объединенный
институт
ядерных
исследований
дубна

E4-87-880

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THE REMARKABLE CAPABILITIES OF RECURSIVE RELATIONS

Submitted to "American Journal of Physics"

1987

1. INTRODUCTION

The recursive relation approach helps one to solve many physical problems. The realisation of this fact is reflected in the appearing of conference proceedings /1/ devoted totally to the recursive relation applications. Here we touch only one aspect of these relations, not considered in /1/.

It was long known, that the recursive relations make it very easy to handle a stepwise potential in wave mechanics or in optics. But it occurs that they are very useful for an analytical solution of such problems as wave propagation in periodic potentials and even in the case of a molecular gas or ultracold neutron diffusion in tubes.

To remind how recursive relations work we first consider the reflection of a particle from a one-dimensional rectangular potential in quantum mechanics. Then we shall consider a more general stepwise potential and a periodic potential. After that we shall go to diffusion problems and show how to solve them for the case of particle diffusion in linear and branched guides. And in the last section we shall consider wave problems with a three dimensional spherically symmetric potential.

2. SCATTERING OF A SCALAR PARTICLE FROM A ONE-DIMENSIONAL STEP POTENTIAL IN QUANTUM MECHANICS

Let us consider a step potential

$$U = u_0 \theta(x \geq 0), \tag{1}$$

where u_0 is the height of the potential, and θ-function is equal to 1, when its argument inequality is satisfied, and to 0 in the opposite case. Consider a particle with a wave vector k, incident on this step from the left. It is described by the wave $\exp(ikx)$. This particle can enter the region $x \geq 0$ with the amplitude t^+, or it can be reflected back to $x \leq 0$ with the amplitude r^+. The total wave function can be represented by the expression

$$\Psi=\theta(x\leq 0)[\exp(ikx)+r^{+}\exp(-ikx)]+t^{+}\theta(x\geq 0)\exp(ik'x), \tag{2}$$

where k' is the wave-vector inside the potential:

$$k'=\sqrt{k^2-u_0}. \tag{3}$$

Here we take $\hbar^2/2m=1$, where $\hbar$ is the Plank constant and m - is the mass of the particle. Amplitudes t^+ and r^+ must be determined from the matching of the wave function (2) and its first derivative at the point $x=0$. The matching gives two equations

$$\text{a)}\ 1+r^{+}=t^{+}, \qquad \text{b)}\ k(1-r^{+})=k't^{+}, \tag{4}$$

whose solution is

$$\text{a)}\ r^{+}=(k-k')/(k+k'), \qquad \text{b)}\ t^{+}=2k/(k+k'). \tag{5}$$

The sign + over amplitudes r and t means, that the incident wave propagates from left to right. Of course we can consider the opposite propagation as well, when the incident wave comes from the potential region $x\geq 0$. But to obtain the amplitudes r^- and t^- in that case, it is not necessary to solve the equations again. It is enough to replace k with k' and vice versa. So:

$$\text{a)}\ r^{-}=(k'-k)/(k'+k), \qquad \text{b)}\ t^{-}=2k'/(k'+k). \tag{6}$$

3. SCATTERING OF A PARTICLE ON A RECTANGULAR POTENTIAL

Now we are ready to consider the scattering from a rectangular potential $U=u_0\theta(0<x<1)$. Of course we can write down the total wave function with four unknown coefficients:

$$\Psi=\theta(x\leq 0)[\exp(ikx)+a_1\exp(-ikx)]+ \tag{7}$$
$$+\theta(0\leq x\leq 1)[a_2\exp(ik'x)+a_3\exp(-ik'x)]+a_4\theta(x\geq 1)\exp(ikx),$$

and match the wave function and its derivative at two points $x_1=0$ and $x_2=1$. Then we shall have four equations with four unknowns, and it is a very tedious work to solve them. But it is possible to find the reflection and transmission amplitudes for a rectangular potential much more intelligently. It is just here where we meet the recursive relations for the first time.

Let us suppose that the part of the wave function inside the potential, describing the wave going to the right, just before the point x_2, is equal to $\vec{\psi}_2$. It is reflected from the right edge of the potential with the amplitude r_2^+ (which is evidently equal to r^- in (6a)) and spreads to the left with the phase factor $\exp[ik'(1-x)]$. When it reaches the left side of the potential this phase factor becomes $\exp(ik'\ell)$. Here the wave is reflected again with the amplitude r_1^- (which is again equal to r^- (6a)) and spreads to the right acquiring the phase factor $\exp(ik'x)$. Near the right edge of the potential this additional phase factor becomes equal again to $\exp(ik'\ell)$. Now after two reflections the wave $\vec{\psi}_2$ is transformed to $r_1^- r_2^+\exp(2ik'\ell)\vec{\psi}_2$. This makes a part of the original function $\vec{\psi}_2$. The other part is given by the incident wave, which enters the potential region from the left side vacuum with the amplitude t_1^+ equal to (5b) and then spreads to the right edge of the potential, acquiring the phase factor $\exp(ik'x)$, which becomes $\exp(ik'\ell)$ near the point x_2. If we put the value of the incident wave $\vec{\psi}_0$ in the vacuum near the point x_1 to be equal to 1, then for $\vec{\psi}_2$ we can write down the equation:

$$\vec{\psi}_2=\exp(ik'\ell)t^{+}+(r^{-})^2\exp(2ik'\ell)\vec{\psi}_2. \tag{8}$$

The solution of this equation is

$$\vec{\psi}_2=t^{+}\exp(ik'\ell)/[1-r^2\exp(2ik'\ell)]. \tag{9}$$

(Here for brevity we abolished the - sign over r). Now we have everything to calculate transmission and reflection for the rectangular barrier potential. Indeed the transmitted wave has an amplitude equal to $T=t_2^+\vec{\psi}_2$. But t_2^+ evidently is equal to (6b), so, using (9) we have

$$T=t^{-}t^{+}\exp(ik'\ell)/[1-r^2\exp(2ik'\ell)]. \tag{10}$$

To find the reflection amplitude R it is necessary to add the amplitude r^- of the incident wave reflected from the left edge of the potential and the contribution of the wave $\vec{\psi}_2$, that after reflection r_2^+ reaches with the additional phase factor $\exp(ik'\ell)$ the left side of the potential and with the amplitude t_2^- goes over the left side into the vacuum. Since t_2^- is equal to (6b); and r_2^+, to (6a), we easily obtain:

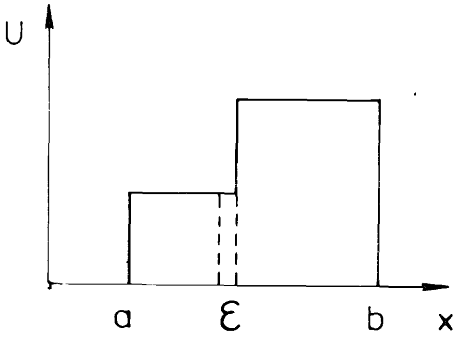
$$R=r^{+}+r^{-}t^{-}t^{+}\exp(2ik'\ell)/[1-r^2\exp(2ik'\ell)]. \tag{11}$$

For the rectangular potential it is possible to simplify this expression. From (5) and (6) it follows, that $t^- t^+ = (1 - r^2)$ and $r^- = -r^+ = -r$, so we insert these relations into (11) and obtain

$$R = r[1 - \exp(2ik'\ell)]/[1 - r^2 \exp(2ik'\ell)]. \tag{12}$$

4. A MORE COMPLICATED RECTANGULAR POTENTIAL

Now, let us consider the potential, shown in the figure. Again we can write down the total wave function with six unknown coefficients and find them from the system of six equations.



But we can proceed more intelligently. Let us introduce an infinitely narrow gap between two rectangular potentials with a zero potential inside it. Such a modification will not lead to any physical cansequences, but drastically simplifies mathematics. Indeed, we know the transmission and reflection amplitudes $T_{1,2}$, $R_{1,2}$ for partial rectangular potentials. Now let us imagine ourselves to be inside the narrow gap. Since the width of the gap is infinitesimal, the phase factor for the wave spreading in it from left to right and vice versa is always equal to unity. So for the wave $\vec{\psi}_2$ incident on the right wall of the gap we can write down by analogy with (8) the equation:

$$\vec{\psi}_2 = T_1 + R_1 R_2 \vec{\psi}_2, \tag{13}$$

whose solution is $\vec{\psi}_2 = T_1/[1 - R_1 R_2]$. To find total transmission and reflection amplitudes we proceed by analogy with (10), (11) and this leads to:

$$\text{a) } T_{12} = T_1 T_2/[1 - R_1 R_2], \qquad \text{b) } R_{12} = R_1 + R_2 T_1^2/[1 - R_1 R_2]. \tag{14}$$

It is easy to see, that T_{12} is symmetric with respect to the indices permutation, that is, it does not depend on the direction of incidence of a primary wave. But R_{12} does depend on this direction. It is not difficult to prove that for the real potential R_{12} differs from R_{21} only by a phase factor. We shall not do that here (see for instance /2/), but in the considered simple case this fact is easy to verify.

5. A PERIODIC POTENTIAL

Now, let us consider a symmetric half infinite one-dimensional periodic potential and introduce infinitesimal gaps with zero potential between the periods. We suppose that the transmission t and reflection r amplitudes for one period are somehow known and find a reflection amplitude R_∞ of the whole potential.

To do that let us look at the relation (13). It can be written here also for the first gap with the replacements: r, t for R_1, T_1, and R_∞ for R_2. Then, to get the reflection amplitude for the whole potential it is only necessary to make the same replacements in (14b):

$$R_\infty = r + t^2 R_\infty/[1 - rR_\infty]. \tag{15}$$

But now it is not the final expression. It is an equation, which is obtained due to the fact, that the reflection from the whole infinitely long potential and the potential (also infinitely long) without one period is the same. The equation (15) is of the type:

$$x^2 - 2px + 1 = 0,$$

and its solutions can be represented in the form

$$x_{1,2} = [\sqrt{p+1} \pm \sqrt{p-1}]/[\sqrt{p+1} \mp \sqrt{p-1}].$$

So the solution of the equation (15) looks like Fresnel coefficients for the reflection of electromagnetic waves from a plane surface:

$$R_\infty = [\sqrt{(r+1)^2 - t^2} - \sqrt{(r-1)^2 - t^2}]/[\sqrt{(r+1)^2 - t^2} + \sqrt{(r-1)^2 - t^2}]. \tag{16}$$

The signs are chosen in such a way, that R_∞ goes over to zero for a fictive potential ($r = 0$).

Not only the reflection amplitude is of interest for a periodic potential but the Bloch wave vector also. It is well known, that the wave function $\psi(x)$ inside a periodic potential can be represented as $\psi(x) = \phi(x)\exp(iqx)$, where $\phi(x)$ is a periodic function with the same period as the potential, and q is the Bloch wave vector. To known the function $\phi(x)$ it is necessary to solve the Shrödinger equation and specify the potential of a period. But here only the reflection and transmission amplitudes of one period are supposed to be known, so we do not pretend to find the total function $\phi(x)$, nevertheless we can easily find the Bloch wave vector, and infinitesimal gaps, introduced between periods, are very useful for this purpose. Indeed, in the gaps the potential is known - it is equal to zero, so the wave function is also known - it is a superposition of two plane waves, going in opposite directions. But due to periodicity this wave function is almost the same in all gaps, differing only by a phase factor $\exp(iq\ell n)$, where $n\ell = x_1 - x_2$ is a distance between two gaps, ℓ - is a period; and n, a number of them.

Now let us find an equation for the phase factor $\exp(iq\ell)$. In the n-th gap let us denote the wave, going to the right, by $\vec{\psi}_n$ and the wave, going to the left, by $\overleftarrow{\psi}_n$. Of course, from periodicity it comes, that

$$\overleftrightarrow{\psi}_n = \exp(iq\ell)\, \overleftrightarrow{\psi}_{n-1} . \tag{17}$$

But $\vec{\psi}_n = r\overleftarrow{\psi}_n + t\vec{\psi}_{n-1}$, and $\overleftarrow{\psi}_n = r\vec{\psi}_n + t\,\overleftarrow{\psi}_{n+1}$. Using the relations (17) we obtain the system of two equations for $\vec{\psi}_n$ and $\overleftarrow{\psi}_n$:

$$\begin{aligned} \vec{\psi}_n &= r\overleftarrow{\psi}_n + t\exp(-iq\ell)\, \vec{\psi}_n , \\ \overleftarrow{\psi}_n &= r\vec{\psi}_n + t\exp(iq\ell)\, \overleftarrow{\psi}_n . \end{aligned} \tag{18}$$

This system has a solution only when

$$[1 - t\exp(-iq\ell)][1 - t\exp(iq\ell)] - r^2 = 0 . \tag{19}$$

This is an equation we wished to have. Its solution can be represented in the same form, as (16):

$$\exp(iq\ell) = [\sqrt{(t+1)^2 - r^2} + \sqrt{(t-1)^2 - r^2}]/[\sqrt{(t+1)^2 - r^2} - \sqrt{(t-1)^2 - r^2}] . \tag{20}$$

From two solutions of (19) we have chosen the one, which for the fictive potential ($r = 0$, $t = \exp(ik\ell)$) goes over to $\exp(ik\ell)$.

It is very easy to check these formulas by applying them to the Kronig-Penney potential. But first it is necessary to decide how to locate the period. If we want to have a symmetric period, its potential must be chosen to be $U = 2p\,\delta(x - 1/2)\ \theta(0 \le x \le 1)$. The reflection and transmission amplitudes for this potential are:

a) $r = -p\exp(ik\ell)/(p - ik)$, b) $t = -ik\exp(ik\ell)/(p - ik)$. (21)

It is not difficult to put (21) in (16) and (20), and we leave it to the reader. We would like only to note, that if we choose asymmetric period, i.e. $U = 2p\cdot\delta(x)\cdot\theta(0 \le x \le 1)$, then it is necessary to discriminate between reflection $\vec{r} = \exp(-ik\ell)\, r$ and $\overleftarrow{r} = \exp(ik\ell)\, r$ for a wave incident on this potential from the left and the right, respectively. For an asymmetric period the total reflection will also be asymmetric, but the phase relation between $\vec{R}_\infty$ and $\overleftarrow{R}_\infty$ is the same as that between $\vec{r}$ and $\overleftarrow{r}$.

Till now we have considered the half infinite potential. What can we say about a potential with a finite number of periods? The answer to this question is amusingly simple. Indeed, suppose, our potential has only N periods, then reflection and transmission amplitudes can be written in full analogy with (10) and (12):

$$\begin{aligned} T_N &= (1 - R_\infty^2)\exp(iqL)/[1 - R_\infty^2\exp(2iqL)] , \\ R_N &= R_\infty[1 - \exp(2iqL)]/[1 - R_\infty^2\exp(2ikL)] , \end{aligned} \tag{22}$$

where $L = N\ell$. To obtain these relations it is only necessary to note that half infinite potential with period ℓ is also periodic with period $N\ell$, so we can write down the expressions (16) and (20) in terms R_N and T_N instead of r and t. After that we can resolve them with respect to R_N and T_N and thus come to expressions (22). For an asymmetric potential the phase relation between $\vec{R}_N$ and $\overleftarrow{R}_N$ is the same as between $\vec{r}$ and $\overleftarrow{r}$, and transmission amplitudes are identical.

6. DIFFUSION EQUATION

In the case of wave problem it is very natural to discuss waves that are going back and forth, are reflected and transmitted. In diffusion problems there are no waves, nevertheless, it is possible to apply to them the same terminology. Indeed, let us go back to the wave reflection, and consider a high rectangular potential, so high, that $k^2 < u_0$ (3). In that case the phase factor becomes a decreasing exponential function. Nevertheless, we can follow the same way, and we come to the correct expressions for reflection and transmission amplitudes.

Let's, for example, consider a molecular gas propagation along pipes (see, for instance[3], chapter 4). It is described by the one dimensional diffusion equation

$$D\, d^2 n/dx^2 = n/\tau , \tag{23}$$

whose solution is $n = \exp(\pm x/\ell)$, where D is diffusion coefficient; n, linear density of a gas; τ, life time of an atom if it can decay or be captured by a wall of the pipe; and ℓ, diffusion length: $\ell^2 = D\tau$. To find transmission of a pipe with length L, it is necessary to find reflection and transmission at one end of a half-infinite pipe. Now we shall do that.

Let us suppose, that from outside on the opening of a half-infinite tube, placed at $x \geq 0$, is incident a gas isotropic in the forward half sphere. The volume density of the gas is n_0, the velocity of its atoms is v, the area of the opening is supposed to be equal to unity, so the incident flow is equal to $n_0 v/2$. A part of this current is reflected back outside. We suppose the reflected particles to be isotropic in the backward half sphere. We denote its volume density as $r^+ n_0$, so r^+ is a reflection coefficient and a reflected flow is equal to $r^+ n_0 v/2$.

The solution of the diffusion equation inside the tube can be represented in the form: $t^+ n_0 \exp(-x/\ell)$. So t^+ plays the role of the transmission coefficient. Since the diffusion flow is defined to be $-D\, dn/dx$, we can match the flow and the density at the opening and thus get two equations to determine r^+ and t^+.

$$\text{a)}\ 1 + r^+ = t^+, \qquad \text{b)}\ 1 - r^+ = q t^+, \qquad \text{where c)}\ q = 2D/\ell v. \tag{24}$$

The solution of these equations is

$$\text{a)}\ r^+ = (1-q)/(1+q), \qquad \text{b)}\ t^+ = 2/(1+q). \tag{25}$$

In the case, when incident flow goes from inside of the tube we write down the density of gas atoms inside the tube in the form: $n = n_0[\exp(x/\ell) + r^- \exp(-x/\ell)]$, and outside $-$ $t^- n_0$. Equations analogous to (24) have the solution:

$$\text{a)}\ r^- = (q-1)/(q+1), \qquad \text{b)}\ t^- = 2q/(q+1). \tag{26}$$

Now, if we repeat all the way, which brought us to formula (10), but instead of the phase factor use the propagation function $\exp(-x/\ell)$, we come to the expression:

$$T = t^+ t^- \exp(-L/\ell)/[1 - r^2 \exp(-2L/\ell)], \tag{27}$$

which gives the transmission of the tube with length L. (Here $r = r^+$). Since $t^+ t^- = 1 - r^2$, the expression (27) can be written in the form similar to (22)

$$T = \exp(-kL)\,(1 - r^2)/[1 - r^2 \exp(-2kL)], \tag{28}$$

where $k = 1/\ell$. The same can be said about the total reflection R of the tube:

$$R = r[1 - \exp(-2kL)]/[1 - r^2 \exp(-2kL)]. \tag{29}$$

7. BRANCHED GUIDES

Let us suppose, that our guide is composed of two different links 1 and 2 (for simplicity we suppose them to have the same cross sections), and imagine a very narrow gap introduced between these parts. The density of atoms N inside the gap can be presented in two ways

$$N = T_1 N_0 + (1 + R_1)\overleftarrow{N}_1 = (1 + R_2)\vec{N}_2 , \tag{30}$$

where $\overleftarrow{N}_1$ represents the part of the density, which corresponds to gas atoms, moving in the gap towards the link 1; and $\vec{N}_2$, towards the link 2, R_i and T_i are total reflection and transmission coefficients of two links, and we denote the density of primary atoms, incident from the left on the begining of the first link, by N_0.

The conservation of the current in the gap leads to a second equation to determine $\overleftarrow{N}_1$ and $\vec{N}_2$

$$T_1 N_0 = (1 - R_1)\overleftarrow{N}_1 + (1 - R_2)\vec{N}_2 . \tag{31}$$

From (30) and (31) it comes out that

$$\vec{N}_2 = (1 - R_1 R_2)^{-1} T_1 N_0 , \qquad \overleftarrow{N}_1 = R_2 \vec{N}_2 = R_2 (1 - R_1 R_2)^{-1} T_1 N_0 . \tag{32}$$

From expressions (32) we directly obtain the reflection and transmission of the two links chain

$$R_{12} N_0 = R_1 N_0 + T_1 \overleftarrow{N}_1 , \qquad R_{12} = R_1 + T_1 R_2 (1 - R_1 R_2)^{-1} T_1 , \tag{33}$$

$$T_{12} N_0 = T_2 \vec{N}_2 , \qquad T_{12} = T_2 (1 - R_1 R_2)^{-1} T_1 . \tag{34}$$

To generalise these expressions to the case of many links guide is trivial. It is not difficult to include here some diafragms or a stepwise change of the guide's radius. We leave this matter to the readers. We would like only to note, that formulas (33), (34) are written in such a way to be correct in the case of a nonscalar molecular gas diffusion, when R and T are not the numbers, but matrices.

Now let us consider the branched guide, i.e. consider a vertex, connecting, for instance, three links, numbered 1, 2 and 3. To find transmission T_{12} in the presence of branch 3 we introduce in the vertex a virtual gap, separating all links. A density N of gas atoms inside this gap can be presented in a three fold way in full analogy with (30)

$$N = T_1 N_0 + (1+R_1)\overleftarrow{N}_1 = (1+R_2)\vec{N}_2 = (1+R_3) N_3^{\downarrow}, \tag{35}$$

where N_i with arrows are the parts of density, giving rise to currents, entering links 1, 2 and 3 from the side of the vertex. (Here for the sake of simplicity we suppose the cross sections of all the links to be the same). One more equation is obtained from the requirement of the current conservation in the gap

$$T_1 N_0 = (1-R_1)\overleftarrow{N}_1 + (1-R_2)\vec{N}_2 + (1-R_3) N_3^{\downarrow}. \tag{36}$$

To find the three N_i from three equations (35), (36) is not the problem. For instance,

$$\vec{N}_2 = T_1 N_0 (1+\eta_1)/(1+R_2)(\eta_1+\eta_2+\eta_3), \quad \eta_i = (1-R_i)/(1+R_i), \tag{37}$$

$$N_1 = T_1 N_0 (1-\eta_2-\eta_3)/(1+R_1)(\eta_1+\eta_2+\eta_3). \tag{38}$$

To find $N_3^{\downarrow}$ it is enough to transmute indices 2 and 3 in (37). Now the transmission T_{12}, for instance, is obtained from the equality: $T_{12} N_0 = T_2 \vec{N}_2$, and reflection of the system from the side of the first link is given by the equality: $R_{1\,{}^2_3} N_0 = R_1 N_0 + T_1 \overleftarrow{N}_1$.

8. THREE-DIMENSIONAL SPHERICALLY SYMMETRIC POTENTIAL

Till now all the formulas were obtained for the simple case when free propagation is described by phase or exponential function. It is pleasant to find, that the structure of all the formulas is the same even in a more complicated case. For instance, in the spherically symmetric wave equation for

free propagation we have two functions: spherical Hankel functions of two kinds. (Of course, one of it can be replaced with the spherical Bessel function).

In order not to mix the radial coordinate with reflection coefficients we denote the former by z.

Let's consider a spherically symmetric potential barrier

$$U = u_0\,\theta(z_1 \le z \le z_2) \tag{39}$$

and a particle, incident on it from the left. First we shall put $z_2 \to \infty$. For ℓ-th harmonics an incident wave is described by $h_\ell^{(1)}(kz)/h_\ell^{(1)}(kz_1)$; reflected, by $r_1^+ j_\ell(kz)/j_\ell(kz_1)$; the wave function inside the barrier, by $t_1^+ h_\ell^{(1)}(k'z)/h_\ell^{(1)}(k'z_1)$, where $k' = (k^2-u_0)^{1/2}$. The amplitudes r_1^+ and t_1^+ are determined from the matching of the wave function and its derivative at the point z_1. Now, if the wave is incident on the step at z_1 from inside of the barrier, then the wave function can be represented as

$$\begin{aligned} f_\ell(z) = {} & \theta(z \ge z_1)\,[\, j_\ell(k'z)/j_\ell(k'z_1) + r_1^- h_\ell^{(1)}(k'z)/h_\ell^{(1)}(k'z_1)] + \\ & + \theta(0 \le z \le z_1)\, t_1^- j_\ell(kz)/j_\ell(kz_1). \end{aligned} \tag{40}$$

After matching this wave and its derivative at point z_1 we find amplitudes r_1^- and t_1^-. The same calculations for the potential (39) with $z_2 \ne \infty$ but $z_1 \to -\infty$ gives us amplitudes $r_2^\pm$ and $t_2^\pm$ at the point z_2. Now after the same reasoning that led us to expressions (10) and (11) we obtain the transmission and reflection from the total nut-shell barrier (39).

$$T_\ell^- = t_1^- e_{12} t_2^- / (1 - e_{12} r_2^+ e_{21} r_1^-), \tag{41}$$

$$R_\ell^- = r_2^- + t_2^+ e_{21} r_1^- e_{12} t_2^- / (1 - e_{12} r_2^+ e_{21} r_1^-), \tag{42}$$

where transmission functions e_{12} and e_{21} describe propagation of the wave from point z_2 to point z_1 and in opposit direction, respectively. In the preceding sections they were exponentials, now they are

$$e_{12} = j_\ell(kz_1)/j_\ell(kz_2), \qquad e_{21} = h_\ell^{(1)}(kz_2)/h_\ell^{(1)}(kz_1). \tag{43}$$

With the help of amplitudes (41), (42) the wave function in the region $z > z_2$ and $0 \le z \le z_1$ can be written in the form

$$f_\ell = \theta(z \geq z_2)[\, j_\ell(kz)/j_\ell(kz_2) + R_\ell^- h_\ell^{(1)}(kz)/h_\ell^{(1)}(kz_2)\,] + \theta(0 \leq z \leq z_1)\, T_\ell^- j_\ell(kz)/j_\ell(kz_1) \,. \quad (44)$$

In the above considerations we didn't pay attention to the fact, that the radial axis is only half infinite, and it is not necessary since it is taken into account automatically in the propagation functions.

Now we can easily investigate resonance scattering and the decay of resonances in the case of a nut-shell potential. For scattering it is seen, that T_ℓ^- is the amplitude of the function inside the nut shell. The position of resonances can be determined from the maximum of T_ℓ^- .

9. CONCLUSION

Recursive relations have a wast range of applications. They give, for instance, a new approach to the solution of one dimensional Schrödinger equation in any potential. They help to solve any second order equation. But they are helpful not only for calculation. They are very helpful for presentation and interpretation of the results. We can refer an interested reader to some additional literature [2-5] , to be more acquainted with them.

Author is very grafeful to V.V.Burov, R.Golub, F.Levchanovskii, M.Popov and A.Steyerl who made his work on this paper so much pleasant.

REFERENCES

1. The Recursion Method and Its Applications: Proc. of the Conf., Imperial College, London, England, September 13-14, 1984. Ed-s: D.G.Pettifor, D.L.Weaire - Berlin etc.: Springer-Verlag, 1985. (Springer Series in Solid-State Sciences. v.58).
2. Ignatovich V.K. - Usp.Fiz.Nauk., 1986, v.150, p.145.
3. Ignatovich V.K. Physics of Ultracold Neutrons. M.: Nauka, 1986.
4. Ignatovich V.K., Nikitenko Yu.V. JINR, P3-87-326, Dubna, 1987.
5. Kiang D. - Am.J.Phys., 1974, v.42, p.785.

Received by Publishing Department
on December 17, 1987.

Игнатович В.К. E4-87-880
Замечательные возможности рекуррентных соотношений

Описывается сущность метода рекуррентных соотношений и иллюстрируется применение этого метода на примере решения одномерного уравнения Шредингера со ступенчатым потенциалом, с полубесконечным и конечным периодическим потенциалом, а также на примере молекулярного течения газа по трубам и рассеяния частиц на сферически симметричном потенциале.

Работа выполнена в Лаборатории нейтронной физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1987

Ignatovich V.K. E4-87-880
The Remarkable Capabilities of Recursive Relations

The essence of the recursive relation method is presented. For illustration of its capabilities it is shown how to use it to get the one-dimensional Schrödinger equation solution with stepwise and periodic potential (halfinfinite and finite). The application of this method to such problems as molecular gas flow in pipes and particle scattering on spherically symmetric potential is also demonstrated.

The investigation has been performed at the Laboratory of Neutron Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1987