объединенный
институт
ядерных
исследований
дубна

E4-88-208

Nguyen Dinh Dang*

DYNAMICAL SYMPLECTIC SYMMETRY IN THE QUASIPARTICLE-PHONON NUCLEAR MODEL

Submitted to "ТМФ"

* Moscow State University - On leave from the Institute of Theoretical Physics - National Center for Scientific Research of Vietnam. - Nghia Dô, Tu Liêm, Hanoi

1988

1. Introduction

Over the past few years the real symplectic group Sp(2d,R) has played an ever-increasing role in many physical problems. In the theory of nuclear collective motion the algebraic Sp(6,R) model has found various applications in light and heavy rotational nuclei[1]. Originating from the pioneering work of various authors as Goshen and Lipkin [2], Cusson[3], Raychev[4], Rosensteel and Rowe[5], Filippov, Vanagas et al.[6], the symplectic model contains as submodels both the Elliott SU(3) model[7] and the Weaver et al. CM(3) model[8].

Recently, in order to consider the electric dipole transitions in the framework of the dynamical group, Quesne has extended the Sp(6,R) model to the WSp(6,R) one[9]. The dynamical group of the latter is the central extension of the inhomogeneous symplectic group or in other words the semidirect product $W(3) \otimes_s Sp(6,R)$ of a Heisenberg-Weyl group W(3) with the Sp(6,R) one.

On the other hand, the success of the phenomenological algebraic models in the description of the nuclear collective properties requires their microscopic foundation based on the use of concrete microscopic nuclear models. One of the latter is the quasiparticle-phonon nuclear model (QPM) suggested and developed by Soloviev and collaborators [10]. The QPM is based on the concrete form of the model Hamiltonian and on the conception of the multicomponent operator wave functions, built on the quasiparticle and phonon operators, by which the model Hamiltonian is expressed. The structure of phonon operators is found by solving a system of RPA equations[9]. In[11,12] it has been shown that under some constraints the QPM phonon operators constitute the closed SU(6) algebra, from which one can derive the microscopic expressions for calculating the parameters of the Interacting Boson Model (IBM). In[13] the generalized conditions under which the QPM Hamiltonian has the SU(m) symmetry and SU(m/n) supersymmetry have been derived.

The aim of the present work is to show that under some conditions the QPM Hamiltonian can be expressed in terms of the generators of the extended symplectic algebra $WSp(2d,R) \supset Sp(2d,R)$. Under these constraints these generators will be constructed directly from the QPM operators. Thus, we will demonstrate that the QPM has the dynamical $WSp(2d,R)$ symmetric limit. We shall also consider the contribution of the symplectic $Sp(2d,R)$ symmetry in the total extended $WSp(2d,R)$ symmetry in this symmetric limit of the QPM.

2. Hamiltonian and properties of the QPM operators

Let us consider the QPM Hamiltonian with the separable multipole-multipole (p-h) interaction in the form[10]

$$H=\sum_{jm} \varepsilon_j \alpha^+_{jm}\alpha_{jm} - \frac{1}{2}\sum_{\lambda\mu}\sum_{\tau,\rho=\pm 1} (\varkappa_0^{(\lambda)}+\rho\varkappa_1^{(\lambda)}) M^+_{\lambda\mu}(\tau) M_{\lambda\mu}(\rho\tau), \qquad (1)$$

where α^+_{jm} (α_{jm}) is the quasiparticle creation (annihilation) operators, ε_j is the quasiparticle energy; $\varkappa_0^{(\lambda)}$ and $\varkappa_1^{(\lambda)}$ denote the isoscalar and isovector constants of the residual interaction of multipolarity λ ; $M^+_{\lambda\mu}(\tau)$ and $M_{\lambda\mu}(\rho\tau)$ are the multipole operators which can be expressed in terms of two-quasiparticle creation $A^+_{\lambda\mu}(jj')$ and annihilation $A_{\lambda\mu}(jj')$ operators, and also, the scattering quasiparticle operator $B_{\lambda\mu}(jj')$ as

$$M^+_{\lambda\mu}(\tau)=\frac{(-)^{\lambda-\mu}}{\hat\lambda}\sum^{\tau}_{jj'} f^{(\lambda)}_{jj'}\left\{\frac{1}{2}u^{(+)}_{jj'}\left[A^+_{\lambda\mu}(jj')+A_{\lambda\tilde\mu}(jj')\right]+v^{(-)}_{jj'}B_{\lambda\mu}(jj')\right\} \qquad (2)$$

with $\quad M_{\lambda\mu}(\rho\tau) = M^+_{\lambda\tilde\mu}(\rho\tau)\ ; \qquad \hat\lambda \equiv \sqrt{2\lambda+1}$

$$\begin{aligned} A^+_{\lambda\mu}(jj') &= \sum_{mm'}(jmj'm'|\lambda\mu)\alpha^+_{jm}\alpha^+_{j'm'} \equiv [\alpha^+_j\otimes\alpha^+_{j'}]_{\lambda\mu} \\ A_{\lambda\mu}(jj') &= \sum_{mm'}(jmj'm'|\lambda\mu)\alpha_{j'm'}\alpha_{jm} \equiv -[\alpha_j\otimes\alpha_{j'}]_{\lambda\mu} \\ B_{\lambda\mu}(jj') &= -\sum_{mm'}(jmj'm'|\lambda\mu)\alpha^+_{jm}\alpha_{\widetilde{j'm'}} \equiv -[\alpha^+_j\otimes\tilde\alpha_{j'}]_{\lambda\mu} \end{aligned} \qquad (3)$$

In Eqs.(2)-(3), $f^{(\lambda)}_{jj'}$ denote the single-particle matrix elements of the multipole operator; $u^{(+)}_{jj'} = u_j v_{j'} + u_{j'} v_j$; $v^{(-)}_{jj'} = u_j u_{j'} - v_j v_{j'}$ are the combinations of the Bogolubov coefficients u, v ; τ denotes the summation over proton (p) and neutron (n) indices: $\tau=\{n,p\}$ and the change $\tau \leftrightarrow -\tau$ corresponds to $n \leftrightarrow p$.

In the QPM the phonon creation Q^+ and annihilation Q operators are introduced[10]

$$\begin{aligned} Q^+_{\lambda\mu i} &= \frac{1}{2}\sum_{jj'}\left\{\psi^{\lambda i}_{jj'}A^+_{\lambda\mu}(jj') - \varphi^{\lambda i}_{jj'}A_{\lambda\tilde\mu}(jj')\right\} \\ Q_{\lambda\mu i} &= (Q^+_{\lambda\mu i})^+, \end{aligned} \qquad (4)$$

where the coefficients ψ and φ are the so-called structures of phonons or the phonon amplitudes. The exact commutation relations for the Q^+, Q and B operators have been derived in [14]. The phonon amplitudes ψ and φ in the QPM are found by solving the RPA equations. These equations have been obtained under some assumptions. First, instead of the exact commutation relation of A and A^+, the quasiboson approximation has been used in which

$$[A_{\lambda\mu}(j_1j_2), A^+_{\lambda'\mu'}(j_1'j_2')] \approx \delta_{\lambda\lambda'}\delta_{\mu\mu'}\left[\delta_{j_1j_1'}\delta_{j_2j_2'} - (-)^{j_1+j_2-\lambda}\delta_{j_1j_2'}\delta_{j_2j_1'}\right], \qquad (5)$$

i.e., the part $\sim\alpha^+\alpha$ has been neglected in the average over the quasiparticle vacuum $|0\rangle$: $\langle 0|\alpha^+\alpha|0\rangle \approx 0$. In other words, it means that the number of quasiparticles in the ground state (the quasiparticle vacuum) is zero. Second, the phonons Q^+ and Q (5) have been considered as ideal bosons and therefore Q^+, Q satisfy the usual boson commutation relations. Under these assumptions the amplitudes ψ and φ satisfy the usual orthonormalized relations given in[10].

If we now use the exact commutation relations from[14], we can easily see that the commutation relations $[Q,Q^+]$, $[Q,Q]$ and $[Q^+,Q^+]$ will be bosonic if the following conditions are satisfied:

$$\sum_p X^{(p-)}_{a,b} B_p = 0 \quad ; \quad \sum_p Y^{(p-)}_{a,b} B_p = 0, \qquad (6)$$

where the notation from[10] is used. Employing the expansion

$$B_p = \sum_{a,b}\left\{X^{(p+)}_{\tilde a,b} Q^+_b Q_a + \frac{1}{2}Y^{(p+)}_{\tilde a,\tilde b} Q_b Q_a + \frac{1}{2}\tilde Y^{(p+)}_{\tilde a,\tilde b} Q^+_b Q^+_a\right\} \qquad (7)$$

we can derive from Eq.(6) the conditions for the phonon amplitudes ψ , φ , under which the phonon Q^+ and Q operators are the ideal bosons. They read

$$\sum_p X^{(p+)}_{a,b} X^{(p-)}_{c,d} = 0 \; ; \; \sum_p X^{(p\pm)}_{a,b} \tilde{Y}^{(p\mp)}_{c,d} = 0 \; ; \; \sum_p Y^{(p+)}_{a,b} \tilde{Y}^{(p-)}_{c,d} = 0. \tag{8}$$

where X, Y are complicated functions of ψ and φ ; $a \equiv \{\lambda_a \mu_a i_a\}$ $p \equiv \{j_1 j_2 ; kq\}$, etc. (see [14]). Eq.(8) is the exact conditions under which Q^+ and Q are ideal bosons whereas the RPA conditions in the QPM are the approximated ones.

In the applications of algebraic models only the collective degrees of freedom are considered. Thus, if we consider the phonons with fixed multipolarity $\lambda_a = \lambda_b = \lambda_c = \lambda_d = \lambda$ and number $i_a = i_b = i_c = i_d = i$ (e.g., the first quadrupole state: $\lambda = 2$, $i = 1$), we can obtain from Eqs.(8) the following conditions for ψ and φ

$$\mathcal{X}_k = 0 \quad ; \quad \mathcal{Y}_k = 0, \tag{9}$$

where

$$\mathcal{X}_k = \hat{\lambda}^4 \sum_{\substack{jj' \\ j_1 j_2}} \left[\psi_{jj_2} \psi_{j_1 j} \psi_{j' j_2} \psi_{j_1 j'} - \varphi_{jj_2} \varphi_{j_1 j} \varphi_{j' j_2} \varphi_{j_1 j'} \right] \begin{Bmatrix} \lambda & \lambda & k \\ j_2 & j_1 & j \end{Bmatrix} \begin{Bmatrix} \lambda & \lambda & k \\ j_2 & j_1 & j' \end{Bmatrix} \tag{10}$$

$$\mathcal{Y}_k = 2\hat{\lambda}^4 \sum_{\substack{jj' \\ j_1 j_2}} \left[\psi_{jj_2} \psi_{j_1 j} \psi_{j' j_1} \varphi_{j_2 j'} - \psi_{j' j_1} \varphi_{j_2 j'} \varphi_{jj_2} \varphi_{j_1 j} \right] (-)^{j_1 - j_2} \begin{Bmatrix} \lambda & \lambda & k \\ j_2 & j_1 & j \end{Bmatrix} \begin{Bmatrix} \lambda & \lambda & k \\ j_2 & j_1 & j' \end{Bmatrix}.$$

The last condition from (8) in this collective approximation always takes place.

Eqs. (10) correspond to the coefficients C_k and D_k from [11,12]. If in (10) we use the RPA amplitudes ψ and φ, the conditions (9) are not fulfilled, as has been demonstrated in [12] by evaluating collective RPA quadrupole ψ, φ amplitudes. In order to fulfill the conditions (8) or (9) one must solve the equations of motion with these conditions taken into account instead of the RPA equations (see the conclusion of [12]). These equations can be derived by the usual variational method. In the present work we however do not pretend to study this problem. Assumming its solution to be known, i.e., Eqs.(8) and (9) to be satisfied, we obtain from the exact commutation relations of [14] the commutators

$$[Q_a, Q_b^+] = \delta_{ab} \quad ; \quad [Q_a, Q_b] = [Q_a^+, Q_b^+] = 0$$

$$\begin{aligned} [B_p, Q_a^+] &= \sum_g X^{(p+)}_{\tilde{a},g} Q_g^+ + \sum_g Y^{(p+)}_{\tilde{a},\tilde{g}} Q_g \\ [Q_a, B_p] &= \sum_g X^{(p+)}_{\tilde{g},a} Q_g + \sum_g \tilde{Y}^{(p+)}_{\tilde{a},\tilde{g}} Q_g^+ \end{aligned} \tag{11}$$

and use them to construct from the QPM operators the generators of the symplectic WSp(2d,R) and Sp(2d,R) algebras and their boson realizations.

3. Sp(2d,R) and WSp(2d,R) algebras in the QPM. Boson realizations for QPM operators

Hereafter, for convenience we use the abbreviation

$$Q_{\lambda\mu i} = Q_\mu. \qquad \text{etc.} \tag{12}$$

to denote phonon operators with fixed multipolarity λ and number i .

Let us construct from operators (12) the biphonon operators

$$\begin{aligned} D^+_{\mu\nu} &= Q^+_\mu Q^+_\nu \quad ; \quad D_{\mu\nu} = Q_\mu Q_\nu \\ C_{\mu\nu} &= Q^+_\mu Q_\nu \quad ; \quad E_{\mu\nu} = C_{\mu\nu} + \tfrac{1}{2}\delta_{\mu\nu} \\ \mu, \nu &= -\lambda, -\lambda+1, \dots, \lambda-1, \lambda. \end{aligned} \tag{13}$$

Under the conditions (10) and by using (11), the commutation relations for (13) can be written as

$$\begin{aligned} [E_{\mu\nu}, E_{\mu'\nu'}] &= \delta_{\nu\mu'} E_{\mu\nu'} - \delta_{\mu\nu'} E_{\mu'\nu} \\ [E_{\mu\nu}, D^+_{\mu'\nu'}] &= \delta_{\nu\mu'} D^+_{\mu\nu'} + \delta_{\nu\nu'} D^+_{\mu\mu'} \\ [E_{\mu\nu}, D_{\mu'\nu'}] &= -\delta_{\mu\mu'} D_{\nu\nu'} - \delta_{\mu\nu'} D_{\nu\mu'} \\ [D^+_{\mu\nu}, D^+_{\mu'\nu'}] &= [D_{\mu\nu}, D_{\mu'\nu'}] = 0 \\ [D_{\mu\nu}, D^+_{\mu'\nu'}] &= \delta_{\mu\mu'} E_{\nu'\nu} + \delta_{\mu\nu'} E_{\mu'\nu} + \delta_{\nu\mu'} E_{\nu'\mu} + \delta_{\nu\nu'} E_{\mu'\mu}. \end{aligned} \tag{14}$$

It is without toil to recognize from Eqs.(14) the realization for the symplectic Sp(2d,R) algebra of the linear canonical transformations in d-dimensional space ($d=2\lambda+1$) /15/. The operators $C_{\mu\nu}$ satisfying the same commutation relations as operators $E_{\mu\nu}$ generate the maximal compact U(d) subgroup of the Sp(2d,R) group.

The operators (13) and Q^+, Q from (12) together with the unit operator I, span an invariant Weyl subalgebra W(d) and are vector operators with respect to Sp(2d,R):

$$
\begin{aligned}
&[Q_\mu, Q^+_\nu] = \delta_{\mu\nu} I \\
&[E_{\mu\nu}, Q^+_\sigma] = \delta_{\nu\sigma} Q^+_\mu \\
&[E_{\mu\nu}, Q_\sigma] = -\delta_{\mu\sigma} Q_\nu \\
&[D_{\mu\nu}, Q^+_\sigma] = \delta_{\mu\sigma} Q_\nu + \delta_{\nu\sigma} Q_\mu \ ; \ [D_{\mu\nu}, Q_\sigma] = 0 \\
&[D^+_{\mu\nu}, Q_\sigma] = -\delta_{\mu\sigma} Q^+_\nu - \delta_{\nu\sigma} Q^+_\mu \ ; \ [D^+_{\mu\nu}, Q^+_\sigma] = 0.
\end{aligned}
\tag{15}
$$

Hence, the operators Q^+, Q and their biphonon combinations (13), together with I, span the semidirect sum W(d) $\oplus_s$ Sp(2d,R), i.e., the central extension WSp(2d,R) of the inhomogeneous symplectic algebra /16/.

In order to obtain the boson realizations for the phonon Q^+, Q operators and their biphonon combinations (13) we employ here the concept of coherent states suggested by Perelomov /17/, Barut and Girardello /18/. This concept has been developed later in /19/ in the formalism of the Perelomov partially coherent states (PPCS) and Barut-Girardello partially coherent states (BGPCS). Thus, we consider PPCS in the form

$$
|\mathcal{P}\rangle = \exp\{\sum_\mu v^*_\mu Q^+_\mu + \sum_{\mu\le\nu}(1+\delta_{\mu\nu})^{-1} u^*_{\mu\nu} D^+_{\mu\nu}\}|(\lambda)\rangle \equiv
$$

$$
\equiv \exp[\mathrm{tr}(v^* Q^+ + \tfrac{1}{2} u^* D^+)]\,|(\lambda)\rangle, \tag{16}
$$

where $|(\lambda)\rangle$ are the basic states in the representation space (RP) of the U(d) irrep $[\lambda_1\lambda_2\ldots\lambda_d]$ /19/. It can easily be built from the lowest weight state $|(\lambda)_{min}\rangle$ /19/. In our case for $|(\lambda)_{min}\rangle$ one can use the vacuum $|\tilde{0}\rangle$

$$
\begin{aligned}
D_{\mu\nu}|\tilde{0}\rangle &= 0 = \langle\tilde{0}|D^+_{\mu\nu} \\
Q_\mu|\tilde{0}\rangle &= 0 = \langle\tilde{0}|Q^+_\mu.
\end{aligned}
\tag{17}
$$

Employing the well-known technique based on the use of the Baker-Campbell-Hausdorff formula, we calculate the matrix elements $\langle\mathcal{P}|X|\Psi\rangle$, where $|\Psi\rangle$ is an arbitrary vector of the RP of the U(d) irrep $[\lambda_1\ldots\lambda_d]$, $X = D^+, D, E, Q^+, Q$. After some transformations we obtain the matrices

$$
\begin{aligned}
\hat{\mathbb{D}}^+ &= (\mathring{\mathbb{C}} + v\Delta_v)u + u(\tilde{\mathring{\mathbb{C}}} + \widetilde{v\Delta_v}) + (u\Delta_u - dI)u\mathbb{I} + \tilde{v}v\mathbb{I} \\
\hat{\mathbb{D}} &= \Delta_u \mathbb{I} + \tilde{\Delta}_v \Delta_v \mathbb{I} \\
\hat{\mathbb{E}} &= \mathring{\mathbb{C}} + \tfrac{1}{2} I\mathbb{I} + u\Delta_u \mathbb{I} + v\Delta_v \mathbb{I} \\
\hat{\mathbb{Q}}^+ &= v + u\tilde{\Delta}_v \ ; \quad \hat{\mathbb{Q}} = \Delta_v,
\end{aligned}
\tag{18}
$$

where $\hat{\mathbb{D}}^+, \hat{\mathbb{D}}, \hat{\mathbb{E}}, \mathring{\mathbb{C}}$ denote dxd matrices with elements $\hat{D}^+_{\mu\nu}, \hat{D}_{\mu\nu}, \hat{E}_{\mu\nu}, \mathring{C}_{\mu\nu}$, respectively. The matrix elements $\mathring{C}_{\mu\nu}$ equal

$$
\mathring{\mathbb{C}}_{\mu\nu} = \|\langle(\lambda)|C_{\mu\nu}|(\lambda')\rangle\| \ ; \quad \tilde{\mathring{\mathbb{C}}}_{\mu\nu} = \mathring{\mathbb{C}}_{\nu\mu}
$$

and $\mathring{C}_{\mu\nu} = 0$ only in the case when all $\lambda_i = 0$, i.e., when $|(\lambda)\rangle = |(\lambda)_{min}\rangle = |\tilde{0}\rangle$. $\hat{Q}^+_\mu$ ($\hat{Q}_\mu$) are the elements of the row matrices $\hat{\mathbb{Q}}^+$ and $\hat{\mathbb{Q}}$

$$
\Delta_{u_{\mu\nu}} \equiv (1+\delta_{\mu\nu})\,\partial/\partial u_{\mu\nu} \ ; \quad \Delta_{v_\mu} \equiv \partial/\partial v_\mu .
$$

Upon the mapping

$$
\begin{aligned}
u &\to a^+ & v &\to b^+ \\
\Delta_u &\to a & \Delta_v &\to b,
\end{aligned}
\tag{19}
$$

where $a^+_{\mu\nu}(a_{\mu\nu})$ and $b^+_\mu(b_\mu)$ ($\mu,\nu = 1,\ldots,d$) are the boson creation and annihilation operators we otain microscopic Dyson boson realization for phonon and biphonon QPM operators in PPCS. It reads

$$
\begin{aligned}
D^+ &= (\mathring{C} + b^+b)a^+ + a^+(\widetilde{\mathring{C} + b^+b}) + (a^+a - dI)a^+ + \tilde{b}^+b^+ \\
D &= a + \tilde{b}b \\
E &= \mathring{C} + \tfrac{1}{2}I + a^+a + b^+b \\
Q^+ &= b^+ + a^+\tilde{b} \ , \quad Q = b,
\end{aligned}
\tag{20}
$$

where $(a^+a)_{\mu\nu} \equiv \sum_\sigma a^+_{\mu\sigma} a_{\sigma\nu}$, etc.

The boson operators a^+ and a are non-normalized whose commutation relations have the simple form

$$[a_{\mu\nu}, a^+_{\mu'\nu'}] = \delta_{(\mu\nu)(\mu'\nu')} \equiv \delta_{\mu\mu'}\delta_{\nu\nu'} + \delta_{\mu\nu'}\delta_{\nu\mu'}$$

$$[a, a] = [a^+, a^+] = 0. \qquad (21)$$

The operators $\mathcal{E}_{\mu\nu} = \sum_\sigma a^+_{\mu\sigma} a_{\nu\sigma}$ are the generators for the U(d) subgroup of the $U(\nu)$ group where $\nu = \frac{1}{2}d(d+1)$ /20/. From the Dyson realization (20) it is easy to obtain the Dyson realization in the BGPCS space, whose basis states are defined as

$$|\mathcal{B}\rangle = \exp[\mathrm{tr}(vQ + \tfrac{1}{2}uD)]\,|W\rangle. \qquad (22)$$

Employing the standard procedure in /19/ we have

$$\begin{aligned}
D^+ &= a^+ + \widetilde{b^+ b^+} \\
D &= a(\mathring{C} + b^+ b) + (\widetilde{\mathring{C} + b^+ b})a + a(a^+ a - dI) + \widetilde{b}b \\
E &= \mathring{C} + \tfrac{1}{2}I + a^+ a + b^+ b \\
Q^+ &= b^+ \\
Q &= b + \widetilde{b^+} a.
\end{aligned} \qquad (23)$$

The realizations (20) and (23), together with the definitions (16) and (22) for the WSp(2d,R) algebra, in the case when $v = 0$ in (16) and (22) transform into the boson realizations for D^+, D and E of the Sp(2d,R) algebra in the PPCS and BGPCS spaces, respectively, which have been already derived in /19/.

Thus, basing on the commutation relations (11), which take place under the conditions (8), we have constructed from the QPM operators the generators D^+, D, E, Q^+, Q for the extended symplectic WSp(2d,R) group which contains Sp(2d,R) with generators D^+, D and E as its subgroup.

4. Dynamical WSp(2d,R) and Sp(2d,R) symmetric limits of the QPM Hamiltonian

Expressing A^+, A through Q^+ and Q by the inverse transformation from Eq.(4) /10/ and utilizing the expansion (7) in the collective approximation ($\lambda_i = \lambda$, $i_i = i$) we transform the QPM Hamiltonian (1) into the form

$$H = h_1 + h_2 + h_3 + h_4$$

$$h_1 = \sum_{\lambda\mu i}\Big(\varepsilon^{(\lambda i)} - \frac{1}{\hat\lambda^2}\sum_{\alpha\beta} \text{æ}^{(\lambda)}(\alpha,\beta) M_i(\alpha) M_i(\beta)\Big) Q^+_{\lambda\mu i} Q_{\lambda\mu i} +$$

$$+ \sum_{\lambda\mu i}\Big(\sigma^{(\lambda i)} - \frac{1}{2\hat\lambda^2}\sum_{\alpha\beta} \text{æ}^{(\lambda)}(\alpha,\beta) M_i(\alpha) M_i(\beta)\Big)\big(Q^+_{\lambda\tilde\mu i} Q^+_{\lambda\mu i} + Q_{\lambda\tilde\mu i} Q_{\lambda\mu i}\big)$$

$$h_2 = \frac{1}{2}\sum_{\substack{\lambda\mu i\\ \lambda'\mu_1 i'\\ \lambda'\mu_2 i'}} \big[U_\lambda(\lambda') Q^+_{\lambda'\mu_1 i'} Q^+_{\lambda'\mu_2 i'} Q_{\lambda\mu i} + h.c. \big]$$

$$h_3 = \sum_{\substack{\lambda\mu i\\ \lambda'\mu_1 i'\\ \lambda'\mu_2 i'}} \big[V_\lambda(\lambda') Q^+_{\lambda'\mu_1 i'} Q^+_{\lambda'\mu_2 i'} Q^+_{\lambda\mu i} + h.c. \big] \qquad (24)$$

$$h_4 = -\frac{1}{2}\sum_{\alpha,\beta}\sum_{\lambda\mu i} \text{æ}^{(\lambda)}(\alpha,\beta)\Big\{\sum_{123}\big[\bar X^{a}_{31}(\alpha)\bar X^{\tilde a}_{32}(\beta) +$$

$$+ 4\bar Y^{a}_{3\tilde 2}(\alpha)\tilde{\bar Y}^{\tilde a}_{3\tilde 1}(\beta)\big] Q^+_1 Q_2 +$$

$$+ \sum_{1234}\big[\bar X^{a}_{\tilde 3 1}(\alpha)\bar X^{\tilde a}_{\tilde 4 2}(\beta) + 2\bar Y^{a}_{\tilde 3\tilde 4}(\alpha)\tilde{\bar Y}^{\tilde a}_{\tilde 1\tilde 2}(\beta)\big] Q^+_1 Q^+_2 Q_3 Q_4 +$$

$$+ 2\sum_{123} \bar X^{a}_{31}(\alpha)\tilde{\bar Y}^{\tilde a}_{3\tilde 2}(\beta)\big[Q^+_1 Q^+_2 + h.c.\big] +$$

$$+ 2\sum_{123}\big[\bar X^{a}_{31}(\alpha)\tilde{\bar Y}^{\tilde a}_{3\tilde 2}(\beta) + \bar Y^{a}_{\tilde 2\tilde 3}(\alpha)\bar X^{\tilde a}_{\tilde 1 4}(\beta)\big]\big[Q^+_1 Q^+_2 Q^+_3 Q_4 + h.c.\big]$$

$$+ 2\sum_{1234} \bar Y^{a}_{\tilde 1\tilde 2}(\alpha)\bar Y^{\tilde a}_{\tilde 3\tilde 4}(\beta)\big[Q^+_1 Q^+_2 Q^+_3 Q^+_4 + h.c.\big].$$

In Eqs.(24)
$$\varepsilon^{(\lambda i)} \equiv \sum_{jj'} \varepsilon_j \left[\left(\psi^{\lambda i}_{jj'}\right)^2 + \left(\varphi^{\lambda i}_{jj'}\right)^2 \right]$$
$$\sigma^{(\lambda i)} \equiv \sum_{jj'} \varepsilon_j \psi^{\lambda i}_{jj'} \varphi^{\lambda i}_{j'j} \tag{25}$$
$$\begin{aligned} \varkappa^{(\lambda)}\left(\tfrac{1}{2},\tfrac{1}{2}\right) &= \varkappa^{(\lambda)}\left(-\tfrac{1}{2},-\tfrac{1}{2}\right) = \varkappa_0^{(\lambda)} + \varkappa_1^{(\lambda)} \\ \varkappa^{(\lambda)}\left(\tfrac{1}{2},-\tfrac{1}{2}\right) &= \varkappa^{(\lambda)}\left(-\tfrac{1}{2},\tfrac{1}{2}\right) = \varkappa_0^{(\lambda)} - \varkappa_1^{(\lambda)} \end{aligned} \tag{26}$$

with $\alpha,\beta = \pm 1/2$ denoting combinations of the isoscalar and isovector constants in different neutron and proton couplings
$$\begin{aligned} U_\lambda(\lambda') &\equiv (\lambda'\mu_1 \lambda'\mu_2 | \lambda\mu)\, U^{\lambda' i'}_{\lambda' i'}(\lambda i) \equiv U^{(2)}_{(1)}(a) \\ V_\lambda(\lambda') &\equiv (\lambda'\mu_1 \lambda'\mu_2 | \lambda\tilde{\mu})\, V^{\lambda' i'}_{\lambda' i'}(\lambda i) \equiv V^{(2)}_{(1)}(\tilde{a}), \end{aligned} \tag{27}$$
where $U^{\lambda' i'}_{\lambda' i'}(\lambda i)$ and $V^{\lambda' i'}_{\lambda' i'}(\lambda i)$ are the complicated functions of the phonon amplitudes ψ and φ. Their explicit forms can be found in /21/. Their expressions in terms of X and Y are given in /14/
$$\begin{aligned} \bar{X}^{a}_{\tilde{1}2}(\alpha) &\equiv \sum_{jj'}^{(\alpha)} f^{(\lambda)}_{jj'} v^{(-)}_{jj'} X^{(jj';\lambda\mu+)}_{\lambda'\tilde{\mu}_1 i', \lambda'\mu_2 i'} \\ \bar{Y}^{\tilde{a}}_{\tilde{1}\tilde{2}}(\alpha) &\equiv \sum_{jj'}^{(\alpha)} f^{(\lambda)}_{jj'} v^{(-)}_{jj'} Y^{(jj';\lambda\tilde{\mu}+)}_{\lambda'\tilde{\mu}_1 i', \lambda'\tilde{\mu}_2 i'}, \end{aligned} \tag{28}$$

where indices 1,2...,etc. denote $(\lambda'\mu_1 i')$, $(\lambda'\mu_2 i')$, etc., while indices a stand for $\{\lambda\mu i\}$. In the form (24) h_1 corresponds to the interaction of phonons. The parts h_2 and h_3 have arisen from the quasiparticle-phonon interaction. The part h_4 contains the terms $\sim BB$, i.e., it corresponds to the higher order in the perturbative treatment. The evaluation performed in /22/ has shown that in the RPA for even-even spherical nuclei the contribution of h_4 to the one-phonon energies is negligibly small.

Using Eq.(13) we can rewrite Hamiltonian (24) as
$$\begin{aligned} h_1 &= \sum_a \omega_{\tilde{a}} C_{aa} + \sum_a \Omega_{\tilde{a}} \left(D^+_{\tilde{a}a} + D_{\tilde{a}a} \right) \\ h_2 &= \tfrac{1}{2} \sum_{12a} U^{(2)}_{(1)}(a) \left\{ D^+_{12} Q_a + h.c. \right\} \\ h_3 &= \sum_{12a} V^{(2)}_{(1)}(a) \left\{ D^+_{12} Q^+_{\tilde{a}} + h.c. \right\} \\ h_4 &= -\tfrac{1}{2} \sum_{\alpha\beta} \sum_a \varkappa^{(\lambda)}(\alpha,\beta) \Big\{ \sum_{12} F_{12} C_{12} + \sum_{12} G_{12} \left[D^+_{12} + D_{12} \right] + \\ &+ \sum_{1234} H_{1234} D^+_{12} D_{34} + \sum_{1234} I_{1234} \left[D^+_{12} C_{34} + h.c. \right] + \\ &+ \sum_{1234} K_{1234} \left[D^+_{12} D^+_{34} + h.c. \right] \Big\}. \end{aligned} \tag{29}$$
where the notation $\omega_{\tilde{a}}$, $\Omega_{\tilde{a}}$, F, G, H, I, K is used to denote the expressions
$\sim Q^+Q$, $(Q^+Q^+ + QQ)$, $Q^+_1 Q_2$, $Q^+_1 Q^+_2 Q_3 Q_4$, $[Q^+_1 Q^+_2 + h.c.]$,

$[Q^+_1 Q^+_2 Q^+_3 Q_4 + h.c.]$ and $[Q^+_1 Q^+_2 Q^+_3 Q^+_4 + h.c.]$ in (24), respectively.

Under the conditions (8) and (9), Eqs.(24) and (29) are the generalized microscopic Hamiltonians having the WSp(2d,R) symmetry. These Hamiltonians contain components
$$H_{Sp} \equiv h_1 + h_4 \subset H \equiv h_1 + h_2 + h_3 + h_4 \tag{30}$$
having the symplectic Sp(2d,R) symmetry. The components $h_2 + h_3$, containing the generators of the Sp(2d,R) algebra together with those of the W(d) Heisenberg-Weyl algebra, have the extended WSp(2d,R) symmetry as the total Hamiltonian (24). Hence, due to the negligible contribution of h_4 discussed above, we can conclude that the symplectic Sp(2d,R) symmetry in the QPM can appear only in some corrections to the one-phonon energies.

To conclude this section we consider two examples which can be represented as the microscopic versions of WSp(6,R) and Sp(6,R) models.

a) The WSp(6,R) model Hamiltonian

Let us consider in Eq.(29) only the dipole interaction in the collective approximation. We also examine only the components
$$(\tilde{H})_{dipole} = h_1 + h_2 + h_3. \tag{31}$$

The Hamiltonian $\tilde{H} = h_1 + h_4$ will be considered separately in b).

By introducing the tensor products

$$\mathcal{D}^+_{LM} = \sum_{\mu_1\mu_2} (1\mu_1 1\mu_2|LM)\, Q^+_{1\mu_1} Q^+_{1\mu_2} \equiv [Q^+_1 \otimes Q^+_1]_{LM}$$
$$\mathcal{D}_{LM} = \sum_{\mu_1\mu_2} (1\mu_1 1\mu_2|LM)\, Q_{1\tilde{\mu}_1} Q_{1\tilde{\mu}_2} \equiv [\tilde{Q}_1 \otimes \tilde{Q}_1]_{LM} \qquad (32)$$
$$\mathcal{E}_{LM} = \sum_{\mu_1\mu_2} (1\mu_1 1\mu_2|LM)\, Q^+_{1\mu_1} Q_{1\tilde{\mu}_2} \equiv [Q^+_1 \otimes \tilde{Q}_1]_{LM}$$

we obtain after some tensor operations the following form for (31):

$$(\tilde{H})_{dipole} = \sqrt{3}\,\omega_1 \mathcal{E}_{00} + \sqrt{3}\,\Omega_1 (\mathcal{D}^+_{00} + \mathcal{D}_{00}) + \qquad (33)$$
$$+ \sum_{L=0,2} (-)^L \hat{L} \left\{ \tfrac{1}{2} U^1_1(L) [\mathcal{D}^+_L \otimes Q_L]_{00} + V^1_1(L) [\mathcal{D}^+_L \otimes Q^+_L]_{00} + h.c. \right\}.$$

The Hamiltonian (33) (valid to the terms $\sim BB$) is the microscopic version of the WSp(6,R)[9] Hamiltonian, in which the coefficients before the WSp(6,R) generators are expressed in terms of the microscopic phonon amplitudes ψ, φ and the matrix elements $f^{(1)}_{jj'}$ of the dipole interaction. Using the Dyson boson realization (20) or (23) we obtain for the scalar products in (33) the following PPCS representations:

$$(\tilde{H})^{PPCS}_{dipole} = \sqrt{3}\,\omega_1 (\mathcal{E}_{00})^{PPCS} + \sqrt{3}\,\Omega_1 [(\mathcal{D}^+_{00})^{PPCS} + (\mathcal{D}_{00})^{PPCS}] + \qquad (34)$$
$$+ \sum_{L=0,2} (-)^L \hat{L} \left\{ \tfrac{1}{2} U^{(1)}_{(1)}(L) [\mathcal{D}^+_L \otimes Q_L]^{PPCS}_{00} + V^{(1)}_{(1)}(L) [\mathcal{D}^+_L \otimes Q^+_L]^{PPCS}_{00} + h.c. \right\}$$

$$(\mathcal{E}_{00})^{PPCS} = \frac{1}{\sqrt{3}} \sum_\mu \left(\mathring{C}_{\mu\mu} + \sum_\nu a^+_{\mu\nu} a_{\nu\mu} + b^+_\mu b_\mu \right) = \frac{1}{\sqrt{3}} \left(\mathring{C} + (a^+ a) + (b^+ b) \right) \qquad (35)$$

$$(\mathcal{D}^+_{00})^{PPCS} = \sum_L \tfrac{2}{3} (-)^L \hat{L} \{ [\mathring{C}_L \otimes a^+_L]^{(0)} + [[b^+_1 \otimes \tilde{b}_1]_L \otimes a^+_L]^{(0)} \} + a^+_{00} + [b^+_1 \otimes b^+_1]^{(0)} +$$
$$+ \frac{1}{\sqrt{3}} \sum_{LL'L''} \hat{L}\hat{L}'\hat{L}'' [[a^+_L \otimes a^+_{L'}] \otimes \tilde{a}_{L''}]^{(0)} \begin{Bmatrix} L & L' & L'' \\ 1 & 1 & 1 \end{Bmatrix} \qquad (36)$$

$$(\mathcal{D}_{00})^{PPCS} = a_{00} + [\tilde{b}_1 \otimes \tilde{b}_1]^{(0)}$$

$$[\mathcal{D}^+_L \otimes \tilde{Q}_L]^{PPCS}_{00} = \sum_{L'L''} 2\hat{L}'\hat{L}'' (-)^L \begin{Bmatrix} L & L' & L'' \\ 1 & 1 & 1 \end{Bmatrix} \{ [[\mathring{C}_{L'} \otimes a^+_{L''}]_L \otimes \tilde{b}_L]^{(0)} + [[[b^+_1 \otimes \tilde{b}_1]_{L'} \otimes a^+_{L''}]_L \otimes \tilde{b}_L]^{(0)} \} +$$
$$+ [[b^+_1 \otimes b^+_1]_L \otimes \tilde{b}_L]^{(0)} + [a^+_L \otimes \tilde{b}_L]^{(0)} + \sum_{L'L''L'''} \sum_\ell \hat{\ell}\hat{L}'\hat{L}''\hat{L}''' (-)^{L'} \begin{Bmatrix} 1 & 1 & L \\ 1 & 1 & L''' \\ L'' & L' & \ell \end{Bmatrix} [[[a^+_{L'} \otimes a^+_{L''}]_\ell \otimes \tilde{a}_{L'''}]_L \otimes \tilde{b}_L]^{(0)} \qquad (37)$$

$$[\tilde{\mathcal{D}}_L \otimes \tilde{Q}_L]^{PPCS}_{00} = [\tilde{a}_L \otimes \tilde{b}_L]^{(0)} + [[\tilde{b}_1 \otimes \tilde{b}_1]_L \otimes \tilde{b}_L]^{(0)} \qquad (38)$$

$$[\mathcal{D}^+_L \otimes Q^+_L]^{PPCS}_{00} = \sum_{L'L''} 2\hat{L}'\hat{L}'' (-)^L \begin{Bmatrix} L & L' & L'' \\ 1 & 1 & 1 \end{Bmatrix} \{ [[\mathring{C}_{L'} \otimes a^+_{L''}]_L \otimes b^+_L]^{(0)} + [[[b^+_1 \otimes \tilde{b}_1]_{L'} \otimes a^+_{L''}]_L \otimes b^+_L]^{(0)} \} + [a^+_L \otimes b^+_L]^{(0)} +$$
$$+ [[b^+_1 \otimes b^+_1]_L \otimes b^+_L]^{(0)} + \sum_{L'L''L'''} \sum_\ell \hat{\ell}\hat{L}'\hat{L}''\hat{L}''' (-)^{L'} \begin{Bmatrix} 1 & 1 & L \\ 1 & 1 & L''' \\ L''' & L & \ell \end{Bmatrix} [[[a^+_{L'} \otimes a^+_{L''}]_\ell \otimes \tilde{a}_{L'''}]_L \otimes b^+_L]^{(0)} +$$
$$+ \sum_{L'L''} \frac{\hat{L}'^2 \hat{L}''}{\hat{L}} \begin{Bmatrix} L & L' & L'' \\ 1 & 1 & 1 \end{Bmatrix} (-)^{L+L'} \{ [[\mathring{C}_{L'} \otimes a^+_{L''}]_L \otimes [a^+_{L'} \otimes \tilde{b}_L]_L]^{(0)} + [[a^+_{L''} \otimes \mathring{C}_{L'}] \otimes [a^+_{L'} \otimes \tilde{b}_L]_L]^{(0)} + \qquad (39)$$
$$+ [[[b^+_1 \otimes \tilde{b}_1]_{L'} \otimes a^+_{L''}]_L \otimes [a^+_{L'} \otimes \tilde{b}_L]_L]^{(0)} + [[a^+_{L''} \otimes [b^+_1 \otimes \tilde{b}_1]_{L'}]_L \otimes [a^+_{L'} \otimes \tilde{b}_L]_L]^{(0)} \} +$$
$$+ \sum_{L'} \frac{\hat{L}'}{\hat{L}} (-)^{L'} \{ [a^+_L \otimes [a^+_{L'} \otimes \tilde{b}_L]_L]^{(0)} + [[b^+_1 \otimes b^+_1]_L \otimes [a^+_{L'} \otimes \tilde{b}_L]_L]^{(0)} \} + \sum_{\ell\ell'} \sum_{L'L''L'''} \frac{\hat{\ell}\hat{\ell}'}{\hat{L}} \hat{L}'\hat{L}''\hat{L}''' (-)^{L'+\ell'} \times$$
$$\times \begin{Bmatrix} 1 & 1 & L \\ 1 & 1 & L''' \\ L'' & L' & \ell \end{Bmatrix} [[[a^+_{L'} \otimes a^+_{L''}]_\ell \otimes \tilde{a}_{L'''}]_L \otimes [a^+_{\ell'} \otimes \tilde{b}_L]_L]^{(0)}$$

$$[Q^+_L \otimes \tilde{\mathcal{D}}_L]^{PPCS}_{00} = [b^+_L \otimes \tilde{a}_L]^{(0)} + [b^+_L \otimes [\tilde{b}_1 \otimes \tilde{b}_1]_L]^{(0)} +$$
$$+ \sum_{L'} \frac{\hat{L}'}{\hat{L}} (-)^{L'} \{ [[a^+_{L'} \otimes \tilde{b}_L]_L \otimes \tilde{a}_L]^{(0)} + [[a^+_{L'} \otimes \tilde{b}_L]_L \otimes [\tilde{b}_1 \otimes \tilde{b}_1]_L]^{(0)} \}. \qquad (40)$$

where

$$a^+_{LM} = \sum_{\mu\nu} (1\mu 1\nu|LM)\, a^+_{\mu\nu}$$
$$\mathring{C}_{LM} = \sum_{\mu\nu} (1\mu 1\nu|LM)\, \mathring{C}_{\mu\nu} \qquad (41)$$
$$\ell', L, L', L'', L''' = 0, 2 \; ; \; \ell = 0, 2, 4.$$

b) The Sp(6,R) model Hamiltonian

We now consider from Eq.(24) the part

$$\tilde{H} = h_1 + h_4 \qquad (42)$$

where D^+, D, C from h_1 and h_4 consist of the dipole phonons. In this case we have the Sp(6,R) symplectic symmetry ($\lambda = 2$ in Eq. (24)):

$$\tilde{H} = (h_1)_{quadrupole} + \sum_{L=0,2,4} \hat{L} \{ \mathcal{H}_L [\mathcal{D}^+_L \otimes \tilde{\mathcal{D}}_L]^{(0)} + \mathcal{J}_L ([\mathcal{D}^+_L \otimes \mathcal{E}_L]^{(0)} +$$
$$+ [\mathcal{E}_L \otimes \tilde{\mathcal{D}}_L]^{(0)}) + \mathcal{K}_L ([\mathcal{D}^+_L \otimes \mathcal{D}^+_L]^{(0)} + [\tilde{\mathcal{D}}_L \otimes \tilde{\mathcal{D}}_L]^{(0)}) \} + \qquad (43)$$
$$+ \frac{5}{\sqrt{3}} \{ F \mathcal{E}_{00} + G (\mathcal{D}^+_{00} + \mathcal{D}_{00}) \},$$

where

$$\mathcal{H}_L = -\frac{5}{2} (-)^L \begin{Bmatrix} 2 & 1 & 1 \\ L & 1 & 1 \end{Bmatrix} \sum_{\alpha,\beta} \varkappa^{(2)}(\alpha,\beta) [\hat{\tilde{X}}^{(2)}_{c,c}(\alpha) \hat{X}^{(2)}_{c,c}(\beta) + 2 \hat{\tilde{Y}}^{(2)}_{c,c}(\alpha) \hat{\tilde{Y}}^{(2)}_{c,c}(\beta)]$$

$$\mathcal{J}_L = -5 (-)^L \begin{Bmatrix} 2 & 1 & 1 \\ L & 1 & 1 \end{Bmatrix} \sum_{\alpha,\beta} \varkappa^{(2)}(\alpha,\beta) [\hat{\tilde{X}}^{(2)}_{c,c}(\alpha) \hat{\tilde{Y}}^{(2)}_{c,c}(\beta) + \hat{\tilde{Y}}^{(2)}_{c,c}(\alpha) \hat{\tilde{X}}^{(2)}_{c,c}(\beta)]$$

$$\mathcal{K}_L = -5(-)^L \left\{ \begin{smallmatrix} 2 & 1 & 1 \\ L & 1 & 1 \end{smallmatrix} \right\} \sum_{\alpha,\beta} \varkappa^{(2)}(\alpha,\beta)\, \hat{Y}^{(2)}_{\varsigma,c}(\alpha)\, \hat{Y}^{(2)}_{\varsigma,c}(\beta) \tag{44}$$

$$F = -\frac{1}{2} \sum_{\alpha,\beta} \varkappa^{(2)}(\alpha,\beta) \left[\hat{X}^{(2)}_{\varsigma,c}(\alpha) \hat{X}^{(2)}_{\varsigma,c}(\beta) + 4 \hat{Y}^{(2)}_{\varsigma,c}(\alpha) \tilde{\hat{Y}}^{(2)}_{\varsigma,c}(\beta) \right]$$

$$G = -\sum_{\alpha,\beta} \varkappa^{(2)}(\alpha,\beta)\, \hat{X}^{(2)}_{\varsigma,c}(\alpha)\, \hat{\tilde{\hat{Y}}}^{(2)}_{\varsigma,c}(\beta)$$

with the notation from [14] and c - denotes the collective approximation (λ =2).

The Hamiltonian (43) has the same structure as the Hamiltonians of the Sp(6,R) model in [5] or its extended version - the Hamiltonian of the Sp(12,R) model from [23]. In these models the parameters for the interactions are chosen phenomenologically to reproduce the experimental data.

5. Conclusion

Thus, we have obtained the conditions under which the QPM Hamiltonian has the WSp(2d,R) symmetry. We have also constructed from the phonon QPM operators and their biphonon combinations the generators of the Sp(2d,R) and WSp(2d,R) algebras and derived for them the Dyson boson realizations. By examining the structure of the QPM Hamiltonian in its WSp(2d,R) symmetric limit we conclude that the symplectic Sp(2d,R) symmetry can appear in the QPM only in some small corrections to the one-phonon energies. In general, by satisfying the conditions (8) the QPM has the WSp(2d,R) symmetric limit. As special cases, we have considered the WSp(6,R) and Sp(6,R) algebras and have derived from the QPM Hamiltonian the microscopic versions for the WSp(6,R) and Sp(6,R) model Hamiltonians. The study of the conditions (8) requires detailed numerical calculations which are planed to perform. On the other hand, it would be interesting to consider the possibility in the construction of the self-functions having symmetric properties without enforcing the QPM Hamiltonian to have a defined symmetry. It is the goal of the future study.

The author would like to thank Prof. V.G.Soloviev and Dr. L.A. Malov for the constant interest and encouragement. The helpful discussions with Prof. I.N.Mikhailov, Dr. V.V.Voronov and their useful comments are gratefully acknowledged.

References

1. D.J.Rowe, Rep.Prog.Phys. 48 (1985) 1419.
2. S.Goshen and H.J.Lipkin, Ann.Phys. (NY) 6 (1959) 301.
3. R.Y.Cusson, Nucl.Phys. A114 (1968) 289.

4. P.P.Raychev, Yad.Fiz. 16 (1972) 1171.
5. G.Rosensteel and D.J.Rowe, Phys.Rev.Lett. 38 (1977) 10; Ann.Phys. (NY) 126 (1980) 343.
6. G.F.Filippov, Fiz.Elem.Chastits At.Yadra 4 (1973) 992. V.V.Vanagas and R.K.Kalinauskas, Yad.Fiz. 18 (1973) 768.
7. J.P.Elliott, Proc. R.Soc.London Ser. A245 (1958) 128, 562.
8. L.Weaver, L.C.Biedenharn and R.Y.Cusson, Ann.Phys. (NY) 77 (1973) 250.
9. C.Quesne, Phys.Lett. B188 (1987) 1.
10. V.G.Soloviev, Theory of Complex Nuclei, Moscow: Nauka, 1971. V.G.Soloviev, Fiz.Elem.Chastits At.Yadra 9 (1978) 580; V.G.Soloviev, Nukleonika, 23 (1978) 1149; L.A.Malov, V.G.Soloviev, Fiz.Elem.Chastits At.Yadra 11(1980) 301; A.I.Vdovin, V.G.Soloviev, Ibid. 14 (1983) 237; V.V.Voronov, V.G.Soloviev, Ibid. 14 (1983) 1381; V.G.Soloviev, Prog.Part.Nucl.Phys. Ed. A.Faessler (NY): Pergamon Press, 19 (1987) 107-165.
11. G.Kyrchev and V.Paar, accepted in Phys.Rev.C (1987).
12. V.V.Voronov and G.Kyrchev, Teor.Mat.Fiz. 69 (1986) 236.
13. Nguyen Dinh Dang, Dynamical SU(m) Symmetry and SU(m/n) Supersymmetry of the Quasiparticle-Phonon Nuclear Model: preprint JINR P4-87-915, Dubna: JINR, 1987.
14. Nguyen Dinh Dang, Z.Phys. A - Atomic Nuclei, 327 (1987) 41.
15. J.Deenen, C.Quesne, J.Math.Phys., 23 (1982) 878; J.Deenen, C.Quesne, Ibid., 23 (1982) 2004.
16. B.G.Wybourne, Classical Groups for Physicists, N.Y.: Wiley (1974) 286.
17. A.M.Perelomov, Commun.Math.Phys., 26 (1972) 222; A.M.Perelomov, Usp.Fiz.Nauk 123 (1977) 23.
18. A.O.Barut, L.Girardello, Commun.Math.Phys. 21 (1971) 41.
19. J.Deenen, C.Quesne, J.Math.Phys., 25 (1984) 2354.
20. J.Deenen, C.Quesne, J.Math.Phys. 26 (1985) 2705.
21. Nguyen Dinh Dang, V.Yu.Ponomarev, Teor.Mat.Fiz., 57 (1983) 154; V.V.Voronov, Nguyen Dinh Dang, Izv.Acad.Nauk SSSR, ser.fiz., 48 (1984) 857; Nguyen Dinh Dang, Teor.Mat.Fiz., 64 (1985) 69.
22. Nguyen Dinh Dang, V.G.Soloviev, A Study of the Corrections to the RPA in Spherical Nuclei: JINR Communication, P4-83-325, Dubna: JINR, 1983.
23. A.Georgieva, P.Raychev, R.Roussev, J.Phys. G: Nucl.Phys., 8 (1982) 1377.

Received by Publishing Department
on March 29, 1988.