 c_3 c_4} \widetilde{W}_{c_1 c_2 c_3 c_4} \mathbf{B}^+_{c_1} \mathbf{B}^+_{c_2} \mathbf{B}_{c_3} \mathbf{B}_{c_4} + \hat{H}^{noncoll}_D(\mathbf{B}^+_n, \mathbf{B}_n) \, . \tag{48}$$

It is seen from (48) that in terms of the new operators $\mathbf{B}^+_\alpha$, $\mathbf{B}_\alpha$, the original boson hamiltonian $\hat{H}_D$ is expressed (up to the two-body terms) as a sum of a collective and a noncollective part without their mutual coupling. Within this approximation we can therefore regard the operators $\mathbf{B}^+_{c_i}$, $\mathbf{B}_{c_i}$ as representing the true collective degrees of freedom and the hamiltonian

$$H_D^{coll} = \sum_c \widetilde{E}_c \mathbf{B}_c^\dagger \mathbf{B}_c + \sum_{c_1c_2c_3c_4} \widetilde{W}_{c_1c_2c_3c_4} \mathbf{B}_{c_1}^\dagger \mathbf{B}_{c_2}^\dagger \mathbf{B}_{c_3} \mathbf{B}_{c_4} \tag{49}$$

as the true collective hamiltonian. The operators $\mathbf{B}_\alpha$, $\mathbf{B}_\alpha^\dagger$ satisfy the ordinary bosonic commutation relations

$$[\mathbf{B}_\alpha , \mathbf{B}_\beta^\dagger] = \delta_{\alpha\beta} , \tag{50}$$

because the transformation (46) is canonical. On the other hand, $\mathbf{B}_\alpha \neq (\mathbf{B}_\alpha^\dagger)^\dagger$, in general, because this transformation is not unitary. Within our approximations, the true collective bosons $\mathbf{B}_c^\dagger$ can be written as

$$\mathbf{B}_c^\dagger = r_c B_c^\dagger + \sum_{d_1d_2d_3} s_{c\,d_1d_2d_3} B_{d_1}^\dagger B_{d_2}^\dagger B_{d_3} . \tag{51}$$

Due to the nonlinear terms, they contain both the collective $(B_c^\dagger)$ and the noncollective $(B_n^\dagger)$ original bosons, thereby incorporating the effect of the latter on the true collective subspace (defined as an invariant subspace of the Dyson hamiltonian $\hat{H}_D$).

The general program we intend to carry out is as follows. First we choose a realistic but simple fermion problem which allows for an exact solution. We map exactly the fermion problem onto the boson one and find the one-boson eigenstates of the mapped hamiltonian. Being guided by the properties of these eigenstates, we choose the "first-round" candidates for the creation operators of the collective degrees of freedom, i.e. we specify the set of indices $c_i = 1, 2, \dots, M_c$ [cf. (14)]. Then, using the procedure described above, we calculate the renormalized parameters $\widetilde{E}_{c_i}$, $\widetilde{W}_{c_1c_2c_3c_4}$ according to the formulae (44a), (44b). With these parameters we construct the collective boson hamiltonian (49) (or equivalently its transformed version (43)), diagonalize it on the respective boson space and compare the resulting eigenvalues with the exact ones. Of course, this comparison makes sense only for the lowest-energy states because the higher-energy ones can hardly be considered as collective. If a good agreement is obtained, one can be reasonably sure that there indeed occurs a decoupling of the selected collective subspace from the rest of the many-body Hilbert space. If the agreement is not good, either some of the approximations involved in the treatment are unjustified or (perhaps more probably) the preliminary structure of the collective subspace inferred from the one-boson system is oversimplified. One then tries to enlarge the assumed "first approximation" collective subspace and to perform the calculations anew. In this way, the present approach is able to provide information not only about the dynamics of the system (which is described by the collective hamiltonian) but also about the structure of the collective subspace. Before carrying out the outlined program in practice, however, another serious obstacle has to be overcome.

2.3. APPROXIMATE REMOVAL OF UNPHYSICAL BOSON STATES

It is well known[1,36-49] that the diagonalization of any boson image of a fermion hamiltonian in the boson basis produces not only the eigenvectors corresponding to actual states of the underlying fermion system but also the unphysical (spurious) ones. These spurious states have no physical meaning because they are associated with the overcompleteness of the boson basis with respect to the space available for fermions. Provided an <u>exact</u> boson image of the fermion hamiltonian is diagonalized (e.g. that given by (10)), the physical and unphysical boson eigenstates are strictly separated from each other and various recipes have been given to identify them[66-70]. However, this is not the case when some truncation of the exact boson hamiltonian is made. Then, the physical and unphysical boson states are no longer well separated, i.e. all eigenstates of the truncated (collective) boson hamiltonian contain in general both physical and unphysical components. The unphysical components can be removed by means of the projection operator onto the physical boson subspace[53,67] but the projected collective boson states become no more involved within the collective boson subspace. This can easily be understood as follows. The above-mentioned projection operator can be written as [1, 43]

$$\hat{\mathcal{P}}_{phys} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \sum_{\substack{i_1 \ldots i_n \\ j_1 \ldots j_n}} \mathcal{P}^{+}_{i_1j_1} \mathcal{P}^{+}_{i_2j_2} \ldots \mathcal{P}^{+}_{i_nj_n} |0)_{BB}(0| b_{i_nj_n} \ldots b_{i_2j_2} b_{i_1j_1} \,, \qquad (52)$$

where $\mathcal{P}^{+}_{ij}$ and b_{ij} are the Dyson boson images of the fermion pair operators $a^{+}_{i}a^{+}_{j}$ and $a_j a_i$, respectively [see eqs. (4a), (4b)]. When expressed in terms of a complete set of collective and non-collective bosons, the "Pauli-corrected" operators $\mathcal{P}^{+}_{ij}$ contain both of them, and thus, the operator $\hat{\mathcal{P}}_{phys}$, acting on a purely collective boson state, leads out of the collective subspace. The situation is very similar to that discussed in sect. 2.2 with respect to the hamiltonian. Now, let us imagine for a moment that we have achieved an exact decoupling of the collective subspace from the rest of the many-body boson space. A necessary and sufficient condition for this to occur is that there exist certain collective operators with closed commutation relations, i.e. such that they do not generate operators other than the collective ones. In this case, however, it was shown on a very general level [71] that all the noncollective boson operators appearing in $\mathcal{P}^{+}_{ij}$ can be omitted without destroying the relevant commutation relations. This is a highly nontrivial result which opens new possibilities for studying the elimination of spurious components from the truncated (collective) boson states. At the same time, this result emphasizes the importance of decoupling for a correct treatment of unphysical components in the collective boson states. Of course, in any realistic situation, where the decoupling is not exact, we have only a more or less accurate elimination of spurious components from the collective boson states. It has been observed that in cases, where the decoupling of the collective subspace is rather poor, serious problems with the unphysical boson components arise [46].

On the other hand, if the decoupling is relatively good (though only on a heuristic level), there are no problems with the spurious admixtures in the truncated boson states[20].

In the case discussed in the present paper we have an exact decoupling in the two-body part of the boson hamiltonian and a relatively suppressed $(\sim \varepsilon, \sim \varepsilon^2, \dots)$ collective-noncollective coupling in the three- and more- body boson terms. We can therefore expect a complete removal of unphysical components from the two-boson collective states and an approximate removal of them from the three- and more- boson states. Let us operate by $\hat{\mathcal{P}}_{phys}$ on the collective boson state (15) and consider only that portion of the physical boson state $\hat{\mathcal{P}}_{phys}|N;\{c\})_B$ which is contained within the collective boson subspace,

$$|N;\{c\};phys)_B = \sum_{\{c'\}} {}_B(N;\{c'\}|\hat{\mathcal{P}}_{phys}|N;\{c\})_B \; |N;\{c'\})_B \; . \tag{53}$$

The fraction

$$\frac{|N;\{c\};phys)_B}{\hat{\mathcal{P}}_{phys}|N;\{c\})_B}$$

is the higher the better is the decoupling of the collective subspace and it reaches the value 1 for $N = 2$, where the decoupling is exact. Unlike the states $|N;\{c\})_B$, the physical boson states $|N;\{c\};phys)_B$ given by (53) are not necessarily linearly independent. In order to remove the possible overcompleteness, it is convenient to introduce a new orthonormal boson basis

$$|N;\sigma))_B = \sum_{\{c\}} \mathcal{X}^{(N,\sigma)}_{\{c\}} |N;\{c\})_B \quad ; \quad \sum_{\{c\}} \mathcal{X}^{(N,\sigma)}_{\{c\}} \mathcal{X}^{(N,\sigma')}_{\{c\}} = \delta_{\sigma\sigma'} \; , \tag{54}$$

in which the projection operator $\hat{\mathcal{P}}_{phys}$ is diagonal,

$$\sum_{\{c'\}} {}_B(N;\{c\}|\hat{\mathcal{P}}_{phys}|N;\{c'\})_B \, \mathcal{X}^{(N,\sigma)}_{\{c'\}} = \mathcal{N}_{N,\sigma} \, \mathcal{X}^{(N,\sigma)}_{\{c\}} \; . \tag{55}$$

Possible linear dependence among the physical boson states will result in a certain number (k) of zero eigenvalues $\mathcal{N}_{N,\sigma_0} = 0, \sigma_0 = 1,2,\dots k$. The corresponding boson states $|N;\sigma_0))_B$ [eq. (54)] must then be excluded from further considerations in order to avoid spurious solutions. It is well known that [1,43])

$$ {}_B(N ; \{c\} | \hat{\mathcal{P}}_{phys} | N ; \{c'\})_B = \frac{1}{(2N-1)!!} {}_F\langle N ; \{c\} | N ; \{c'\} \rangle_F , \qquad (56) $$

where

$$ \left. \begin{aligned} |N ; \{c\}\rangle_F &= A^+_{c_1} A^+_{c_2} \dots A^+_{c_N} |0\rangle_F , \\ A^+_{c_k} &= \frac{1}{\sqrt{2}} \sum_{ij} \phi^{c_k}_{ij} a^+_i a^+_j , \end{aligned} \right\} \qquad (57) $$

$|0\rangle_F$ is the fermion vacuum and the coefficients $\phi^{c_k}_{ij}$ are the same as those defining the collective boson operators $B^+_{\kappa=c_k}$ [see (6a), (7)]. In practice we therefore construct the fermion states (57) first, then calculate the boson matrix elements of $\hat{\mathcal{P}}_{phys}$ according to (56), solve eq. (55) and after excluding the zero-eigenvalue solutions $\chi^{(N,\sigma_0)}_{\{c\}}$ ($\mathcal{N}_{N,\sigma_0} = 0$) , we obtain the boson basis

$$ |N ; \sigma))_B = \sum_{\{c\}} \chi^{(N,\sigma)}_{\{c\}} |N ; \{c\})_B , \qquad \sigma \neq \sigma_0 , \qquad (58) $$

in which the boson collective hamiltonian with renormalized parameters is diagonalized. We may thus be sure that, within the approximation explained above, the resultant boson eigenvectors lie to a large extent in the physical collective boson subspace.

3. Application to the multi-level pairing model

The complexity of any realistic many-particle problem rules out, in general, attempts at an exact treatment based on an "ab initio" calculation. The single-particle energies and the residual particle-particle interaction are usually chosen phenomenologically with parameters fitted to the experimental data. Therefore, it is important to be certain that the theoretical method used to study the problem is reliable before a comparison with the data is carried out. For this reason we consider it sensible to apply our method to a solvable "toy" model at first. Such models have become very popular and valuable as tests for the validity of new approximation methods. In order to be useful in practice, the model hamiltonian has to be simple enough to be solved exactly and yet it should include the most essential features of the realistic problem that we are interested in. Here we use the multi-level pairing model described by the hamiltonian

$$\hat{H} = \sum_{am} \varepsilon_a a^+_{am} a_{am} - G \sum_{aa'} A^+_a A_{a'} , \tag{59}$$

where

$$A^+_a = \sum_{m>0} (-)^{j_a-m} a^+_{am} a^+_{a-m} , \qquad A_a = (A^+_a)^+ , \tag{60}$$

with the index $a \equiv (n_a, \ell_a, j_a)$. The quantities ε_a are the single-particle energies and G is the pair coupling constant. The operators A^+_a, A_a and $\sum_m a^+_{am} a_{am}$ satisfy the algebra $SU(2) \times SU(2) \times \ldots$,

$$\begin{aligned} [A_a , A^+_{a'}] &= \delta_{aa'} \left(\Omega_a - \sum_m a^+_{am} a_{am} \right) , \\ [A_{a'} , \sum_m a^+_{am} a_{am}] &= 2 \delta_{aa'} A_a , \end{aligned} \tag{61}$$

where $2\Omega_a = 2j_a + 1$. Due to the simplicity of the commutation relations (61), the hamiltonian (59) can be solved exactly by a numerical diagonalization. At the same time, it has been rather successful [42] in describing the energetically lowest collective states in semi-magic nuclei with only one kind of valence nucleons, such as Ni, Sn, Pb. The Dyson representation of the operators A^+_a, A_a, $\sum_m a^+_{am} a_{am}$ is realized through

$$\begin{aligned} A^+_a &\to \left(\sqrt{\Omega_a}\, b^+_a - \sqrt{\tfrac{1}{\Omega_a}}\, b^+_a b^+_a b_a \right) , \\ A_a &\to \sqrt{\Omega_a}\, b_a , \\ \sum_m a^+_{am} a_{am} &\to 2 b^+_a b_a , \end{aligned} \tag{62}$$

where

$$[b_a , b^+_{a'}] = \delta_{aa'} , \qquad [b_a , b_{a'}] = [b^+_a , b^+_{a'}] = 0 . \tag{63}$$

With the use of (63) it is easy to verify that the boson operators on the right-hand side of (62) fulfill the commutation relations (61).

The boson image of the pairing hamiltonian (59) then reads

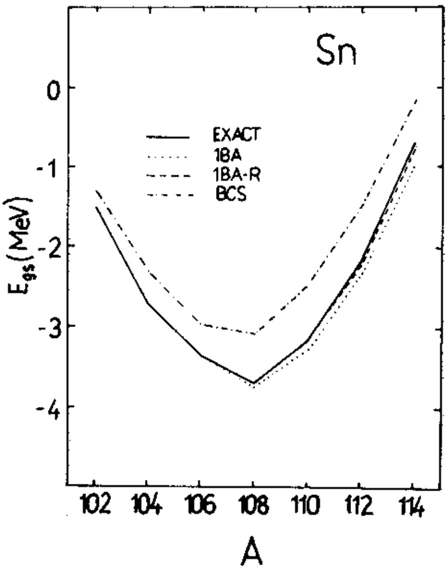
$$\hat{H}_D = \sum_{aa'} \left(2\varepsilon_a \delta_{aa'} - G\sqrt{\Omega_a \Omega_{a'}} \right) b_a^+ b_{a'} + G \sum_{aa'} \sqrt{\frac{\Omega_{a'}}{\Omega_a}}\, b_a^+ b_a^+ b_a b_{a'} \,. \qquad (64)$$

This is a special case of the hamiltonian (5) and it is further treated in the framework of the method described in sect. 2.

In the present paper we study the low-lying collective states (the ground state and the lowest excited 0^+ - states) of the even isotopes of Sn, Ni and Pb. For Sn isotopes the neutrons outside the Z=50, N=50 inert core are restricted to occupy five ($2d_{5/2}$, $1g_{7/2}$, $3s_{1/2}$, $2d_{3/2}$, $1h_{11/2}$) valence levels having 0.0, 0.22, 1.90, 2.20 and 2.80 MeV unperturbed single-particle energies, respectively. The strength of the pairing interaction is taken to be G = 0.187 MeV. In the case of Ni isotopes the neutrons outside the ^{56}Ni inert core (Z=28, N=28) are restricted to occupy four ($2p_{3/2}$, $1f_{5/2}$, $2p_{1/2}$ and $1g_{9/2}$) valence levels with the energies 0.0, 0.78, 1.56 and 4.52 MeV, respectively, and the value G = 0.331 MeV has been used. The Pb isotopes considered have the neutron valence shell more than half-filled. Therefore, we exploit the particle-hole symmetry [73] and describe them as hole systems with the Z=82, N=126 inert core and five ($3p_{1/2}$, $2f_{5/2}$, $3p_{3/2}$, $1i_{13/2}$ and $2f_{7/2}$) levels with energies 0.0, 0.57, 0.90, 1.63 and 2.35 MeV, respectively. The pairing interaction strength is G = 0.111 MeV. All the values given above are taken to be the same as in ref. [72].

In figs. 1, 2 and 3 we show the ground-state energies of the Sn, Ni and Pb isotopes, respectively, calculated in different approximations. The solid lines are the exact results obtained by the diagonalization of the hamiltonian (59). The dotted lines come from the boson hamiltonian (10) truncated to the single (most collective) degree of freedom $B^+_{\kappa=1}$. We shall refer to this truncation as the one-boson approximation (1BA). The dashed lines are obtained within the same truncation from the boson hamiltonian (43)

Fig.1. Ground-state energies of Sn isotopes, calculated in various approximations using the pairing hamiltonian(59). The parameters ε_α and G are given in the text. The solid line represents exact results, the dotted line is obtained within the one-boson approximation (1BA) from the hamiltonian (10), the dashed line gives the results obtained within the same 1BA from the renormalized hamiltonian (49) and the dash-dotted line shows the standard BCS results.



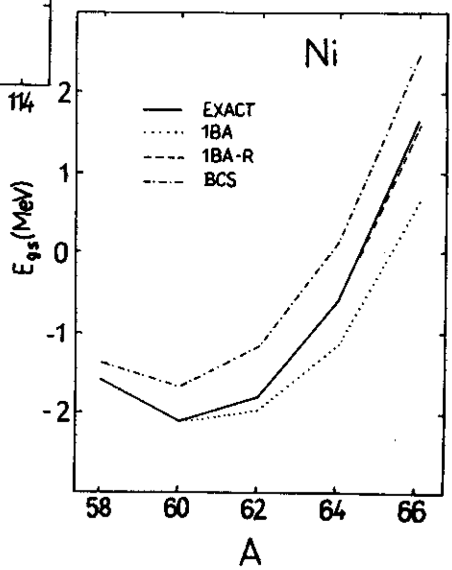


Fig.2. The same as in fig. 1 for Ni isotopes.

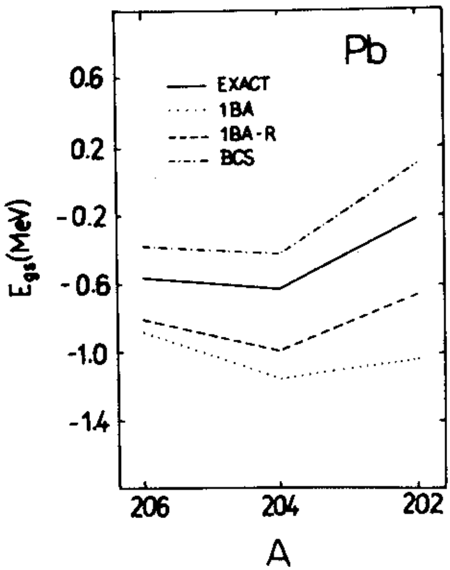


Fig.3. The same as in fig. 1 for Pb isotopes.

with the renormalized parameters $\tilde{E}_1$, $\tilde{W}_{1111}$ calculated according to eqs. (44a) and (44b), respectively. Since it is useful to compare new approximation schemes with the older ones that already have an established power, we also display (as dash-dotted lines) the results of the quasiparticle BCS approach [1] . From figs. 1-3 two important conclusions can be drawn. First, the renormalization of parameters of the boson collective hamiltonian improves the agreement with exact results considerably. Second, in the case of Sn and Ni the "renormalized" 1BA results are very close to the exact ones, in particular, much closer than the BCS values. However, this is not true for the Pb isotopes, which are known not to show such strong pairing correlations as Sn and Ni. The failure of the 1BA in Pb simply means that the structure of the Pb ground state is more complicated and cannot be described in terms of only one boson. A similar observation has been made in ref. [21] . If we take into

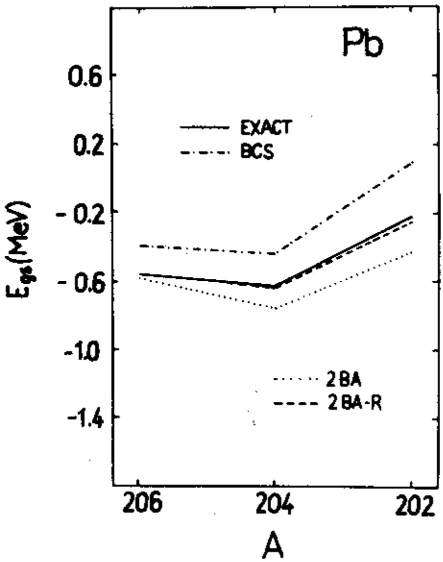
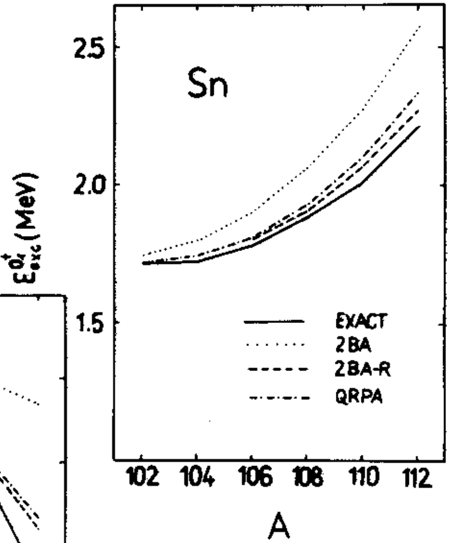


Fig.4. Ground-state energies of Pb isotopes calculated within the two-boson approximation (2BA) . The dotted line is the "bare" result obtained from (10) , the dashed line represents the "renormalized" result obtained from (49) . The exact and BCS results (solid and dash-dotted line, respectively) are the same as in fig. 3.

account two collective bosons B_1^+ , B_2^+ (two-boson approximation - 2BA) , we obtain the results shown in fig. 4. As is seen, the "renormalized" two-boson approximation then works very well.

Now we turn our attention to the first excited 0^+- states. As an initial "guess" for the structure of the collective subspace we take the 2BA. The results are displayed in figs. 5-7, where the solid lines are again the exact results, the dotted lines are obtained by diagonalizing the boson hamiltonian (10) in the framework of the 2BA ($\alpha = 1,2$) , the dashed lines represent the "renormalized" 2BA results [see (43) with $c_i = 1,2$ and (44a,b)] and the dash-dotted lines are obtained using the quasiparticle RPA [1)]. We again observe an impressive influence of the renormalization and

Fig.5. Excitation energies of the first excited 0^+ - states in Sn isotopes. The exact results (solid line) are compared with those obtained within the 2BA from (10) (dotted line) and those obtained within the renormalized 2BA (dashed line) . The dash-dotted line represents the results obtained in the quasiparticle RPA.



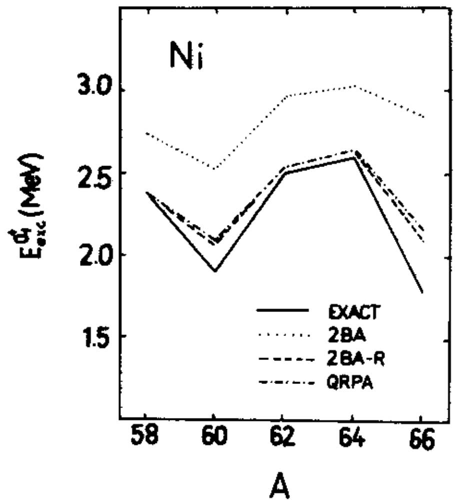
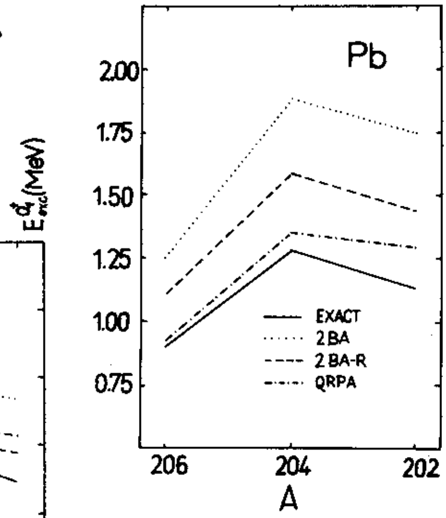


Fig.6. The same as in fig. 5 for Ni isotopes.

a good agreement of the renormalized results with the exact values in the case of Sn and Ni, but much a worse one in the case of Pb. The results for the Pb isotopes, however, are significantly improved if we include the third boson B_3^+ (the three-boson approximation - 3BA) . This is shown in fig. 8. It is also worthwhile to notice that, unlike the ground-state case, the quality of our boson approximation is roughly comparable with that of the quasiparticle approach.

Fig.7. The same as in fig. 5 for Pb isotopes.



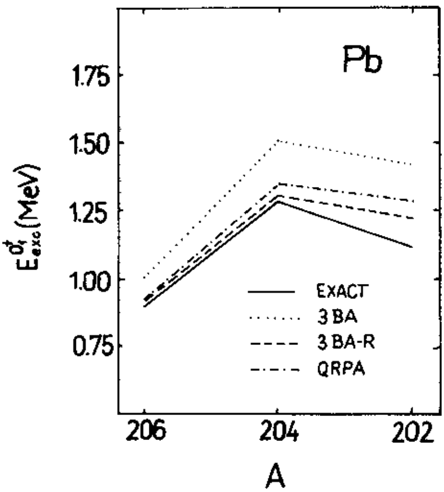


Fig.8. Excitation energies of the first excited 0^+- states in Pb isotopes, calculated in the three-boson approximation (3BA) using the "bare" hamiltonian (10) (dotted line) and the "renormalized" hamiltonian (49) (dashed line) . The solid and dash-dotted lines, representing the exact and QRPA results, respectively, are the same as in fig. 7.

Concluding this section, we have shown that the low-energy collective states of nuclei exhibiting strong pairing correlations (Sn, Ni) can be microscopically described with the help of at most two bosonic degrees of freedom (one boson for the ground state and two bosons for the first excited state) . By the term "microscopically" we mean that the structure of the relevant bosons has been determined completely by the single-particle nucleon energies (ε_a) and the two-particle nucleon-nucleon pairing interaction (G)– no adjustment of the many-particle quantities has been performed. We have also seen that in case of nuclei with weaker pairing correlations (Pb), a satisfactory reproduction of the ground-state energies requires two different bosons. At the same time, three types of bosons are needed to describe the properties of the first excited 0^+-states. The last but not least conclusion which can be drawn from our investigation is that within our method the ground-state energies are reproduced better than within the standard BCS approach, and the agreement for the energies of the excited states is at least as good as in the framework of the quasiparticle RPA.

4. Summary and perspectives

In this paper, we have proposed and tested a method for determining an optimally decoupled collective subspace of the many-body Hilbert space appropriate to the description of nuclei. The basic ingredients of our approach are the Dyson boson representation of the shell-model hamiltonian and a canonical transformation in the boson space which eliminates (up to the two-body interaction) the coupling between the collective and noncollective one-boson eigenstates of the Dyson hamiltonian. The method has been applied to the solvable multi-level pairing model and the results have been compared with the exact ones, as well as with the usual quasiparticle approach BCS + QRPA . The renormalization of parameters of the boson collective hamiltonian due to the coupling with the noncollective degrees of freedom has been found to play an important role in describing the lowest collective states of the Sn, Ni and Pb isotopes. The method offers a very good approximation to the energies of the ground and first excited states of the above nuclei (in general better than the quasiparticle approach) . Its peculiar feature consists in that it is able to provide information (though only heuristically) on the structure of the collective subspace, i.e. not only to optimize variationally the preselected structure of this subspace as, e.g., in the BCS approach.

We have paid special attention to the problem of identifying and removing the unphysical components from the collective boson states. We would like to emphasize that this problem is intimately related to the decoupling of the collective subspace from the rest of the boson space.

In summary, we (believe to) have presented a sound and flexible method for extracting the collective degrees of freedom from the microscopic shell-model description of nuclei. In this paper we have restricted our discussion to nuclei with strong pairing correlations. A direct extension to other types of nuclei (especially the deformed ones with a strong proton-neutron quadrupole-quadrupole interaction) is connected with three kinds of problems:

1) an explicit introduction of proton and neutron degrees of freedom ;
2) the angular momentum algebra complications ;
3) an estimate of the importance of the three-body terms in the collective boson hamiltonian, because these terms are very likely to be non-negligible in deformed nuclei [14,15]) .

Work on this subject is now in progress.

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Received by Publishing Department
on March 20, 1989.