ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

ДУБНА

E4 - 8818

V.A.Zagrebnov, I.G.Brankov, N.S.Tonchev

A RIGOROUS STATEMENT ABOUT SYSTEMS INTERACTING WITH BOSON FIELD

1975

E4 - 8818

V.A.Zagrebnov, I.G.Brankov, N.S.Tonchev

A RIGOROUS STATEMENT ABOUT SYSTEMS INTERACTING WITH BOSON FIELD

Submitted to ДАН СССР

A system consisting of two coupled subsystems B,and L contained in a volume V is investigated. The space of states $\mathcal{H}$ of such a system is tensor product of the corresponding separable Hilbert spaces: $\mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_L$. Consider the case when B is a free boson field with M modes having energies $\omega_k \geq \omega_0 > 0 (k=1,2,\ldots,M)$ and the N-particle subsystem L is described by a Hamiltonian $H_{L,N}$ which obeys the following conditions:

$$\frac{1}{N} \| H_{L,N} \| \leq c_1, \quad \left| \frac{1}{\beta N} \ln \mathrm{Tr}_{\mathcal{H}_L} \exp\{-\beta H_{L,N}\} \right| \leq c_2. \qquad (1)$$

Here the trace is over the Hilbert space $\mathcal{H}_L(N)$ for the N particles.

In the present note we confine our attention to a class of models described on $\mathcal{H}(M,N) = \mathcal{H}_B(M) \otimes \mathcal{H}_L(N)$ by a Hamiltonian $H_N = T + U$ with

$$T = \sum_{k=1}^{M} \omega_k b_k^+ b_k \otimes I$$

$$U = \frac{1}{\sqrt{N}} \sum_{k=1}^{M} \lambda_k [b_k \otimes L_k^+ + b_k^+ \otimes L_k] + I \otimes H_{L,N} . \qquad (2)$$

T h e o r e m . If a system is described by a Hamiltonian of the form (2) satisfying conditions (1) and if

$$\frac{1}{N} \| [H_{L,N}, L_k] \|_{\mathcal{H}_L(N)} \leq c_3, \quad \frac{1}{N} \| L_k \|_{\mathcal{H}_L(N)} \leq c_4, \qquad (3)$$

then (i) $f_N[H_N] = -\frac{1}{\beta N}\ln \mathrm{Tr}\exp\{-\beta H_N\}$ exists and is bounded uniformly in N,

(ii) $|f_N[H_N] - \min_{(\eta)} f_N[H_{0,N}(\eta)]| \le \max_{(k)}|\lambda_k|\cdot M\cdot\epsilon_N$,

where $\epsilon_N \underset{N\to\infty}{\to} 0$

$$H_{0,N}(\eta) = \sum_{k=1}^{M} \omega_k \left(b_k^+ + \frac{\lambda_k}{\omega_k}\sqrt{N}\,\eta_k^*\right)\left(b_k + \frac{\lambda_k}{\omega_k}\sqrt{N}\,\eta_k\right)\otimes I - \tag{4}$$
$$- I\otimes\sum_{k=1}^{M} \frac{\lambda_k^2}{\omega_k}\left(L_k^+\eta_k^+ + L_k\eta_k^*\right) + I\otimes H_{L,N} + I\otimes \sum_{k=1}^{M} N\frac{\lambda_k^2}{\omega_k}|\eta_k|^2 .$$

We shall present here the main points of the proof. Short arguments for the validity of the first part of our statement have been given by Hepp and Lieb in Ref. [1].

(i) One can easily verify that the densely defined self-adjoint operators $T > 0$ and U satisfy the conditions of the Kato-Rellich theorem [2], i.e., $D(T) \subset D(U)$ and U is a T-bounded operator

$$|U\psi| \le a|\psi| + b|T\psi|, \qquad \psi\in D(T), \tag{5}$$

with $b < 1$ if

$$a > 4c_4^2\,\frac{\lambda^2_{max}}{\omega_0}\,NM^2 + \sqrt{2}\,(c_4\lambda_{max}\sqrt{N}\cdot M + c_1 N) .$$

Thus operator H_N is self-adjoint and bounded from below on the domain $D(T)$. Put H_N in the form $H_N = H_0 + V_1$ where

$$H_0 = \frac{3}{4}\sum_{k=1}^{M} \omega_k b_k^+ b_k\otimes I + I\otimes H_{L,N},$$

$$V_1 = \frac{1}{4}\sum_{k=1}^{M} \omega_k\left(b_k^+\otimes I + \frac{4\lambda_k}{\omega_k\sqrt{N}}\, I\otimes L_k^+\right)\times \tag{6}$$
$$\times\left(b_k\otimes I + \frac{4\lambda_k}{\omega_k\sqrt{N}}\, I\otimes L_k\right) - \sum_{k=1}^{M}\frac{4\lambda_k^2}{N\omega_k}\, I\otimes L_k^+L_k .$$

Since $D(H_0) = D(V_1) = D(T)$ the self-adjoint operator H_N is represented as a sum of two self-adjoint operators bounded from below: $H_0 \ge -c_1 N$, $V_1 \ge -4\max_{(k)}\lambda_k^2\cdot\omega_0^{-1}\cdot c_4\cdot M\cdot N \equiv -A\cdot N$. Thus $H_N = H_0 + V_1$ defined on $D(T)$, obeys the conditions of a theorem proved by Ruskai [3] and having in mind that $\exp\{-\beta H_0\}$ is a trace-class operator we can make use of the Golden-Thompson inequality

$$\mathrm{Tr}\exp\{-\beta H_N\} \le \mathrm{Tr}\{\exp[-\beta H_0]\exp[-\beta V]\} \le$$

$$\le \exp[\beta(A + c_2)N]\cdot\left[1 - \exp\left(-\frac{3}{4}\beta\omega_0\right)\right]^{-M}.$$

This completes the proof of the first part of the theorem.

(ii) Let us write now H_N in the form $H_N = H_{0,N}(\eta) + H_{1,N}(\eta)$ where $H_{0,N}(\eta)$ (see (4)) is bounded from below,

$$D(H_{0,N}(\eta)) \subset D(H_{1,N}(\eta)),$$

and

$$H_{1,N}(\eta) = \sum_{k=1}^{M}\frac{\lambda_k}{\sqrt{N}}\left(b_k + \frac{\lambda_k}{\omega_k}\sqrt{N}\,\eta_k\right)\otimes\left(L_k^+ - \sqrt{N}\,\eta_k^*\right) + \text{h.c.} \tag{7}$$

Using the explicit form of the symmetric operator $H_{1,N}(\eta)$ one can easily prove that it is $H_{0,N}(\eta)$ – bounded (see (5)) with $b < 1$ and $a > \text{const.}\,N$. Therefore, by the Kato-Rellich theorem [2] for all $|t| < b^{-1}$ $H_N(t) = H_{0,N}(\eta) + tH_{1,N}(\eta)$ is self-adjoint bounded from below operator defined on $D(H_N(t)) = D(H_{0,N}(\eta))$.

Further, operator $H_N(t)$ satisfies the conditions of the Maison theorem /4/. Hence for $|t| < b^{-1}$, $\exp[-\beta H_N(t)]$ is a trace-class operator since $\exp[-\beta H_{0,N}(\eta)]$ is trace-class. Furthermore, $Z_\beta(t) = \mathrm{Tr}\exp\{-\beta[H_{0,N}(\eta)+tH_{1,N}(\eta)]\}$ is an analytic function of t in the domain $\{|t|<b^{-1}\}\times\{\beta > 0\}$. This makes it possible to prove the Bogolubov inequality by direct differentiation with respect to $t \in [0,1]$ /5/:

$$\frac{1}{N}\langle H_{1,N}(\eta)\rangle_{t=1} \le f_N[H_N] - f_N[H_{0,N}(\eta)] \le \frac{1}{N}\langle H_{1,N}(\eta)\rangle_{t=0}. \quad (8)$$

Using inequality (8) and the results of a previous work of the authors /6/ we obtain the estimate (ii) which proves the thermodynamic equivalence of the Hamiltonians H_N and $H_{0,N}(\eta)$.

Note that in the cases when the limit $\lim_{N\to\infty} \min_{(\eta)} f_N[H_{0,N}(\eta)]$ exists /6,7/ inequality (ii) implies the existence of the thermodynamic limit for the free energy density of the original Hamiltonian (2).

References

1. *K.Hepp and E.H.Lieb. Phys.Rev., A8, 2517 (1973).*
2. *T.Kato. Perturbation Theory for Linear Operators. (Berlin, Heidelberg, New York, Springer, 1966).*
3. *M.B.Ruskai. Commun. Math. Phys., 26, 280 (1972).*
4. *H.D.Maison. Commun. Math. Phys., 22, 166 (1971).*
5. *N.N.Bogolubov (Jr.). A Method for Studying Model Hamiltonians (Pergamon Press, Oxford, 1972).*
6. *I.G.Brankov, V.A.Zagrebnov and N.S.Tonchev. Teor. i Mat. Fiz., 22, 20 (1975).*
7. *I.G.Brankov, V.A.Zagrebnov and N.S.Tonchev. Preprint JINR, P4-7917, Dubna, 1974 (to be published in Physica).*

Received by Publishing Department
on April 21, 1975.