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SEPARABLE EXPANSION OF POTENTIAL IN MULTI-DIMENSIONAL SPACE. THREE-BODY PROBLEM

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In the last years a considerable progress has been achieved in the description of a system consisting of three particles interacting via a short-range potential. In this case, as is well known, the equations /1/ with Fredholm kernels have been derived and rather rapidly convergent regular procedures have been developed for solving three-particle equations /2/.

However, calculations of the properties of such a system with realistic interaction potentials as the Reid potential are still rather complicated and for positive energies of three-body systems these only begin to develop /3/. This situation is due to the following. First, the realistic potentials, being very complicated, generate many components in the wave function of three-particle system. Second, the solution of the Faddeev equations even with central local potentials are still more complicated than that of two particle system with the same interaction.

Thus, if the first difficulty cannot, generally, be avoided, one may try to simplify the solution of three-particle equations with central potentials.

In this note we propose a version of such an attempt based on the separable expansion of the potential $V = V_{12} + V_{23} + V_{13}$ in the six-dimensional space.

Let us take the Lippmann-Schwinger equation describing a bound state of three identical spinless particles:

$$|\Psi\rangle = G_0(E)\, V |\Psi\rangle . \qquad (1)$$

In the momentum representation it has the form:

$$\Psi(\vec{q}\vec{p}) = \langle \vec{q}\vec{p} | \Psi \rangle = \frac{G_0(qpE)}{(2\pi)^6} \int d\vec{q}'\, d\vec{p}' \langle \vec{q}\vec{p} | V | \vec{q}'\vec{p}' \rangle \Psi(\vec{q}'\vec{p}') \qquad (2)$$

where $G_0(qpE) = \frac{m}{mE - q^2 - \frac{3}{4}p^2}$ is the Green function of noninteracting particles, m is the mass of the particle.

By introducing the functions with the given total angular momentum $Y^{LM}_{\ell\lambda}(\hat{q}\hat{p})$

$$Y^{Lm}_{\ell\lambda}(\hat{q}\hat{p}) = \sum_{m\mu} (\ell\lambda m\mu | LM)\, Y_{\ell m}(\hat{q})\, Y_{\lambda\mu}(\hat{p}), \qquad (3)$$

(here $\hat{q}$, $\hat{p}$ are the unit vectors directed along the vectors $\vec{q}$ and $\vec{p}$, resp., $(\ell\lambda m\mu|LM)$ is the Clebsch-Gordan coefficient), from (2) we get the equations for partial waves:

$$\Psi^{L}_{\ell\lambda}(qp) = \frac{G_0(qpE)}{(2\pi)^6} \int_0^\infty q'^2 dq' p'^2 dp' \sum_{\ell'\lambda'} K^{L}_{\ell\lambda,\ell'\lambda'}(qp, q'p')\, \Psi^{L}_{\ell'\lambda'}(q'p'). \qquad (4)$$

The kernel K has the form:

$$K^{L}_{\ell\lambda,\ell'\lambda'}(qp, q'p') = \int Y^{LM*}_{\ell\lambda}(\hat{q}\hat{p}) \langle \vec{q}\vec{p} | V | \vec{q}'\vec{p}' \rangle Y^{LM}_{\ell'\lambda'}(\hat{q}'\hat{p}')\, d\hat{q} d\hat{p} d\hat{q}' d\hat{p}'. \qquad (5)$$

As we are solving the bound-state problem of three particles, the solutions of equations (2) and (4) are square integrable functions. This, in principle, allows us to change the square nonintegrable function $\langle \vec{q}\vec{p} | V | \vec{q}'\vec{p}' \rangle$ by a more regular function. It is clear that the scattering problem should be handled with more carefully.

Hence, instead of the exact potential $V = V_{12} + V_{23} + V_{13}$ we introduce the approximate one $\tilde{V}$

$$\langle \vec{q}\vec{p} | \tilde{V}^N | \vec{q}'\vec{p}' \rangle = \sum_{ij=1}^{N} \langle \vec{q}\vec{p} | V | \chi_i \rangle d^{-1}_{ij} \langle \chi_j | V | \vec{q}'\vec{p}' \rangle, \qquad (6)$$

where $d_{ij} = \langle \chi_i | V | \chi_j \rangle$, functions χ_i are defined in the six-dimensional space, i.e., expression (6) realizes the complete separation. As a result, the problem becomes algebraic. The functions χ_i can, clearly, be introduced in the $n < 6$ dimensional space. For example, if we introduce them in five-dimensional space equation (4) reduces to a set of one-dinemsional integral equations. Let

us note that expansion (6) is a direct multidimensional analog of the separable expansions of a two-body potential which have been applied extensively to various problems of nuclear physics.

Inserting (6) into (5) we get the approximate kernel

$$\tilde{K}^{L}_{\ell\lambda,\ell'\lambda'}(qp,q'p') = \sum_{ij} {}^{i}R^{L*}_{\ell\lambda}(qp)\, d^{-1}_{ij}\, {}^{j}R^{L}_{\ell'\lambda'}(q'p'), \qquad (7)$$

where

$${}^{i}R^{L}_{\ell\lambda}(qp) = \int Y^{LM}_{\ell\lambda}(\hat{q}\hat{p}) < \vec{q}\vec{p} \mid V \mid \chi_i > d\hat{q}\, d\hat{p}. \qquad (8)$$

Using the approximate kernel instead of exact one in the equation (4) we get the expansion in projected matrix elements of the potential for the wave function:

$$\psi^{L}_{\ell\lambda}(qp) = \frac{G_0(qpE)}{(2\pi)^6} \sum_{i} {}^{i}R^{L*}_{\ell\lambda}(qp)\, C^{L}_{i}(E). \qquad (9)$$

Coefficients $C^{L}_{i}(E)$ satisfy the system of algebraic equations:

$$C^{L}_{i}(E) = \frac{1}{(2\pi)^6} \sum_{j} A^{L}_{ij} C^{L}_{j}(E). \qquad (10)$$

Here the matrix A^{L}_{ij} is defined as

$$A^{L}_{ij_1} = \sum_{j\,\ell'\lambda'} d^{-1}_{ij}\, {}^{jj_1}I^{L}_{\ell'\lambda'}(E), \qquad (11)$$

where

$${}^{jj_1}I^{L}_{\ell'\lambda'}(E) = \int_0^{\infty} q^2 dq p^2 dp\, G_0(qpE)\, {}^{j}R^{L}_{\ell'\lambda'}(qp)\, {}^{j_1}R^{L}_{\ell'\lambda'}(qp). \qquad (12)$$

For the sake of simplicity consider states with $L = 0$, and besides, restrict ourselves to values $\ell = \lambda = 0$, as we have the short-range potential. In this case expression (8) takes the form:

$${}^{i}R^{0}_{00}(qp) = \frac{1}{4\pi} \int < \vec{p}\vec{q} \mid V \mid \chi_i > d\hat{q} d\hat{p}. \qquad (13)$$

Introducing the Jacobian coordinates $\vec{\xi}, \vec{\eta}$ for the matrix elements $<\vec{qp}|\,V|\,\chi_i>$ we have

$$<\vec{qp}\,|\,V|\,\chi_i> = \int d\,\vec{\xi}\,d\vec{\eta}\; e^{i\vec{\xi}\vec{q} + i\vec{\eta}\vec{p}}\; <\vec{\xi}\vec{\eta}|\chi_i>[V_{12} + V_{23} + V_{13}]. \quad (14)$$

Here $<\vec{\xi}\vec{\eta}\,|\,\chi_j>$ are χ_i functions in the coordinate representation. Expanding in (14) exponents in spherical functions

$$e^{i\vec{\xi}\vec{q}} = 4\pi \sum_{\ell m} i^{\ell} j_{\ell}(\xi q)\, Y_{\ell m}(\hat{q})\, Y^{*}_{\ell m}(\hat{\xi}),$$

and inserting (14) into (13) we get:

$$^{i}R^{0}_{00}(qp) =$$

$$= 4\int_0^{\infty} \xi^2 d\xi\, r^2\, dr\, dx\, j_0(\xi q) j_0(\eta p) <\vec{\xi}\vec{\eta}\,|\chi_i>[V_{12}(|\vec{\xi}|) + V_{23}(|\vec{\eta} + \frac{\vec{\xi}}{2}|) + V_{13}(|\vec{\eta} - \frac{\vec{\xi}}{2}|)].$$

(15)

Further calculation will be made for the two-body potential

$$V(\xi) = V_0\, e^{-\beta \xi^2}. \quad (16)$$

Parameters V_0 and β are chosen to reproduce the experimental value of binding energy of deuteron and of triplet np scattering length. For this potential there are known very exact estimations of lower and upper bounds of the binding energy of the three-particle system:

$$-E_T = 9.7813 \pm 0.0024. \quad (17)$$

The functions χ_i will be chosen aiso in the Gaussian form:

$$\chi_i = e^{-\alpha_i \xi^2 - \beta_i \eta^2}. \quad (18)$$

For the application of expansion (6) only the linear independence of functions χ_i is required. The energy of the system is obtained from the condition:

$$\det\left(1 - \frac{A^L(E)}{(2\pi)^6}\right) = 0 . \qquad (19)$$

Expressions for matrix elements ${}^{i}R^{0}_{00}(qp)$ and integrals (12) are given in the appendix. By using the potential (16) and functions (18) integral (15) can be calculated explicitly. Coefficients ${}^{jj}I^{0}_{00}(E)$ are expressed by one-dimensional integrals which were calculated numerically.

The results of binding energy calculations are given in the table:

	1	2	3	4	5	6	7
a_i	0.45	0.4	0.35	1	0.35	0.45	0.95
b_i	0.55	0.4	0.30	1	0.35	0.4	0.95
$-E_T$	8.86	8.90	9.46	9.51	9.64	9.68	9.72

From this table it is clear that even three terms in the expansion of the potential given by expression (6) enable us to find the tritium binding energy with an error not higher than 4%*.

From the table it is also clear that all the functions χ_i have appeared to depend, in practice, only on the six-dimensional radius $\rho^2 = \xi^2 + \eta^2$. The wave function $\Psi^0_{00}(qp)$ of bound state, as follows from formula (1A) of the appendix, depends on the six-dimensional radius $R^2 = q^2 + p^2$ only in the limit $\alpha_i = \beta_i \gg \beta$, i.e., when the approximate potential $\tilde{V}$ reproduces well the exact one at distance small as compared with the range of the two-body potential.

In conclusion the authors express their gratitude to the participants of the seminar of sector 3 in the Laboratory of Theoretical Physics for useful discussions and interest in this work.

*The choice of the parameters α_i and β_i was realized by minimizing energy E_i.

Appendix

Here we will give the explicit form of integrals (12) and matrix elements (15), which determine the functional dependence of the bound state wave function.

$$^{i}R^{0}_{00}(qp) = \frac{4\pi V_0}{\beta^3}\left\{2\sqrt{A_i}\, e^{-B_i y^2 - C_i x^2}\frac{\mathrm{sh}(A_i xy)}{xy} + D_i e^{-E_i y^2 - F_i x^2}\right\}, \tag{1A}$$

$$x = \frac{q}{\sqrt{\beta}}; \quad y = \frac{p}{\sqrt{\beta}}, \quad a_i = \frac{\alpha_i}{\beta}, \quad b_i = \frac{\beta_i}{\beta},$$

$$A_i^{-1} = 4a_i(1 + b_i) + b_i, \qquad D_i^{-1} = 8[b_i(1 + a_i)]^{3/2},$$

$$B_i = (1 + b_i)A_i, \qquad E_i^{-1} = 4(1 + a_i),$$

$$C_i = \frac{1 + A_i}{4(1 + b_i)}, \qquad F_i^{-1} = 4b_i.$$

$$^{ij}I^{0}_{00}(E) = -\frac{16 V_0^2 \pi^2 m}{\beta^4}\left\{4\,^{ij}I_{11}(E) + 2[^{ij}I_{12}(E) + {}^{ij}I_{12}(E)] + {}^{ij}I_{22}(E)\right\}, \tag{2A}$$

$$^{ij}I_{11}(E) = -\frac{\sqrt{A_i A_j}}{4\sqrt{3}}\int_{-\pi/2}^{\pi/2}\left\{e^{E^{ij}f_1^{+}(\phi)}E\left[-F^{ij}f_1^{+}(\phi)\right] - \right. \tag{3A}$$

$$\left. - e^{E^{ij}f_1^{-}(\phi)}E\left[-E^{ij}f_1^{-}(\phi)\right]\right\}d\phi,$$

$$^{ij}f_1^{\pm}(\phi) = {}^{ij}B_1 \sin^2\phi + \frac{2}{\sqrt{3}}\,{}^{ij}A_1^{\pm}\sin\phi\cos\phi + \frac{4}{3}\,{}^{ij}C_1\cos^2\phi, \tag{4A}$$

$$^{ij}B_1 = B_i + B_j, \qquad {}^{ij}A_1^{\pm} = A_i \pm A_j, \qquad {}^{ij}C_1 = C_i + C_j,$$

$$
{}^{ij}I_{12} = \frac{2\pi}{3\sqrt{3}}\sqrt{A_j}D_i\left\{\frac{A_j}{\frac{4}{3}A_j^2+({}^{ij}B_2-\frac{4}{3}{}^{ij}C_2)^2}\left(\frac{{}^{ij}B_2+\frac{4}{3}{}^{ij}C_2}{\frac{2}{\sqrt{3}}\sqrt{4^{ij}B_2{}^{ij}C_2-A_j^2}}-1\right)-\right.
$$

$$
\left.-E\frac{\sqrt{3}}{2\pi}\int_{-\pi/2}^{\pi/2}e^{E^{ij}f_2(\phi)}E_i\left[-E^{ij}f_2(\phi)\right]\sin\phi\cos\phi d\phi\right\}, \qquad (5A)
$$

$$
{}^{ij}f_2(\phi) = {}^{ij}B_2\sin^2\phi+\frac{2}{\sqrt{3}}A_j\sin\phi\cos\phi+\frac{4}{3}{}^{ij}C_2\cos^2\phi, \qquad (6A)
$$

$$
{}^{ij}B_2 = B_i + F_j, \qquad {}^{ij}C_2 = C_i + F_j,
$$

$$
{}^{ij}I_{22} = \frac{2\pi}{3\sqrt{3}}\frac{D_iD_j}{\frac{4}{3}{}^{ij}C_3-{}^{ij}B_3}\left\{\frac{1}{\frac{4}{3}{}^{ij}C_3-{}^{ij}B_3}\left(\frac{\frac{4}{3}{}^{ij}C_3+{}^{ij}B_3}{\frac{4}{3}\sqrt{{}^{ij}B_3{}^{ij}C_3}}-1\right)-\right.
$$

$$
\left.-E\int_0^{\infty}e^{-(\frac{4}{3}{}^{ij}C_3+{}^{ij}B_3)\zeta}I_1\left[(\frac{4}{3}{}^{ij}C_3-{}^{ij}B_3)\zeta\right]\frac{d\zeta}{2\zeta+E}\right\}, \qquad (7A)
$$

$$
{}^{ij}B_3 = E_i + E_j, \qquad {}^{ij}C_3 = F_i + F_j.
$$

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