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NUCLEAR COLLECTIVE MOTION WITHIN THE O(N-1) INVARIANT DYNAMICS

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I. Introduction

Investigation of the nuclear collective motion, which is a very important field of research of today, is approached in two very different directions.

The first approach springs from the time dependent Hartree--Fock theory. Thus, the cranking model is successfully applied to study the yrast states of nuclei. The random-phase approximation built on the rotating mean field allows one to describe the properties of numerous states of a relatively simple structure.

During the last decade or two an important progress was achieved using as a standpoint the concept of nuclear fluid, whose dynamics could be described in terms of the distribution function[1] or the quantities related to it.[2-5] Due to these investigations the nuclear hydrodynamic model has found a sound theoretical basis and is greatly enriched in its context.

On the other hand, the very important mathematical development took place in the field of the unified model by Bohr and Mottelson. The original idea of the adiabaticity of the motion of nuclear inertia tensor with respect to motion of nucleons in the intrinsic frame led to the construction of an elegant theory based on the introduction of a complete set of nuclear coordinates including the six components of the nuclear inertia tensor.[6-12] The intrinsic motion is then described using the concept of the $O(N-1)$ group of rotations.

Both approaches have their merits and their drawbacks. The first way offers a possibility of using the physical intuition and is, in fact, quite effective in digesting the wealth of recent experimental data. Its systematic application led to the creation of a coherent picture of nuclear structure where the mean field plays the central role.

The second approach is very transparent in its mathematical context which is very important in formulating the quantum theory of nuclei free from the ad-hoc suggestions typical of the first approach.

Much work is done to clear up the algebraic structure of the theory describing the evolution of the inertia tensor: the dynamical group related to the collective motion is found to be the Sp(6,R) group (sometimes called also Sp(3,R)) and analysed in much detail in its relation to the nuclear structure.[11,12]

By now it became clear that more or less direct application of both approaches demands a great computer power to obtain physical results. The theory formulated in terms of the Wigner distrubution function meets difficulties when the restrictions arising from the antisymmetry of the wave-functions become important.
The second way demands further developements especially in studying the collective motion wchich is not related to the evolution of the inertia tensor.

Much progress is achieved in relating the approach based on the dynamical symmetry Sp(6,R) group of transformations and the nuclear shell model (see review 13). The parallels between this theory and the nuclear hydrodynamics are also established.[12]

The aim of the cycle of papers starting from this one is to go further in studying the relations between these theories. In this paper we arrive at equations of motion for the O(N-1) invariant system and establish the relation between the theory based on the introduction of the full set of coordinates. This is done by making the quasiclassical approximation in the equation of motion for the components of the inertia tensor while treating quantum mechanically the motion in the space of intrinsic coordinates. For

the O(N-1) invariant system a closed set of equations is obtained describing the time evolution of the inertia tensor influenced by the forces of potential nature and by the kinetic energy stresses (Fermi-surface deformations).

The classical stationary solutions of these equations of motion are found. They represent a rather simple generalization of the so-called S-Riemann ellipsoids giving the solution of the Dedekind problem of a gravitating liquid droplet.[14] The generalized S-Riemann ellipsoids found in this paper give the stationary solution for the equations of the motion of a droplet of Fermi-liquid where the pressure tensor is anysotropic.

In the future publications we are going to discuss a number of prolems including

(1) the relation between the solution to the equations of motion formulated here given by the generalized S-Riemann ellipsoids and the solutions of the cranking model for the deformed harmonic oscillator potential;

(2) the small vibrational perturbations of the stationary configurations and

(3) applications of the theory.

II. The Hamiltonian and the equations of the motion within the O(N-1) invariant model

The leading role of the quadrupole (ellipsoidal) deformation in determining the nuclear shape is well established. One expects that the main axes of inertia tensor in heavy nuclei do not change much due to the quantum oscillations, i.e. that dynamics of the six components of this tensor may be studied using the concept of the classical mechanics. On the other hand, the intrinsic motion could not be studied without taking into account the Pauli principle. For this reason, we start with the quantum many-body Hamiltonian and bring it to the form in which the components of the nuclear inertia tensor play the role of collective coordinates. This is achieved using the following parametrization of the elements of a 6N-dimensional phase space including the coordinates x_k and the momenta p_k of N nucleons $(k=1,2,\dots,N)$[6-8,11]:

$$x_{\alpha k} = X_\alpha + \sum_{X} \mathcal{D}(\theta)_{X\alpha}\, \lambda_X\, \mathcal{R}(\varphi_1,\varphi_2,\dots,\varphi_{3N-9})_{Xk}\,, \tag{2.1}$$

$$p_{\alpha k} = \frac{1}{N} P_\alpha + \sum_X \mathcal{D}(\theta)_{X\alpha} \Big\{ \lambda_X^{-1}\pi_{Xk} + \mathcal{R}_{Xk} p_X + \sum_{Y(\neq X)} \frac{(\tilde{L}_{XY}\lambda_Y + \lambda_X \tilde{J}_{XY})}{\lambda_X^2 - \lambda_Y^2}\, \mathcal{R}_{Yk} \Big\}\,. \tag{2.2}$$

The 6N+27 variables (operators on the quantum mechanical level) satisfy the 27 conditions

$$\sum_{\alpha=1}^{3} \mathcal{D}_{X\alpha}\mathcal{D}_{Y\alpha} = \delta_{XY}\,, \tag{2.3a}$$

$$\sum_{k=1}^{N} \mathcal{R}_{Xk} = 0, \qquad \sum_{k=1}^{N} \pi_{Xk} = 0\,, \tag{2.3b,c}$$

$$\sum_{k=1}^{N} \mathcal{R}_{Xk}\mathcal{R}_{Yk} = \delta_{XY}\,, \qquad \sum_{k=1}^{N} \mathcal{R}_{Xk}\pi_{Yk} = 0\,. \tag{2.3d,e}$$

Equations (2.1)-(2.3) allow for the inverse transformation

$$X_\alpha = \frac{1}{N}\sum_{k=1}^{N} x_{\alpha k}\,, \qquad P_\alpha = \sum_{k=1}^{N} p_{\alpha k}\,, \tag{2.4a,b}$$

$$\sum_\beta Q_{\alpha\beta}\, \mathcal{D}_{A\beta} = \lambda_A \mathcal{D}_{A\alpha}\,, \qquad \text{cycl } A,B,C, \tag{2.4c}$$

$$\mathcal{R}_{An} = \frac{1}{\lambda_A} \sum_\alpha \mathcal{D}_{A\alpha}\, \bar{x}_{\alpha k}\,, \qquad \text{cycl } A,B,C, \tag{2.4d}$$

$$p_A = \sum_{\alpha,k} \mathcal{D}_{A\alpha}\, \mathcal{R}_{Ak}\, p_{\alpha k}\,, \qquad \text{cycl } A,B,C, \tag{2.5a}$$

$$\tilde{L}_{AB} \equiv L_C = \sum_{X,Y,\alpha,k} \varepsilon_{CXY}\, \lambda_X\, \mathcal{R}_{Xk}\, \mathcal{D}_{Y\alpha}\, p_{\alpha k} = \sum_{X,Y,k} \varepsilon_{CXY}\, \bar{x}_{Xk} p_{Yk}\,, \qquad \text{cycl } A,B,C \tag{2.5b}$$

$$\tilde{J}_{AB} \equiv J_C = \sum_{X,Y,\alpha} \varepsilon_{CXY}\, \lambda_X\, \mathcal{R}_{Yk}\, \mathcal{D}_{X\alpha}\, p_{\alpha k} = \sum_{X,Y,k} \varepsilon_{CXY}\, \frac{\lambda_X}{\lambda_Y}\, \bar{x}_{Yk}\, p_{Xk}\,, \qquad \text{cycl } A,B,C, \tag{2.5c}$$

$$\pi_{Am} = \lambda_A \times \sum_{\alpha,n} \mathcal{D}_{A\alpha}\, f_{mn}\, p_{\alpha n}\,, \qquad \text{cycl } A,B,C, \tag{2.5d}$$

where

$$\bar{x}_{\alpha n} = x_{\alpha n} - X_\alpha\,, \tag{2.6}$$

$$Q_{\alpha\beta} = \sum_{k=1}^{N} \bar{x}_{\alpha k}\, \bar{x}_{\beta k}\,, \tag{2.7}$$

are the coordinates of the N-th particle in the center of mass system and the quadrupole tensor, respectively, and

$$f_{kl} = \delta_{kl} - \frac{1}{N} - \sum_{A=1}^{3} \mathcal{R}_{Ak}\, \mathcal{R}_{Al}\,, \tag{2.8}$$

appear to be the projecting matrices $(N \geq 4)$. One can easily check the expressions (2.5a-d) noticing that

$$\sum_n \mathcal{R}_{Xn} f_{nk} = 0, \qquad \text{for } X=A,B,C;\ k=1,\dots,N.$$

The matrix $\mathcal{D}_{A\alpha}$ depends on the three Euler angles $\theta_1, \theta_2, \theta_3$ and describes the SO(3) transformation from the laboratory to the intrinsic frame of references, the latter being defined so that the quadrupole tensor is diagonal in it: $Q = \mathrm{diag}(\lambda_A^2, \lambda_B^2, \lambda_C^2)$. Here we assume that $\lambda_A > \lambda_B > \lambda_C$, then the matrix $\mathcal{D}(\theta)$ is determined up to the transformations of the group D_2 of rotations through the angle π around the principal axes of the quadrupole tensor.
Throughout the paper we denote by α, β, γ the indices of SO(3) ten-

sors in the laboratory frame, and we use the characters A,B,C for their indices in the intrinsic frame. When convenient, we use the pseudovectors instead of the antisymmetric tensors ($F_A = \frac{1}{2}\varepsilon_{ABC}\tilde{F}_{BC}$) and the sum convention for the repeated indices.

The quantities $\lambda_A, \lambda_B, \lambda_C$ characterize the size and the quadrupole (ellipsoidal) deformation of the system. The quantity L is just the angular momentum, while J has a name of the vortex spin vector. The conditions (2.3a-d) are valid both in classical and quantum mechanics. The quantum generalization of (2.3,e) takes the form

$$\sum_{k=1}^{N} [\mathcal{R}_{Xk}\pi_{Yk} + \pi_{Yk}\mathcal{R}_{Xk}] = 0 , \qquad \text{for } X,Y=A,B,C,$$

where the momenta $\Pi(\varphi_1,\varphi_2,\dots,\varphi_{3N-6})_{Yk}$ and J_A, J_B, J_C are the differential operators with respect to the factor space $M \equiv O(N-4)\backslash\backslash O(N-1)$ which can be parametrised by the set of angles $\varphi_1,\varphi_2,\dots,\varphi_{3N-9}$ representing the intrinsic degrees of freedom. On the other side, one can think of the quantities $\vec{\mathcal{R}}_k \equiv (\mathcal{R}_{Ak},\mathcal{R}_{Bk},\mathcal{R}_{Ck})$, and $\vec{\pi}_k = (\pi_{Ak},\pi_{Bk},\pi_{Ck})$, J as of the functions on some 3N-9 independent coordinates and 3N-9 momenta defined in such a way that the relations (2.3b-e) be satisfied identically.

In the following we use extensively the bracket notation ⟨a, b⟩. The latter could be understood either in the classical sence as the Poisson bracket, or quantum mechanically as the commutator times the factor $(\hbar i)^{-1}$ The name "Poisson bracket" is used in the paper independently of the meaning provided the clarification is not needed. The calculations yield

$$\langle \pi_{Ak}, \pi_{Bl} \rangle = -f_{kl}\tilde{J}_{AB} + (\mathcal{R}_{Al}\,\pi_{Bk} - \mathcal{R}_{Bk}\,\pi_{Al}) , \qquad (2.9a)$$

$$\langle \pi_{Ak}, \mathcal{R}_{Bl} \rangle = -\delta_{AB}\, f_{kl} , \qquad (2.9b)$$

$$\langle J_A, \pi_{Bk} \rangle = -\varepsilon_{ABC}\,\pi_{Ck} , \qquad \langle J_A, \mathcal{R}_{Bk} \rangle = -\varepsilon_{ABC}\,\mathcal{R}_{Ck} , \qquad (2.9c,d)$$

and[12]

$$\langle J_A, J_B \rangle = -\varepsilon_{ABC}\, J_C , \qquad (2.10a)$$

$$\langle L_A, L_B \rangle = -\varepsilon_{ABC}\, L_C , \qquad \langle L_A, \mathcal{D}_{B\alpha} \rangle = -\varepsilon_{ABC}\,\mathcal{D}_{C\alpha} , \qquad (2.10b,c)$$

$$\langle \lambda_A, P_B \rangle = \delta_{AB} , \qquad (2.10d)$$

$$\langle X_\alpha, P_\beta \rangle = \delta_{\alpha\beta} . \qquad (2.11)$$

In arriving at (2.9a-d) we use expressions (2.26) and the well known Poisson bracket relation for the j_{mn} generators of the so(N) algebra. All the other omitted brackets ⟨...⟩ are equal to zero. In the new coordinates the Hamiltonian has the form

$$H = \frac{1}{2Nm}\vec{P}\cdot\vec{P} + T^{vib} + T^{rot} + T^{intr} + U , \qquad (2.12)$$

where

$$T^{vib} = \frac{1}{2m}\sum_B P_B^2 , \qquad T^{intr} = \sum_B \sum_k \frac{\pi_{Bk}^2}{\lambda_B^2} , \qquad (2.13a,b)$$

$$T^{rot} = \frac{1}{2m}\sum_{A>B} \frac{1}{(\lambda_A^2 - \lambda_B^2)^2} \Big\{ (\lambda_A^2 + \lambda_B^2)(\tilde{L}_{AB}^2 + \tilde{J}_{AB}^2) + + 4\lambda_A\lambda_B\tilde{L}_{AB}\tilde{J}_{AB} \Big\} , \qquad (2.14)$$

$$U = U^0(\lambda_A,\lambda_B,\lambda_C) + U^{tens} , \qquad (2.15a)$$

$$U^{tens} = \sum_{p=2}^{\infty} U^{(2p)} , \qquad (2.15b)$$

$$U^{(m)} = \sum_{\Sigma p_X = m} f(\lambda_A,\lambda_B,\lambda_C)^{[p_A p_B p_C]}\, R_{[p_A p_B p_C]} , \qquad (2.15c)$$

$$R_{[p_A p_B p_C]} = \sum_{k=1}^{N} \prod_{A,B,C} (R_{Ak})^{p_A} , \qquad (2.15d)$$

The potential U contains the collective part U^0 which is a scalar with respect to the O(N-1) group and the tensor part U^{tens}. In principle, all terms in eq.(2.15a,b) may be calculated once the particle-particle interaction is known. Both parts of the potential can be generalized for spin-isospin forces.[15-16] In ref.15 some detailed calculation of U^0 generalized in this way for central nucleon-nucleon forces is made. The role of the tensor two-body interactions in forming the collective potential $U(\lambda_A \lambda_B \lambda_C)$ is given in.[16]

Describing the motion in the collective space in a classical way we introduce the pseudovector of the frequency of rotation $\vec{\Omega}$ and another pseudovector $\vec{\Lambda}$ which we call the **vortex** velocity[12]:

$$\begin{bmatrix} \Omega_C \\ \Lambda_C \end{bmatrix} = \begin{bmatrix} \partial H/\partial L_C \\ \partial H/\partial J_C \end{bmatrix} = \frac{1}{m(\lambda_A^2 - \lambda_B^2)^2} \begin{bmatrix} (\lambda_A^2+\lambda_B^2) & 2\lambda_A\lambda_B \\ 2\lambda_A\lambda_B & (\lambda_A^2+\lambda_B^2) \end{bmatrix} \begin{bmatrix} L_C \\ J_C \end{bmatrix} . \qquad (2.16)$$

We introduce also the symmetric tensor

$$Y_{AB} = \frac{1}{2} \sum_k^N (\pi_{Ak}\pi_{Bk} + \pi_{Bk}\pi_{Ak}) , \qquad (2.17)$$

which is called the Fermi-energy tensor, and also another tensor

$$\tilde{\mathcal{L}}_{AB} = \delta_{AB} \mathcal{L}_A = \frac{\delta_{AB}}{m\lambda_A^2} . \qquad (2.18)$$

Using the Poisson bracket relations one finds the following system of equations:

$$\dot{\lambda}_A - \frac{1}{m} P_A = 0 , \qquad (2.19a)$$

$$\dot{P}_A - \frac{1}{2m} \sum_{B(\neq A)} [(\lambda_A-\lambda_B)^{-3}(\tilde{L}_{AB}+\tilde{J}_{AB})^2 + (\lambda_A+\lambda_B)^{-3}(\tilde{L}_{AB}-\tilde{J}_{AB})^2] -$$
$$- \frac{\mathcal{L}_A}{\lambda_A} Y_{AA} + \frac{\partial U^0}{\partial \lambda_A} = - \frac{\partial U^{tens}}{\partial \lambda_A} , \qquad (2.19b)$$

$$\dot{J}_A + \sum_{B,C} \varepsilon_{ABC}(\Lambda_B J_C + \mathcal{L}_B Y_{BC}) = 0 , \qquad (2.19c)$$

$$\dot{L}_A + \sum_{B,C} \varepsilon_{ABC} \Omega_B L_C = 0 , \qquad (2.19d)$$

$$\frac{d}{dt}\begin{bmatrix} \vec{\mathcal{R}}_k \\ \vec{\pi}_k \end{bmatrix} - \begin{bmatrix} \tilde{\Lambda} & \tilde{\mathcal{L}} \\ -Y\tilde{\mathcal{L}} & \tilde{\Lambda}-\tilde{J}\tilde{\mathcal{L}} \end{bmatrix} \begin{bmatrix} \vec{\mathcal{R}}_k \\ \vec{\pi}_k \end{bmatrix} = -\begin{bmatrix} 0 \\ DU^{tens}/d\vec{R}_k \end{bmatrix} , \qquad (2.19e,f)$$

$$\dot{\mathcal{D}}_{A\alpha} + \sum_{B,C} \varepsilon_{ABC} \Omega_B \mathcal{D}_{C\alpha} = 0 , \qquad (2.19g)$$

$$\dot{X}_\alpha = \frac{1}{N m} P_\alpha , \qquad \dot{P}_\alpha = 0 . \qquad (2.19h,i)$$

The r.h.s. of eqs.(2.19e,f) contains the tensorial part of the interaction. It is calculated using eqs.(2.15b-d) and the formula

$$\frac{D\mathcal{R}_{[p]}}{D\mathcal{R}_{Ak}} = \sum_{m=1}^N f_{km} \frac{\partial \mathcal{R}_{[p]}}{\partial \mathcal{R}_{Am}} . \qquad (2.20)$$

Calculating the time derivative of the expressions on the l.h.s. of eqs.(2.3a-e) it is easy to check that these are constants of the motion. Of course, this signifies only the consistency of the equations of motion (2.19a-i) with the chosen parametrization of the nuclear coordinates and momenta.

Equations (2.19a-i) may be understood both on the level of the quantum mechanics and in the spirit of classical mechanics. In particular, eqs.(2.19a-i) can be regarded as operator relations in the Heisenberg representation. Formally at least, one may take another point of view on the same equations understanding them as differential equations determining the coordinates of the N-body system in the phase space. Then, these equations define the trajectory specified by some initial conditions which must be consistent with (2.3a-e). The center of mass motion (eqs.(2.19h,i)) and the time dependence of the Euler angles describing the orientation of the main axes of the nuclear inertia tensor in the space (eqs. (2.19g)) are separated from the rest of the equations. Thus, one is left with the closed set of 6N-9 equations. Of course, the energy and total angular momentum $L^2 = L_A^2 + L_B^2 + L_C^2$ are also the integrals of motion and the system of equations may be reduced to two more units.

The drastic reduction of the system takes place when the $O(N-1)$ tensorial part of the interaction vanishes ($U^{tens}=0$). The following part of the paper is devoted to this particular case: to the $O(N-1)$ invariant model of the nucleus. Using equations (2.19f) one easily finds that the time derivative of Y_{AB} (see eq.(2.17)) satisfies the equations

$$\dot{Y}_{AB} - \sum_{Y} [(\tilde{\Lambda}_{AY} - \tilde{J}_{AY}\mathscr{L}_{Y})Y_{YB} + (\tilde{\Lambda}_{BY} - \tilde{J}_{BY}\mathscr{L}_{Y})Y_{AY}] = 0 . \qquad (2.19e')$$

The latter equations and eqs.(2.19a-d) form a closed set of differential equations. The fact that the subsystem of equations for eighteen collective variables $(q_1,q_2,\ldots,q_{18}) \equiv (\lambda_X,p_X,L_X,J_X,Y_{XY})$ $(X,Y=A,B,C)$ is closed has a very simple algebraic explanation. It is so because the coordinates (q), the three Euler angles $\theta_1,\theta_2,\theta_3$ and also the Hamiltonian H belong to the enveloping field of observables in the noncompact $sp(6,R)$ algebra.[17] It means that the above-mentioned dynamical variables are functions depending only on the $sp(6,R)$ generators. These variables satisfy the Poisson bracket relations of the form

$$\{q_k,q_l\} = e_{kl} + c^{m}_{kl}\, q_m + d^{mn}_{kl}\, q_m\, q_n , \qquad (2.21)$$

where e_{kl}, c^{m}_{kl} and d^{mn}_{kl} are constant. In particular, one has

$$\{L_A,Y_{BC}\} = \{\lambda_A,Y_{BC}\} = \{p_A,Y_{BC}\} = 0 , \qquad (2.22)$$

$$\{J_A,Y_{BC}\} = -(\varepsilon_{ABX}Y_{XC} + \varepsilon_{ACX}Y_{BX}) , \qquad (2.23)$$

$$\{Y_{AB},Y_{CD}\} = -\tilde{J}_{AC}Y_{BD} - \tilde{J}_{AB}Y_{BC} - \tilde{J}_{BC}Y_{AD} - \tilde{J}_{BD}Y_{AC} . \qquad (2.24)$$

Relations (2.10a-d) together with (2.23-2.24) introduce the very transparent structure in the space of the collective variables (q) $(q)\equiv (\lambda_A,\lambda_B,\lambda_C;P_A,P_B,P_C)\times(L)\times(J,Y)$.

This means that quantization of the collective motion in these coordinates is also nontrivial. The problem is greatly simplified in the two-dimensional case. In Appendix A we show that four elements $Y_{AA},Y_{BB},Y_{AB},\tilde{J}_{AB}$ can be expanded in terms of the four elements of the $u(2)$ algebra.

It is obvious that the algebraic structure (2.21) leads to additional integrals of the motion. In fact we find ($U^{tens}=0$):

$$\{j_{kl},H\} = 0 , \qquad (2.25)$$

$$j_{kl} \overset{df}{=} (\mathcal{R}^{T}\pi - \pi^{T}\mathcal{R} + \mathcal{R}^{T}\tilde{J}\,\mathcal{R})_{kl} . \qquad (2.26)$$

The above integrals of the motion are not linearly independent and we introduce six new functions:

$$C_{2p} = \mathrm{trace}\,(j)^{2p} , \qquad p=1,2,3 \qquad (2.27a,b,c)$$

$$C_2' = \mathrm{trace}\,(2Y + K) , \qquad (2.28a)$$

$$C_4' = \mathrm{trace}\,((2Y + K)^2 - 2Y^2) , \qquad (2.28b)$$

$$C_6' = \mathrm{trace}\,((Y+K)^3 + 3YK^2 - 3YJYJ + 3Y^2K + Y^3) , \qquad (2.28c)$$

where we use the notation

$$K_{AB} = \sum_{X=1}^{3} \tilde{J}_{AX}\tilde{J}_{BX} ,$$

$$\mathrm{trace}\, F^1F^2\ldots F^n = \sum_{(x_1\ldots x_n)} F^1_{x_1x_2}F^2_{x_2x_3}\ldots F^n_{x_nx_1} .$$

The integrals C_2,C_4,C_6 are independent Casimir invariants of the $so(N-1)$ algebra of operators defined in the space of variables $\varphi_1,\varphi_2,\ldots,\varphi_{3N-9}$ (see eq.(2.1)) representing the intrinsic degrees of freedom. The ingredients of C_2',C_4',C_6' depend on the generators of the sp(6,R) algebra associated with the nuclear collective dynamics. Using eq.(2.17) and the constraints (2.3d,e) one immediately finds that

$$C_{2p} = C_{2p}{}' . \tag{2.29}$$

Hence, $C_{2p}{}'$ are also the integrals of the motion (the Casimir invariants of the sp(6,R) algebra). It is obvious that expressions (2.28a-c) may be formulated on the quantum level too. Then, eqs. (2.29) are the operator relations between the Casimir invariants of the algebra sp(6,R) and so(N-1) which are operating on the quantum N-body space $\mathcal{H}^N$. From the other[18] studies of the group chain decomposition

$\mathcal{H}^N \times Sp(6N-6,R) \times Sp(6,R) \times O(N-1)$

it is known that the reduction of the $\mathcal{H}^N$ which contains only two representations of the sp(6N-6,R) (the centrum mass of the motion being eliminated from) is multiplicity free and may by written in the form

$$\mathcal{H}^N \times \sum_{f_1 \geq f_2 \geq f_3 \geq 0} \sum \sum \mathcal{H}(f_1 f_2 f_3)^N , \tag{2.30a}$$

with

$$\mathcal{H}(f_1 f_2 f_3)^N = \langle f_1' f_2' f_3' \rangle^{sp(6,R)} \otimes (f_1 f_2 f_3 0^{(n-7)/2})^{so(N-1)} , \tag{2.30b}$$

$$f_\iota{}' = f_\iota + \frac{N-1}{2} .$$

The eigenvalues of the Casimir invariants of the so(N-1) algebras are well known.[19] Approximation which can be applied even for rather light nuclei is given by the formulas

$$C_{2p}{}' \cong 2 \sum_{\iota=1}^{3} ((f_\iota + \frac{N-5}{2})\hbar)^{2p} \equiv 2(f_A^{2p} + f_B^{2p} + f_C^{2p}) . \tag{2.31}$$

The domain of validity of eqs.(2.31) is examined in Appendix B.

It is rather obvious that the spin-isospin space must be included to apply the above results for the nuclear system, and the selection of the states allowed by the Pauli principle must by done. We say that the sp(6,R) representation $\langle f_1 + \frac{N-1}{2}, f_2 + \frac{N-1}{2}, f_3 + \frac{N-1}{2} \rangle$ is allowed when the set of the antisymmetric wafe functions with the O(N-1) orbital structure $(f_1 f_2 f_3 00 \ldots 0)^{N-1}$ is not empty. Due to the group reductions $O(N-1) \supset S_N^{orb}$ where S_N^{orb} is the symmetric group (the group of the permutations of the space particle vectors) the last problem may be solved in the pure group-theoretical way.[20] We say also that the O(N-1) invariant classical model is prequantized when the initial conditions are chosen in such a way that the part of the phase space belonging to the forbidden values of the invariants $C_{2p}{}'$ is truncated.

III. Semiclassical description of the nuclear collective dynamics

The closed system of 18 equations containing the dynamical variables $\langle q \rangle$, i.e. λ_X, p_X, L_X, J_X, Y_{XY} (X,Y=A,B,C) describes the nuclear collective dynamics. One may think that in heavy nuclei as in all large systems in the states which are not too far from the yrast line the time evolution of these quantities consists in small oscillations around some mean values well defined in the intrinsic frame of references. Intuitively one may think also that the dynamics of collective variables may be studied in the semiclassical approximation. In other words, one may consider these equations as differential equations defining the trajectory in the collective part of the phase space.

It is appropriate to notice that eqs.(2.21a-d), (2.22e') may be deduced from the Liouville equation for the distribution function $f(\vec{r},\vec{p},t)$ giving the probability to find a nucleon at a point

$\vec{r}$ having the linear momentum $\vec{p}$ at a given time t.[5] In the procedure described in ref.(5) the main role is played by the assumptions on the nuclear shape and velocity field of a collective flow: the shape is assumed to remain ellipsoidal all the time, and the velocity field

$$\vec{u} \overset{df}{=} \int dp^3 f(\vec{r},\vec{p},t)\, \frac{\vec{p}}{m} \,,$$

is parametrized as follows

$$(\vec{u}_{body})_A = \sum_{B,C} \varepsilon_{ABC} \frac{a_A}{a_B} x_B \Lambda_C + \frac{1}{a_A} \frac{da_A}{dt} x_A \,.$$

The consistency of the assumption on the ellipsoidal shape of nuclear surface and on the so-defined velocity field was proven by Riemann.[21]

The following mapping of the variables makes the equations obtained in this way identical with eqs.(2.19a-d),(2.19e')

– $\lambda_A^2 = N \langle x_A^2 \rangle = \frac{N}{5} a_A^2$ with a_A being the semiaxes of the ellipsoid describing the surface,

– Ω_A is the angular frequency of rotation of the nuclear inertia tensor projected on its main axes,

– $\Lambda_A = - \frac{a_B a_C}{a_A^2 + a_B^2} (\mathrm{rot}\, \vec{u}_{body})_A$, $(A \neq B \neq C)$,

– $Y_{AB} = a_A \Pi_{AB} a_B$,

where the intergrated pressure tensor Π is defined as follow

$$\Pi_{AB} = \frac{1}{m} \int\int dr^3 dp^3 (p_A - mu_A)(p_B - mu_B)\, f(\vec{r},\vec{p},t) \,.$$

Among the solutions of the differential equations (2.19a-d), (2.19e') of special importance are the stationary ones describing the state of equilibrium of atomic nuclei. The search for the stationary solutions is facilitated by the analogy of these equations with the equations describing the motion of classical liquid droplet with ellipsoidal shape.[14] This motion is described by a special case of eqs.(2.19a-e') with $Y_{AB} = \lambda_A \lambda_B \Pi\, \delta_{AB}$ (Π=const). For the Fermi liquid from the well known families of stationary solutions (S- and P-Riemann ellipsoids) there remain only those for which $\vec{L} \parallel \vec{J}$ and both are directed along one of the principal axis of the inertia tensor. We call these states as the generalized S-Riemann ellipsoids. Hence, denoting the values of the collective quantities at the stationary points by $\langle \mathring{q}_1, \ldots, \mathring{q}_{18} \rangle \equiv \langle \mathring{\lambda}_A, \mathring{\lambda}_B, \ldots \rangle$ we arrive at the following equations:

$$\mathring{L}_A = \mathring{L}_B = \mathring{J}_A = \mathring{J}_B = 0 \,, \qquad (3.1a\text{-}d)$$

$$\dot{Y}_{AB} = \dot{Y}_{BC} = \dot{Y}_{CA} = 0 \,, \qquad (3.2a\text{-}d)$$

$$\mathring{P}_A = \mathring{P}_B = \mathring{P}_C = 0 \,, \qquad (3.3a\text{-}c)$$

$$\mathring{\Lambda}_C = \frac{\mathring{J}_C}{m} \times \frac{\mathring{Y}_{AA} / \mathring{\lambda}_A^2 - \mathring{Y}_{BB} / \mathring{\lambda}_B^2}{\mathring{Y}_{AA} - \mathring{Y}_{BB}} \,, \qquad (3.4)$$

$$\omega_X^2 = (m\, \mathring{\lambda}_X)^{-1} \left\{ \frac{\partial U^0}{\partial \lambda_X} \right\}_{[\lambda]=[\mathring{\lambda}\ ]} \,, \qquad (X=a,B,C) \qquad (3.5a,b,c)$$

$$\omega_A^2 = \frac{\mathring{Y}_{AA}}{m^2\, \mathring{\lambda}_A^4} + \mathring{\Omega}_C^2 + \mathring{\Lambda}_C^2 - 2\frac{\mathring{\lambda}_B}{\mathring{\lambda}_A} \mathring{\Omega}_C \mathring{\Lambda}_C \,, \qquad (3.6a)$$

$$\omega_B^2 = \frac{\mathring{Y}_{BB}}{m^2\, \mathring{\lambda}_B^4} + \mathring{\Omega}_C^2 + \mathring{\Lambda}_C^2 - 2\frac{\mathring{\lambda}_A}{\mathring{\lambda}_B} \mathring{\Omega}_C \mathring{\Lambda}_C \,, \qquad (3.6b)$$

$$\omega_C^2 = \frac{\mathring{Y}_{CC}}{m^2\, \mathring{\lambda}_C^4} \,. \qquad (3.6c)$$

Equations (3.5) and (3.6a,b,c) express the balance of all the forces acting on the nuclear matter in the state of equilibrium corresponding to the extended S-Riemann ellipsoid. The r.h.s. of eq.(3.5) is determined by the potential and depends on the nuclear shape (i.e. on $\mathring{\lambda}_A, \mathring{\lambda}_B, \mathring{\lambda}_C$), while the l.h.s. depends also on other characteristics of the motion of nucleons: the centrifugal and Coriolis forces and the stresses produced by the intrinsic motion. The latter relate the vortex spin $\mathcal{J}_C$ to the total angular momentum $\mathring{L}_C$ in accordance with eq.(3.4) (see also eq.(2.16) in which $\mathring{\Omega}_C$, $\mathring{\Lambda}_C$ are given as functions $\mathcal{J}_C$, $\mathring{L}_C$). The elements $\mathring{Y}_{XX}$ appear in equations (3.4)-(3.a.b,c) as parameters representing the equation of the state of the nuclear matter. If they are known the above algebraic system is closed for given values of $\mathcal{J}_C$. The selection of real solutions of this system leads to the stationary rotational bands of the system and will be discussed in the next paper.

In the present approach the equation of state of the nuclear matter is determined unambiguously by the quantum numbers f_1, f_2, f_3 characterising the dependence of the wave-function on the intrinsic coordinates. The central role in fixing the elements of the pressure tensor is played by the symmetry conditions (2.31) relating the values of the Casimir invariants of the so(N-1) and sp(6,R) algebras. When the equilibrium conditions (3.1a-d) and (3.2a-c) are satisfied, the integrals of motion C'_{2p} can easily be calculated and presented in the form

$$C'_{2p} = 2\sum_{X} \mathring{\Sigma}_X^{2p} , \qquad (3.7)$$

where

$$\mathring{\Sigma}^2_{A(B)} = \frac{1}{2}(\mathring{Y}_{AA} + \mathring{Y}_{BB} + \mathcal{J}^2_C) \pm \frac{1}{2}\left[(\mathring{Y}_{AA} - \mathring{Y}_{BB})^2 - 2\mathcal{J}^2_C(\mathring{Y}_{AA} + \mathring{Y}_{BB}) + \mathcal{J}^4_C\right]^{1/2}, \qquad (3.8a,b)$$

$$\mathring{\Sigma}^2_C = \mathring{Y}_{CC} . \qquad (3.8c)$$

On the strenght of eq.(2.31) one concludes that

$$\mathring{Y}_{AA(BB)} = \frac{1}{2}(f_A^2 + f_B^2 - \mathcal{J}^2_C) \pm$$

$$\pm \frac{1}{2}\left[[(f_A + f_B)^2 - \mathcal{J}^2_C][(f_A - f_B)^2 - \mathcal{J}^2_C]\right]^{1/2} , \qquad (3.9a,b)$$

$$\mathring{Y}_{CC} = f_C^2 . \qquad (3.9c)$$

These relations close the set of equations determining the nuclear shape in the rotational bands. For given f_A, f_B, f_C there exist three classes of equilibrium solutions with the vortex spin values $\mathcal{J}_C$ lying in the interval

$$0 \le \mathcal{J}^2_C \le (f_A - f_B)^2 ,$$

where f_A and f_B represent one of the three pairs of the quantum numbers $\hbar(f_1 + \frac{N-5}{2})$, $\hbar(f_2 + \frac{N-5}{2})$, $\hbar(f_3 + \frac{N-5}{2})$. The shape and the angular momentum of the state are determined by eqs.(3.4)-(3.6c), and the total energy can easily be found from the equations presented in Section II. It is useful to rewrite this system of equations (3.4), (3.5a-c) in a somewhat different way. Equation (3.4) may be written in the form

$$\mathring{L}_C = -\mathcal{J}_C \frac{1}{(1-z^2)^{1/2}} \left[1 - \frac{2z^2}{1-z^2}(1 - z\mathring{Y}/\Delta\mathring{Y})\right] \qquad (3.10)$$

where

$$z = \frac{\mathring{\lambda}_A^2 - \mathring{\lambda}_B^2}{\mathring{\lambda}_A^2 + \mathring{\lambda}_B^2} , \qquad (3.11)$$

$$\mathring{Y} = \mathring{Y}_{AA} + \mathring{Y}_{BB} , \qquad \Delta\mathring{Y} = \mathring{Y}_{AA} - \mathring{Y}_{BB} , \qquad (3.12a,b)$$

while eqs.(3.4a-c) read

$$(m\mathring{\lambda}_A\mathring{\lambda}_B)^{-1}(1-z^2)^{-5/2} \sum_{n=0}^{5} A_n z^n = \left(\lambda_A \frac{d}{d\lambda} - \lambda_B \frac{d}{d\lambda}\right)U^0\Big|_{[\lambda]=[\mathring{\lambda}]}, \qquad (3.13a)$$

$$\mathring{T}_{AA} + \mathring{T}_{BB} - 2\mathring{T}_{CC} = \frac{1}{2}\left(\lambda_A \frac{d}{d\lambda_A} + \lambda_B \frac{d}{d\lambda_B} - 2\lambda_C \frac{d}{d\lambda_C}\right)U^0\Big|_{[\lambda]=[\mathring{\lambda}]}, \qquad (3.13b)$$

$$\mathring{T}_{AA}+\mathring{T}_{BB}+\mathring{T}_{CC} = \frac{1}{2}\sum_{x} \lambda_x \frac{d}{d\lambda_x} U^o \Big|_{[\lambda]=[\mathring{\lambda}]} . \qquad (3.13c)$$

In eqs.(3.13a) the coefficients A_n are determined as follows

$$A_o = \Delta\mathring{Y} , \qquad A_1 = \frac{5}{2}\mathring{J}^2-\mathring{Y} , \quad A_2 = 2\Delta\mathring{Y}-6B_1 , \quad A_3 = -2A_1+4B_2 ,$$

$$A_4 = \Delta\mathring{Y}+2B_1 , \quad A_5 = A_1-2B_2 , \quad B_1 = \mathring{Y}/\Delta\mathring{Y} , \; B_2 = [1+(\mathring{Y}/\Delta\mathring{Y})^2]\times \mathring{J}^2_C ;$$

The quantities $\mathring{T}$ in eqs.(3.13b), (3.13c) are the components of the kinetic energy tensor corresponding to the stationary solution

$$\mathring{T}_{AA}+ \mathring{T}_{BB} = (m\mathring{\lambda}_A\mathring{\lambda}_B)^{-1}\times\Big\{(1-z^2)^{-1/2}[\tfrac{1}{2}(\mathring{Y}-z\Delta\mathring{Y})+ \tfrac{1}{4}\mathring{J}^2_C] +$$

$$+ \mathring{J}^2_C(1-z^2)^{-5/2}z^3(1-z\mathring{Y}/\Delta\mathring{Y})(z-\mathring{Y}/\Delta\mathring{Y})\Big\} , \qquad (3.13)$$

$$\mathring{T}_{CC} = \frac{1}{2m}\frac{\mathring{Y}_{CC}}{\mathring{\lambda}^2_C} . \qquad (3.14)$$

The left-hand side of eq.(3.13c) is just the total kinetic energy of the nucleus.

Appendix A

Let us introduce the four dimensional algebra u(2) $(\sigma_o,\sigma_1,\sigma_2,\sigma_3)$. Hence we assume that the structural constants for this algebra are given by the equations

$$f_{ij}^{\;k} = 2\varepsilon_{ijk} \qquad \text{for } i,j,k = 1,2,3,$$

$$f_{oi}^{\;j} = f_{io}^{\;j} = f_{ij}^{\;o} = 0 \qquad \text{for } i,j = 1,2,3.$$

The realization of the quantities $\tilde{J}_{XY}$, Y_{XY} $(X,Y\in(A,B))$ reads

$$Y = Y_{AA}+ Y_{BB} = \frac{1}{2}(\sigma_o^2 +\sigma_1^2 -\sigma_2^2 +\sigma_3^2) , \qquad (A.1a)$$

$$\Delta Y = Y_{AA}- Y_{BB}= \sigma_1\times (\sigma_o^2 -\sigma_2^2)^{1/2} , \qquad (A.1b)$$

$$Y_{AB}= \sigma_3\times (\sigma_o^2 -\sigma_2^2)^{1/2} , \qquad (A.1c)$$

$$\tilde{J}_{AB}= \sigma_2 . \qquad (A.1d)$$

In fact, using the relation

$$(F,G) = (\partial^i F)(\partial^j G)\; f_{ij}^{\;k}\,\sigma_k ,$$

we recover the relations

$$(\tilde{J}_{AB}, Y) = 0, \qquad (\tilde{J}_{AB},\Delta Y) = -4Y_{AB}, \qquad (\tilde{J}_{AB},Y) = \Delta Y, \qquad (A.2a-c)$$

$$(Y,\Delta Y) = 8Y_{AB}, \qquad (Y,Y_{AB}) = 2\tilde{J}_{AB}\Delta Y, \qquad (Y_{AB},\Delta Y) = 2\tilde{J}_{AB}Y, \qquad (A.2d-f)$$

identical with those in eqs.(2.23-2.24).

The relations (A.1a-d) open the way of studying the quantum O(N-1) invariant model. In the two-dimensional case it is achieved by mapping (q) on to the set of the self adjoint operators of the algebra u(2). The remark should be added concerning the generalization of the theory to the three-dimensional case. In[22] the formulas are developed giving the matrix elements for the operators $\tilde{J}$, **Y** defined as the operators in the many body subspaces $\mathscr{P}(f_1 f_2 f_3)^N$ (2.30a,b). It is found that the dimension of the matrix representation of the $\tilde{J}$, **Y** obtained in this way coincides with that of the irreducible representation of the u(3) algebra. However, no explicit relations are known for these operators in terms of the generators of the u(3) algebra.

Appendix B

Some comments are necessary to establish the domain in which expressions (2.31) for the eigenvalues of the Casimir invariants C_{2p} and the semiclassical relations (3.9a-c). We are going to show that the approximation (2.31) works much better than one can think judging from the exact formulas for these invariants and that eqs. (3.9a-c) can be applied even for the system with relatively few nucleons. In order to discuss this problem we apply the quantum expressions for the Hamiltonian[11]

$$\hat{H}^{vib} = -\frac{\hbar^2}{2m}\sum_{A}\left\{\frac{\partial^2}{\partial\lambda_A^2} + \frac{N-4}{\lambda_A}\frac{\partial}{\partial\lambda_A} + 2\sum_{B(\neq A)}\frac{\lambda_A}{\lambda_A^2-\lambda_B^2}\frac{\partial}{\partial\lambda_A}\right\}, \qquad (B.1)$$

The unitary transformation

$$\hat{F}' = \hat{f}^{-1}\,\hat{F}\,\hat{f}\,, \qquad (B.2)$$

$$\hat{f} = (\lambda_A\lambda_B\lambda_C)^{-(N-4)/2}[(\lambda_A^2-\lambda_B^2)(\lambda_B^2-\lambda_C^2)(\lambda_A^2-\lambda_C^2)]^{-1/2}, \qquad (B.3)$$

affects only the term $\hat{H}^{vib}$ which transforms into

$$(\hat{H}^{vib})' = -\frac{\hbar^2}{2m}\sum_{A}\frac{\partial^2}{\partial\lambda_A^2} + \hat{U}_1^{quan} + \hat{U}_2^{quan}, \qquad (B.4)$$

$$\hat{U}_1^{quan} = \frac{\hbar^2}{2m}\frac{(N-5)^2-1}{4}\sum_{A}\frac{1}{\lambda_A^2}, \qquad (B.5)$$

$$\hat{U}_2^{quan} = -\frac{\hbar^2}{2m}\sum_{A}\left\{\frac{\lambda_A^2+\lambda_B^2}{(\lambda_A^2-\lambda_B^2)^2} + \frac{\lambda_A^2+\lambda_C^2}{(\lambda_A^2-\lambda_B^2)^2}\right\} - $$

$$+ \frac{\hbar^2}{2m}\sum_{A}\lambda_A^2\left\{\frac{1}{(\lambda_A^2-\lambda_B^2)} + \frac{1}{(\lambda_A^2-\lambda_C^2)}\right\}^2, \qquad (B.6)$$

The term $\hat{U}_1^{quan}$ proportional to N^2 is much more important than $\hat{U}_2^{quan}$. Neglecting $\hat{U}_2^{quan}$ and adding $\hat{U}_1^{quan}$ to the intrinsic kinetic energy one obtains

$$(\hat{T}^{intr})' = \frac{1}{2m}\sum_{A}\frac{Y'_{AA}}{\lambda_A^2}, \qquad (B.7)$$

with Y' being given by

$$Y'_{AB} = Y_{AB} + \tfrac{1}{4}[(N-5)^2-1]\hbar^2\delta_{AB}\,\mathbb{I}\,. \qquad (B.8)$$

The Y' tensor has slightly different meaning as compared with the definition given in the principal part of this paper. Now (2.28a) takes the form

$$\sum_{X}(Y'_{XX} + J_X^2) = \tfrac{1}{2}C_2 + \tfrac{3}{4}((N-5)^2-1)\hbar^2 = \sum_{i}(f_i + \tfrac{N-5}{2})^2\hbar^2 +$$

$$+ (-\tfrac{3}{4} + 2(f_1-f_3))\hbar^2 \approx f_A^2+f_B^2+f_C^2\,. \qquad (B.9)$$

For the same reason the eigenvalues of the quantities C_4, C_6 should be renormalized too. We notice that the transformation (B.8) only approximately conserves the commutation relations, and we obtain some noninvariant terms in C'_4, C'_6. More important for us is the fact that the classical analogues of the operators C_4, C_6 are not uniquely defined. Some other order of the j-operators in eqs.(2.27 b,c) leads to the quantum-mechanically knew invariant operators, whose eingenvalues depend on the ordering of the j-operators. One may use this ambiguity and choose undefined terms in such a way that the relations (2.31) become valid also for C_4', C_6'.

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