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HAMILTONIAN THEORIES' QUANTIZATION BASED ON A PROBABILITY OPERATOR

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1. Introduction

More than 60 years of quantum theory existence are full of unusual paradoxical situations. The arguments on the most principle questions of quantum theory, starting from the birth of quantum mechanics, are still under studies.

Among the discussed aspects of the generally accepted quantum mechanics the so-called "incompleteness of ..." occupies one of the most important places. One ought to distinguish at least three types of the "incompleteness":

A. The incompleteness of the quantum-mechanical description represents the statement that quantum mechanics does not give a complete description of physical reality. The most elegant attempt to prove this statement is based on the paradox which follows from the Einstein-Podolsky-Rosen gedankenexperiment [1]. It is important to underline that the Einstein-Podolsky-Rosen paradox arises from the questions like whether the physical variables reflect any simultaneously existing physical realities, whether their values exist before a measuring. 50 years of arguments and deep studies of these problems have not yet led to any definitive answer (see, for example [2-5]).

B. The incompleteness of the probability interpretation consists in the fact that quantum mechanics, despite of its obvious and generally acknowledged statistical character, is not a theory of the consistent probability nature [6-14]. It does not make use of joint probability distributions for physical variables, for example for coordinate and momentum, it defines no conditional probabilities. The numerous attempts (see, for example [15-23]) to introduce joint coordinate-momentum probability distributions $F_{\psi}(q,p,t)$ (quantum distribution functions) for quantum states ψ showed [13,24,25] that this programme cannot be applied to the generally accepted quantum mechanics.

C. The incompleteness of the mathematical formalism means the absence of an univocal and generally accepted law (correspondence rule) according to which the quantum operator $\hat{A}(t)$ is set for the classical function $A(q,p,t)$. The problem of the correspondence rule which is considered to be one of the major processes of quantization arose simultaniously with the quantum mechanics birth [26-29] and is still under investigation [30-33].

The number of the articles on the above-mentioned problems is now enormous. Nevertheless the current of the investigations does not decrease. One has to notice that a considerable part of the investigators admit the necessity of some theory which would turn to be more general (more complete) than the existing quantum mechanics. What is needed is not a new reinterpretation or a reformulation of the generally accepted quantum theory but a construction of some generalized theory free from the paradoxes and logical problems of the orthodox quantum mechanics.

In the present paper we are going to consider and to discuss one method of such generalization which eliminates the "incompleteness B". This method is based on the classical theory's quantization leading to a correspondent quantum theory in which any quantum state $\psi(t)$ is connected with a normalized non-negative coordinate-momentum-time function $F_\psi(q,p,t)$ treated as the joint coordinate-momentum probability density.

We shall demonstrate that the elimination of the "incompleteness B" gives a serious limitation to the arbitrariness for the correspondence rule of the "incompleteness C" and eliminates all the problems connected to the "incompleteness A".

2. General approach to the problem

It is possible to eliminate the "incompleteness of the probability interpretation" by an appropriate modification of the generally accepted quantum theory based on different grounds. But the results of the last years investigations [34-36] have showed that such modification can be always formulated with the help of a probability operator introduced into the procedure of the classical theory's quantization.

We understand the procedure of quantization as the transition from the known classical theory(C-theory) to the correspondent quantum theory (Q-theory). While studing this procedure we shall be based on the following grounds.

C-theory:

1(C). The state of a physical system in the moment t is given by a vector $X(t) = (q(t), p(t))$ in the phase-space R_{qp} of the coordinates $q = (q_1, \dots, q_N)$ and momenta $p = (p_1, \dots, p_N)$ values.

2(C). Any physical variable A characterizing the physical system in the moment t is given by a coordinate-momentum-time function $A(q,p,t)$.

3(C). The expectation value $\langle A \rangle$ of a physical variable

in the state X is determined by its function, i.e.

$$\langle A \rangle_X = A(q,p,t) \big|_{(q,p)=X} . \qquad (1a)$$

4(C). The evolution of the state $X(t)$ in time is determined by the equations of Hamilton:

$$\left. \begin{aligned} d_t q(t) &= \partial_p H(q,p,t) \big|_{(q,p)=X(t)} , \\ d_t p(t) &= -\partial_q H(q,p,t) \big|_{(q,p)=X(t)} , \end{aligned} \right\} \qquad (1b)$$

where $H(q,p,t)$ -function corresponding to the hamiltonian of the system.

Q-theory:

1(Q). The state of a physical system at the moment t is determined by a vector $\psi(t)$ of a complex space $\mathcal{L}$ with some scalar product $(\cdot/\cdot)$.

2(Q). To every physical variable A, which characterizes the physical system at the moment t, there corresponds a self-adjoint operator $\hat{A}(t)$, which belongs to an algebra $\mathcal{A}$ of linear operators defined in the space $\mathcal{L}$.

3(Q). The expectation value $\langle A \rangle$ of a physical variable A in the state ψ is determined by the formula:

$$\langle A \rangle_\psi = (\psi/\hat{A}(t)\psi) / (\psi/\psi) . \qquad (2a)$$

4(Q). The evolution of the state $\psi(t)$ in time is determined by the equation of Schrödinger

$$i\hbar\, \partial_t \psi(t) = \hat{H}(t)\psi(t) , \qquad (2b)$$

where $\hat{H}(t) = \hat{H}^+(t)$ is the operator which corresponds to the hamiltonian of the considered system.

Providing the comparison of the requirements 1-4 of the C-theory with the corresponding requirements of the Q-theory we can see that for constructing a Q-theory, if we know a C-theory, it is necessary to solve two problems :

Problem 1. To choose a space $\mathcal{L}$ of quantum states ψ and to fix an algebra $\mathcal{A}$ of linear in $\mathcal{L}$ operators.

Problem 2. To point out the reflection (the correspondence)

$$\hat{A}(t) = O(A(q,p,t)) \in \mathcal{A} , \qquad (3)$$

which would allow to set the operators of all physical variables characterizing the system which are under consideration.

It is quite clear that problems 1 and 2 are interdependent and the search of their solutions can be provided by different methods.

It also should be noted that when the problem 1 is in some way solved the solution of the problem 2 can be formulated as some correspondence table, including only H and the set of physical variables $\{A\}_{exp}$, which values can be measured experimentally.

In the present article we confine ourselves by the procedure of quantization, based on the following demands, claimed to the correspondence rule (3):

Demand 1. The reflection $O(\dots)$ permits an analytical formulation, which has sense for a wide enough multitude of functions

$$M = \{A(q,p,t)\} \supset \{A(q,p,t)\}_{exp} = M_{exp}$$

and is linear one, i.e.:

$$O(1) = \hat{1}, \qquad O(\alpha A(q,p,t)) = \alpha O(A(q,p,t)), \tag{4a}$$

$$O(A_1(q,p,t) + A_2(q,p,t)) = O(A_1(q,p,t)) + O(A_2(q,p,t)). \tag{4b}$$

Here $\hat{1}$ is the unit operator in $\mathcal{L}$, α is a complex number.

Demand 2. The reflection $O(\dots)$ is such, that in the obtained Q-theory for any $\psi \in \mathcal{L}$ there exists the coordinate-momentum distribution $F_\psi(q,p,t)$, which might be treated as the phase-space probability density, i.e.:

$$\int F_\psi(q,p,t)\,dq\,dp = 1, \qquad F_\psi(q,p,t) \geq 0, \tag{5a}$$

$$\langle A\rangle_\psi = \int A(q,p,t)\, F_\psi(q,p,t)\,dq\,dp. \tag{5b}$$

Here and in what follows the integration is performed over the whole classical space R_{qp}.

Thus we are going to consider only such procedures of quantization which lead to Q-theories with the correspondence $\psi \to F_\psi \geq 0$.

It should be noted, that the only demand of linearity (4) leads to the quantization with the quaziprobability operator $\hat{f}(q,p,t) \in \mathcal{A}$ in the terms of which one can write all the known unique correspondence rules. The normalized quazidistribution $f_\psi = (\psi/\hat{f}\psi)/(\psi/\psi)$ for which (5b) is fulfiled automaticaly, appears in Q-theory in such case (for more details see, for example [36,37]).

3. Coordinate-momentum probability operator

As it is shown in the works [35,36] the only method of quantization which leads to a quantum theory with the consistent probabilistic interpretation is the method, based on the following correspondence rule:

$$\hat{A}(t) = O_{\hat{F}}(A(q,p,t)) =$$
$$= \int A(q,p,t)\,\hat{F}(q,p,t)\,dq\,dp \in \mathcal{A}, \tag{6a}$$

where operator $\hat{F}$ has the properties

$$\int \hat{F}(q,p,t)\,dq\,dp = \hat{1} \in \mathcal{A}, \qquad (\psi/\hat{F}(q,p,t)\psi) \geq 0, \tag{6b}$$

for any $q, p \in R_{qp}$ and any $\psi \in \mathcal{L}$.

The sufficiency of the operator $\hat{F}$ existence follows from the next reasoning. Constructing quantum operators due to rule (6) we can see that correctness of relations (4) is obvious. Having determined the values (2a) of operators (6a) in any state $\psi \in \mathcal{L}$, we obtain (5b), where

$$F_\psi(q,p,t) = (\psi/\hat{F}(q,p,t)\psi)/(\psi/\psi). \tag{7}$$

Finally from the properties (6b) of the operator $\hat{F}$ follow the properties (5a) of joint coordinate-momentum probability density(7).

In order to prove the necessity of the operator $\hat{F}(q,p,t)$ existence let us consider the characteristic function of distribution F_ψ, which exists according to the demand 2:

$$\tilde{F}(u,v,t) = (2\pi)^{-2N} \int F_\psi(q,p,t)\, e^{-i(uq+vp)}\,dq\,dp.$$

Using the relation (5a) and reflection (3) let us rewrite it in the form:

$$\tilde{F}(u,v,t) = \langle (2\pi)^{-2N} e^{-i(uq+vp)} \rangle_\psi =$$
$$= (\psi / O((2\pi)^{-2N} e^{-i(uq+vp)})\psi)/(\psi/\psi).$$

Reproducing the starting distribution from $\tilde{F}$ we have:

$$F_\psi(\xi,\rho,t) = (\psi / \int du\,dv\, e^{i(u\xi+v\rho)}\, O((2\pi)^{-2N} e^{-i(uq+vp)})\psi)/(\psi/\psi).$$

Taking into consideration the properties (4) of representation $O(\dots)$ we finally obtain (7), where

$$\hat{F}(\xi,\rho,t) = O(\delta(\xi-q)\,\delta(\rho-p)) \in \mathcal{A}. \tag{8}$$

The representation (3) now may be written as

$$\hat{A}(t) = O(A(q,p,t)) = O\left(\int A(\xi,\rho,t)\,\delta(\xi-q)\,\delta(\rho-p)\,d\xi\,d\rho\right),$$

from which, making use of linearity (4), we receive the rule (6a) with the operator (8). Finally, the properties (6b) of operator (8) follow from the relation (7) and conditions (5a) of the demand 2, and this proves the statement.

Let us point out, that according to the relation (7) the values of $\hat{F}$ understood in the sence of quantum values (2a), in any $\psi \in \mathcal{L}$ coincide for the whole space R_{qp} with the values of the joint coordinate-momentum probability density F_ψ ,i.e.,from the point of view of Q-theory $\hat{F}$ is the probability operator. As this takes place from (8) and (6) the probability operator $\hat{F}$ himself corresponds to the δ -function in the classical phase-space.

4. Principles of the probability operator constructing

In accordance with the statement of the previous paragraph, any linear procedure of quantization, which leads to a Q-theory with a correspondence $\psi \rightarrow F_\psi(q,p,t) \geqslant 0$, may be uniquely defined by a coordinate-momentum probability operator $\hat{F}(q,p,t)$.

The probability operator fixes correspondence (6a) between objects $A(q,p,t)$ and $\hat{A}(t)$, which describe the same physical variable A in C- and Q-theories accordingly.

However for establishing the evident form of operator $\hat{F}$ (in a given algebra) the properties (6b) of normalization and non-negativity are not enough. So there appears a task to reduce the arbitrariness of $\hat{F}$ by some demands on the correspondence (6a) in addition to the properties (6b).

For this purpose,following the article [37],one may consider the evolution of physical variable values with time. In C-theory, because of the relation (1),

$$d_t \langle A \rangle_x = \left(\partial_t A(q,p,t) + \{H(q,p,t), A(q,p,t)\}\right)\Big|_{(q,p)=x}, \quad (9)$$

where $\{\cdot,\cdot\}$ is the classical Poisson brackets. In Q-theory due to the relations (2),

$$d_t \langle A \rangle_\psi = \left(\psi / \left(\partial_t \hat{A}(t) + \tfrac{i}{\hbar}[\hat{H}(t), \hat{A}(t)]_-\right)\psi\right)/(\psi/\psi), \quad (10)$$

where $[\cdot,\cdot]_-$ is a commutator. On comparing C-relations (1a) and (9) with the corresponding Q-relations (2a) and (10) it is quite logically assumed that the reflection (3) should be extended on the objects,which determine the evolution of the physical variables values with time, i.e.

$$O\left(\partial_t A(q,p,t) + \{H(q,p,t), A(q,p,t)\}\right) = \\ = \partial_t \hat{A}(t) + i\hbar^{-1}[\hat{H}(t), \hat{A}(t)]_- \quad . \quad (11)$$

The equations (11) represent the dynamical correspondence of the quantization procedure (in addition to the statistical correspondence, represented by equations (6)).

At the constructing of a Q-theory with a correspondence $\psi \rightarrow F_\psi \geqslant 0$ the possibility to write down the reflection (3) in the form (6a)

brings about the set of conditions, claimed to the probability operator:

$$O_{\hat{F}}\left(\{H(q,p,t), A(q,p,t)\}\right) - \int A(q,p,t)\cdot\partial_t \hat{F}(q,p,t)\,dq\,dp = \\ = i\hbar^{-1}\left[O_{\hat{F}}(H(q,p,t)),\ O_{\hat{F}}(A(q,p,t))\right]_- \quad (12a)$$

for all $A \in H \cup \{A\}_{exp}$, where $O_{\hat{F}}(\ldots)$ is the reflection (6).

The conditions (12a) demand in particular a specific correlation of the H and $\hat{F}$ evolutions with time. Thus,it follows from (6a) and (12a) for $A = H$:

$$\int H(q,p,t)\cdot\partial_t \hat{F}(q,p,t)\,dq\,dp = 0\,, \quad (12b)$$

$$\partial_t \hat{H}(t) = \int \partial_t H(q,p,t)\cdot\hat{F}(q,p,t)\,dq\,dp\,,$$

and $\partial_t \hat{H}(t) = 0$, if $\partial_t H(q,p,t) = 0.$

It is also notable that if for some $A \in \{A\}_{exp}$

$$\int A(q,p,t)\cdot\partial_t \hat{F}(q,p,t)\,dq\,dp = 0\,, \quad (13a)$$

then the correspondent condition (12a) means the correspondence between C- and Q- Poisson brackets for A and H . Such correspondence might be generalized for variables A and B by the conditions to the probability operator

$$O_{\hat{F}}\left(\{A(q,p,t), B(q,p,t)\}\right) = \\ = i\hbar^{-1}\left[O_{\hat{F}}(A(q,p,t)),\ O_{\hat{F}}(B(q,p,t))\right]_- \quad (13b)$$

with any pair $A, B \in \{A\}_{exp}$, both satisfying the requirement (13a). The set of equations (13) represents the canonical correspondence, which is extensivly used in the modern procedures of quantization (canonical quantization). It should be underlined,that the principle of canonical correspondence may be apparently fulfiled only for quantizations of C-theories invariant with respect to time translation and only with noncommutative algebras $\mathcal{A}$.

5. Quantization procedure based on a probability operator

According to the results of the previons paragraphs the procedure of transition from a C-theory to a Q-theory with the correspondence $\psi \rightarrow F_\psi \geqslant 0$ can be formulated in the following way.

Let a)the classical functions $H(q,p,t)$ and $\{A(q,p,t)\}_{exp}$, which describe the hamiltonian and the set of experimentally measured physical variables be known, and let b) some complex vector space $\mathcal{L}$ and algebra $\mathcal{A}$ of linear in it operators be chosen.

Then operators $\hat{H}(t)$ and $\{\hat{A}(t)\}_{exp}$ of the Q-theory with the state-space $\mathcal{L}$ are defined by the correspondence rule:

$$\hat{A} = O_{\hat{F}}(A(q,p,t)) = \int A(q,p,t)\, \hat{F}(q,p,t)\, dq\, dp \in \mathcal{A}, \qquad (14)$$

where $A \in H \cup \{A\}_{exp}$, $\hat{F}(q,p,t) \in \mathcal{A}$ is a non-negative in $\mathcal{L}$ operator of the coordinate-momentum probability density, i.e.,

$$\hat{F}(q,p,t) = \sum_n \hat{f}^+_n(q,p,t)\, \hat{f}_n(q,p,t) \qquad (15)$$

$\hat{f}_n(q,p,t) \in \mathcal{A}$, n is some collective index of summation.

Operator $\hat{F}$ satisfies the normalization condition

$$\int \hat{F}(q,p,t)\, dq\, dp = \hat{1} \in \mathcal{A} \qquad (16)$$

and the system of the integral equations

$$\int \{H(q,p,t), A(q,p,t)\}\cdot \hat{F}(q,p,t)\, dq\, dp - \int A(q,p,t)\cdot \partial_t \hat{F}(q,p,t)\, dq\, dp =$$

$$= i\hbar^{-1} \int H(q,p,t)\cdot A(\xi,\eta,t)\cdot [\hat{F}(q,p,t), \hat{F}(\xi,\eta,t)]_-\, d\xi\, d\eta\, dq\, dp, \qquad (17)$$

which follow from (12a) and represent the dynamical correspondence.

Therefore to fulfil the procedure of quantization based on the rule (14) with a probability operator it is necessary, on the first hand, to solve the system of equations (17) with respect to the operator $\hat{F}$, having the structure (15), satisfying the normalization (16) and conserving the mathematical sense of the integrals (14).

In this case three different cases may occur:

1. The problem has no solution. It means that the C-theory under consideration cannot be quantizated with any probability operator within the chosen algebra $\mathcal{A}$ (analogue to the Pauli theorem in quantum field theory).

2. The problem has a unique solution, i.e. the C-theory is quantizated and the sought for Q-theory is constructed.

3. The problem has a set of solutions. In this case it can be attempted to narrow the set of solutions either with the system of the additional to (17) equations

$$\int A(q,p,t)\cdot \partial_t \hat{F}(q,p,t)\, dq\, dp = 0, \qquad (18a)$$

$$\int \{A(q,p,t), B(q,p,t)\}\cdot \hat{F}(q,p,t)\, dq\, dp =$$

$$= i\hbar^{-1} \int A(q,p,t)\cdot B(\xi,\eta,t)\cdot [\hat{F}(q,p,t), \hat{F}(\xi,\eta,t)]_-\, d\xi\, d\eta\, dq\, dp, \qquad (18b)$$

with $A, B \in \{A\}_{exp}$, following from (13 a,b) and representing the canonical correspondence, or by broading the set $\{A\}_{exp}$ in (17) and at the same time in (18) up to a certain multitude $\{A\} \supset \{A\}_{exp}$.

In conclusion we will note that the quantization procedure with a probability operator

a) always conserves all the dynamic invariants of the initial C-theory in the set of dynamic invariants of the constructed Q-theory and

b) profoundly depends on the hamiltonian of the quantizated system, i.e. even within the same algebra $\mathcal{A}$ it may appear to be different for two different physical systems being described in one and the same C-theory.

6. Probability operator in commutative algebras

If the problem (14)-(17) has a certain solution $\hat{F}(q,p,t)$ in a commutative algebra $\mathcal{A}_c$ than it cannot satisfy the system of equations (18). In fact, for $A = q_j$ and $B = p_k$ the relation (18b) due to the general commutativity contradicts the normalization (16). So, while quantizating with a probability operator within a commutative algebra the canonical correspondence cannot take place.

Broadening the set $\{A(q,p,t)\}_{exp}$ up to the multitude of analytical in R_{qp} functions, from (17), after integrating by parts and taking into consideration the arbitrariness of $A(q,p,t)$ and commutativity of $\mathcal{A}_c$, we find:

$$\partial_t \hat{F}(q,p,t) + \{H(q,p,t), \hat{F}(q,p,t)\} = 0. \qquad (19)$$

Introducing now the distribution $F_\psi(q,p,t) = \langle \hat{F}(q,p,t) \rangle_\psi$ for its derivative with respect to t we will find, taking into consideration the evolution equations (2b) and (19), the classical Liouville equation

$$\partial_t F_\psi(q,p,t) + \{H(q,p,t), F_\psi(q,p,t)\} = 0. \qquad (20)$$

Thus, the quantization of a C-theory based on a probability operator in a commutative algebra brings us to a Q-theory which coincides with the classical statistical theory.

7. Coordinate-momentum probability operator in the Bose algebras

The permutation relations,

$$[X_j, Y_k]_- = -i\delta_{jk}\hat{1}, \qquad [X_j, X_k]_- = [Y_j, Y_k]_- = 0, \qquad (21)$$

for the self-conjugate generators $X = (X_1, \dots, X_N)$ and $Y = (Y_1, \dots, Y_N)$ of the Bose algebra $\mathcal{A}_B(N)$ allow us to write down the operators $\hat{F}$ and $\hat{f}_n$ from (15) in some arranged form, for example:

$$\hat{F}(q,p,t) = \int F(q,p,\xi,\eta,t)\cdot e^{i(\xi X + \eta Y)} d\xi d\eta \in \mathcal{A}_B(N), \qquad (22a)$$

$$\hat{f}_n(q,p,t) = \int f_n(q,p,\xi,\eta,t)\cdot e^{i(\xi X + \eta Y)} d\xi d\eta \in \mathcal{A}_B(N). \qquad (22b)$$

Here $\xi, \eta \in R^N$; ξX and ηY are scalar products. Putting (22) in (15) and (16) and using the Weyl identity

$$\exp(A+B) = \exp\left(\tfrac{1}{2}[B,A]_-\right)\cdot \exp(A)\cdot \exp(B),$$

we obtain

$$F(q,p,\xi,\eta,t) = \sum_n \int f_n^*(q,p,\xi',\eta',t)\cdot f_n(q,p,\xi+\xi',\eta+\eta',t)\cdot e^{\frac{i}{2}(\xi\eta' - \xi'\eta)} d\xi' d\eta' \qquad (23)$$

with the normalization requirement

$$\int F(q,p,\xi,\eta,t)\, dq\, dp = \delta(\xi)\delta(\eta). \qquad (24)$$

Nonnegativety of the operator (22a) was investigated [35-37] in a concrete representation of the algebra $\mathcal{A}_B(N)$ and the space $\mathcal{L}$ connected with it:

$$\psi(\tau) \in \mathcal{L}, \quad \tau = (\tau_1, \dots, \tau_N) \in R^N, \quad X_j = -i\partial_{\tau_j}, \quad Y_j = \tau_j .$$

It was shown that any nonnegative operator $\hat{F}(q,p,t)$ can be given by a set of functions $\mu_k(q,p,\xi,t)$. A comparison of the representation [35-37] and the representation (22a) gives:

$$F(q,p,\xi,\eta,t) = (2\pi)^{-N} e^{-\frac{i}{2}\xi\eta} \sum_k \int \mu_k(q,p,\xi',t)\cdot \mu_k^*(q,p,\xi+\xi',t)\cdot e^{-i\xi'\eta} d\xi', \qquad (25)$$

$$f_n(q,p,\xi,\eta,t) = (2\pi)^{-N} e^{-\frac{i}{2}\xi\eta} \int U_m(q,p,\xi',t)\cdot \mu_k^*(q,p,\xi-\xi',t) e^{i\xi'\eta} d\xi', \qquad (26)$$

where $n = (m,k)$, the functions U_m and μ_k satisfy the normalization condition

$$\sum_m \int |U_m(q,p,\xi,t)|^2 d\xi = 1, \qquad (27a)$$

$$\sum_k \int \mu_k(q,p,\xi,t)\cdot \mu_k^*(q,p,\xi',t)\, dq\, dp = \delta(\xi - \xi'). \qquad (27b)$$

The set of square integrable functions U_m does not influence the results of the quantization: a transition from the set $\hat{f}_n \in \mathcal{A}$ to a new set $\hat{f}_n' = \hat{U}\hat{f}_n \in \mathcal{A}$, where $\hat{U}$ is any unitary in $\mathcal{L}$ operator, does not change the probability operator (15).

In order to fulfil a concrete quantization with $\hat{F}(q,p,t) \in \mathcal{A}_B(N)$ it is necessary to put the probability operator (22a) into the dynamical (17) and canonical (18) equations. It brings us to a system of integral equations which must be satisfied by some choice of functions $\mu_k(q,p,\xi,t)$.

The most interesting(from the point of vew of the generally accepted quantum mechanics) quantization is such when between pairs of classical generators (q_j, p_j) and pairs of generators (X_l, Y_l) of the Bose algebra exists one-to-one correspondence. For example

$$q_j \rightleftarrows Y_j, \quad p_j \rightleftarrows X_j, \quad j = \overline{1,N}, \qquad (28)$$

in the sense that operator $\hat{A}$ depends on operator X_j (on operator Y_j) then and only then, if the classical function $A(q,p,t)$ depends on the component of momentum p_j (of coordinate q_j).

Replacement $X \rightleftarrows Y$ in (28) is insignificant because it brings about the replacement $\xi \rightleftarrows \eta$ in the kernel (22a).

Putting operator (22a) in (14) and demanding, so that operators for functions not depending on p_j (on q_j) do not depend on operator X_j (on operator Y_j), we get the chain of relations:

$$\int F dp_j \sim \delta(\xi_j), \quad \int F dq_j \sim \delta(\eta_j), \quad \int F dp_j dq_l \sim \delta(\xi_j)\delta(\eta_l), \dots$$

and so on up to the normalization (24). From this it follows with necessity:

$$F(q,p,\xi,\eta,t) = v(\xi,\eta,t)\cdot \exp\left\{-i\sum_{j=1}^{N}\left(\frac{\eta_j q_j}{\alpha_j} + \frac{\xi_j p_j}{\beta_j}\right)\right\}, \qquad (29)$$

where α_j and β_j are real constants. Putting the kernal (29) in (22a) and further in (18) with $A = q_j$, $B = p_l$ and using (16) we get: $\alpha_j\beta_j = \hbar$, $j = \overline{1,N}$, $\hat{q}_j = \alpha_j Y_j + q_j^0[v]$ and $\hat{p}_j = \beta_j X_j + p_j^0[v]$ (for more details see [39]).

The last relations allow us to introduce new self-conjugate generators $(\hat{q}_j, \hat{p}_j)$, $j = \overline{1,N}$, having the physical sense of the coordinate and momentum operators, which satisfy the usual permutation relations:

$$[\hat{q}_j,\hat{p}_\ell]_- = i\hbar\delta_{j\ell}\hat{1}, \qquad [\hat{q}_j,\hat{q}_\ell]_- = [\hat{p}_j,\hat{p}_\ell]_- = 0. \tag{30a}$$

Now from (29) and (22a) after a substitution of variables it follows:

$$\hat{F}(q,p,t) = (2\pi\hbar)^{-2N}\int \mathcal{U}(\xi,\eta,t)\cdot e^{\frac{i}{\hbar}[\xi(\hat{p}-p)+\eta(\hat{q}-q)]}d\xi d\eta, \tag{30b}$$

where $R_{\xi\eta}$ is a space isomorphical to R_{qp}. Finally, writing down the nonnegative operator (30b) in the form (22a) with a kernel like (25), we have:

$$\mathcal{U}(\xi,\eta,t) = e^{\frac{i}{2\hbar}\xi\eta}\sum_\kappa\int\varphi^*_\kappa(\xi',t)\cdot\varphi_\kappa(\xi+\xi',t)e^{\frac{i}{\hbar}\xi'\eta}d\xi', \qquad \sum_\kappa\int|\varphi_\kappa(\xi,t)|^2 d\xi = 1. \tag{30c}$$

Thus, the quantization of a C-theory on the base of a probability operator in the Bose algebras with one-to-one correspondence between classical and quantum generators brings us to the Q-theory investigated in [34,38-47].

But in this case the set of "subquantum" [38-47] functions $\{\varphi_\kappa\}$ is related with the functions $\{A(q,p,t)\}_{exp}$ and $H(q,p,t)$ of the initial C-theory through the system of the integral equations obtained from (30) and the conditions of dynamical (17) and canonical (18) correspondences.

8. Intrinsic moment probability operator in the Fermi algebra

Let us consider an example of quantization of the C-theory of a point three-dimensional particle with mass m and charge e in a magnetic field $\vec{\mathcal{H}} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$. The correspondent Q-theory with $\psi \to F \geq 0$ is then defined by a probability operator $\hat{F}(\vec{q},\vec{p},t) \in \mathcal{A}_B(3)$.

Let the particle under consideration has intrinsic mechanical $\vec{S} = (S_1, S_2, S_3)$ and magnetic $\vec{M} = ge\vec{S}/2mc$ moments, where g is the giromagnetic ratio, while $|\vec{S}| = s = Const$. In this case to the classical generators $(\vec{q},\vec{p})$ two more should be added, for example the spherical angles θ and φ of $\vec{S}$. Accordingly to the generators $(\hat{\vec{q}},\hat{\vec{p}})$ of algebra $\mathcal{A}_B(3)$ should be also added two generators.

Let us suppose that the operators of the moments belong to the Fermi algebra $\mathcal{A}_F(1)$, the two self-conjugate generators σ_1 and σ_2 of which satisfy the permutation relations:

$$[\sigma_\kappa, \sigma_\ell]_+ = 2\delta_{\kappa\ell}\hat{1}, \qquad \kappa,\ell = \overline{1,2}. \tag{31}$$

For a simplification we shall consider a stationary quantization $\partial_t\hat{F} \equiv 0$ and not the whole probability operator $\hat{F}(\vec{q},\vec{p},\theta,\varphi) \in \mathcal{A}_B(3)\times\mathcal{A}_F(1)$ but its integral

$$\hat{F}(\theta,\varphi) = \int\hat{F}(\vec{q},\vec{p},\theta,\varphi)d\vec{q}d\vec{p} \in \mathcal{A}_F(1), \tag{32}$$

defining $\hat{A}$ for functions $A(\vec{S}) = A(s,\theta,\varphi)$ according to the correspondence rule

$$\hat{A} = O_{\hat{F}}(A(s,\theta,\varphi)) = \int A(s,\theta,\varphi)\hat{F}(\theta,\varphi)d\Omega \in \mathcal{A}_F(1). \tag{33}$$

The integration here is carried out over the whole spatial angle, i.e. $d\Omega = \sin\theta d\theta d\varphi$, $\theta\in[0,\pi]$, $\varphi\in[0,2\pi]$.

To construct the probability operator (32) let us write down the conditions of nonnegativity and normalization

$$\hat{F}(\theta,\varphi) = \sum_n \hat{f}^+_n(\theta,\varphi)\hat{f}_n(\theta,\varphi), \qquad \int\hat{F}(\theta,\varphi)d\Omega = \hat{1}, \tag{34}$$

and also the condition of dynamical correspondence for the moment $\vec{S}$ and the part of H depending on the intrinsic moment:

$$O_{\hat{F}}(\{S_\alpha,(\vec{M}\vec{\mathcal{H}})\}) = \frac{i}{\hbar}[O_{\hat{F}}(S_\alpha), O_{\hat{F}}((\vec{M}\vec{\mathcal{H}}))]_-. \tag{35a}$$

Putting in (35a) the evident form of $\vec{M}$, the classical Poisson brackets and the reflection (33) we come to the system of equations:

$$\int[\vec{S}\times\vec{\mathcal{H}}]\cdot\hat{F}(\theta,\varphi)d\Omega = \frac{i}{\hbar}\int\vec{S}\cdot(\vec{\mathcal{H}}\vec{S}')\cdot[\hat{F}(\theta,\varphi),\hat{F}(\theta',\varphi')]_- d\Omega d\Omega', \tag{35b}$$

Permutation relations (31) of the algebra $\mathcal{A}_F(1)$ allow us to write down the probability operator in an arranged form, for example:

$$\hat{F} = F_0\cdot\hat{1} + F_1\sigma_1 + F_2\sigma_2 + iF_3\sigma_1\sigma_2, \qquad F_\mu = F^*_\mu = F_\mu(\theta,\varphi). \tag{36a}$$

In this case from (34) we have:

$$F_0 + F_3 \geq 0, \quad F_0^2 - F_3^2 \geq F_1^2 + F_2^2, \quad \int F_0 d\Omega = 1, \quad \int F_\alpha d\Omega = 0. \tag{36b}$$

Introducing the following designations

$$\varphi_{\alpha0} = \int S_\alpha F_0 d\Omega, \qquad \varphi_{\alpha\beta} = \int S_\alpha F_\beta d\Omega, \qquad \alpha,\beta = \overline{1,3}, \tag{36c}$$

putting (36a) in (35b) and taking into account arbitrariness of the magnetic field strength $\vec{\mathcal{H}}$ we get the system of equations

$$\varphi_{\alpha0} = 0, \qquad \hbar\varphi_{\gamma\mu} = 2(\varphi_{\alpha\rho}\varphi_{\beta\lambda} - \varphi_{\alpha\lambda}\varphi_{\beta\rho}), \tag{36d}$$

where $(\alpha\beta\gamma), (\mu\lambda\rho) = (123), (231), (312)$.

It is naturally to demand, as in the previous paragraph, one-to-one correspondence between the classical and the quantum generators, for example in the form:

$$S_1 = s\sin\theta\cos\varphi \rightleftarrows \sigma_1, \qquad S_2 = s\sin\theta\sin\varphi \rightleftarrows \sigma_2 . \tag{37}$$

The replacement $\sigma_1 \rightleftarrows \sigma_2$ in the right-hand sides of (37) is not significant because it is equivalent to the substitution $F_1 \rightleftarrows F_2$, $F_3 \rightleftarrows -F_3$ in (36).

The simplest consequences of the correspondence (37), $\varphi_{12} = \varphi_{13} = \varphi_{21} = \varphi_{23} = 0$, allow us to solve the system of equations (36d):

$$\varphi_{j0} = \varphi_{31} = \varphi_{32} = 0, \qquad \varphi_{11} = \varphi_{22} = -\varphi_{33} = \frac{\hbar}{2} . \tag{38}$$

Using (38), while setting the operator $\hat{S}$ according to the rule (33) with the probability operator defined by (36), we get:

$$\hat S_1 = \frac{\hbar}{2}\sigma_1, \qquad \hat S_2 = \frac{\hbar}{2}\sigma_2, \qquad \hat S_3 = -i\frac{\hbar}{2}\sigma_1\sigma_2 . \tag{39}$$

Relations (39) allow us to go over to new selfconjugate generators of $\mathcal{A}_F(1)$, having the physical sense of the intrinsic moment component operators ($\hat S_1$, $\hat S_2$ and related with them $\hat S_3$, $\hbar\hat S_3 = -2i\hat S_1\hat S_2$) and satisfying the permutation relations

$$[\hat S_\alpha, \hat S_\beta]_- = i\hbar \hat S_\gamma, \qquad \hat S_\alpha^2 = \frac{\hbar^2}{4}\hat 1 , \tag{40}$$

where $(\alpha\beta\gamma) = (123), (231), (312)$.

Thus, the quantization of the C-theory of a particle with an intrinsic moment on the base of a probability operator in the Fermi algebra $\mathcal{A}_F(1)$ with one-to-one correspondence between the classical and the quantum generators bring us to a Q-theory of a particle with spin 1/2.

The other concrete properties of the functions $F_\mu(\theta,\varphi)$ in the probability operator (36a) have to be determined from the set of requirements (36). The simplest solution of the problem is:

$$\hat F(\theta,\varphi) = \frac{1}{4\pi}\left\{\hat 1 + \frac{3\hbar}{2s}\left[\sin\theta\left(\cos\varphi\,\sigma_1 + \sin\varphi\,\sigma_2\right) - i\cos\theta\,\sigma_1\sigma_2\right]\right\} \tag{41}$$

where $s \geq 3\hbar/2$ plays the role of an auxiliary ("subquantum") parameter of the quantization.

9. Nonrelativistical quantum mechanics with a probability operator

Let us consider the mathematical formulation of the Q-theory, arising as the result of the nonrelativistical classical mechanics' quantization based on a probability operator.

Following the generally accepted quantum mechanics we shall admit:

a) All the quantum operators belong to the Bose algebra $\mathcal{A}_B$.

b) There exists a one-to-one correspondence between the classical variables $(q,p) = (q_1,\dots,q_N,p_1,\dots,p_N)$ and the generators $(X,Y) = (X_1,\dots,X_N,Y_1,\dots,Y_N)$ of the algebra $\mathcal{A}_B(N)$.

To simplify the comparison of the considered Q-theory with the generally accepted quantum mechanics we shall choose the usual coordinate representation of the state space $\mathcal{L}$ and of the generators (X,Y), i.e.:

$$\psi = \psi(x) \in \mathcal{L}, \qquad X_j = -i\partial_{x_j}, \qquad Y_j = x_j, \qquad j = \overline{1,N}, \tag{42a}$$

where $x \in R_x$, which is a space isomorphical to R_q.

Then, in accordance with the results of the previous paragraphs, we have to write the coordinate-momentum probability operator as follows:

$$\hat F(q,p,t) = (2\pi\hbar)^{-2N}\int U(\xi,\eta,t)\cdot e^{\frac{i}{\hbar}[\eta(x-q) - \xi(p+i\hbar\nabla_x)]}\, d\xi\, d\eta . \tag{42b}$$

The kernel $U(\xi,\eta,t)$ here is defined (see (30 c)) by a set of auxiliary functions $\{\varphi_\kappa(\xi,t)\}$ ($\xi \in R_\xi$, isomorphical to R_q). The auxiliary functions satisfy the normalization requirement

$$\sum_\kappa \int |\varphi_\kappa(\xi,t)|^2 d\xi = 1, \tag{43}$$

and the set of integral equations, which follow after the substitution of the probability operator (42b) into the requirements of the dynamical (17) and canonical (18) correspondences.

The probability operator (42) and the correspondence rule (14), determined by it, can be written down in different equivalent differential and integral forms. In particular, with the help of the relations (30 c) and the Weyl identity one may rewrite the probability operator (42b) in the regulated differential form:

$$\hat F(q,p,t) = (2\pi\hbar)^{-N}\sum_\kappa \int \varphi^*_\kappa(q-x-\xi,t)\,\varphi_\kappa(q-x,t)\, e^{-\frac{i}{\hbar}\xi(p+i\hbar\nabla_x)}\, d\xi . \tag{44a}$$

From this one obtains

$$\hat{F}(q,p,t)\psi(x,t) = (2\pi\hbar)^{-N}\sum_k \int \varphi_k^*(q-x',t)\varphi_k(q-x,t)e^{\frac{i}{\hbar}(x-x')p}\psi(x',t)dx', \quad (44b)$$

that reflects the action of the probability operator on a vector $\psi(x,t)$ of the quantum state space $\mathcal{L}$.

The correspondence rule (14) with the probability operator (44) defines the quantum operators of all physical variables:

$$\hat{A}(t) = (2\pi\hbar)^{-N}\sum_k \int A(q+x,p,t)\,\varphi_k^*(q-\xi,t)\varphi_k(q,t)e^{-\frac{i}{\hbar}\xi(p+i\hbar\nabla_x)}d\xi dq dp \quad (45)$$

Here, as it was before, $A(q,p,t)$ is the function, representing a physical variable $A \in HU\{A\}_{exp}$ in the initial C-theory.

To finish the formulation of the nonrelativistical quantum mechanics with a probability operator we have to note once more that here, as in the generally accepted quantum mechanics, the state vector evolution satisfies the equation

$$i\hbar\partial_t\psi(x,t) = \hat{H}(t)\psi(x,t) \quad (46)$$

and the value $\langle A\rangle$ of a physical variable A in a state ψ is given by the formula

$$\langle A\rangle_\psi = (\psi/\hat{A}\psi)/(\psi/\psi), \qquad (\psi_1/\psi_2) = \int \psi_1^*(x,t)\psi_2(x,t)dx. \quad (47)$$

Thus, the mathematical formalism of the quantum mechanics with a probability operator differs from that of the generally accepted quantum mechanics only by the dependence of the physical variable operators (45) from some auxiliary functions $\{\varphi_k(\xi,t)\}$ which have no analogues neither in the classical nor in the generally accepted quantum mechanics. Following the works [34-45] we shall call these functions and all the other notions related with them "subquantum".

It shoud be noted that with the help of the construction

$$\varphi(\xi,\eta,t) = (2\pi\hbar)^{-N/2}e^{-\frac{i}{\hbar}\xi\eta}\sum_k \varphi_k(\xi,t)\cdot\tilde{\varphi}_k^*(\eta,t) \quad (48a)$$

of the "subquantum" functions, where

$$\tilde{\varphi}_k(\eta,t) = (2\pi\hbar)^{-N/2}\int \varphi_k(\xi,t)e^{-\frac{i}{\hbar}\xi\eta}d\xi, \quad (48b)$$

the action of the operators (45) on a vector $\psi(x,t) \in \mathcal{L}$ may be written in the integral formulation (equivalent to (45)):

$$\hat{A}(t)\psi(x,t) = (2\pi\hbar)^{-N}\int \varphi(\xi,\eta,t)A(x+\xi,p+\eta,t)e^{\frac{i}{\hbar}(x-x')p}\psi(x',t)dx'dp d\xi d\eta. \quad (48c)$$

The last relations with an arbitrary set of $\{\varphi_k\}$, normalized as (43), coincide with the correspondence rule of the "quantum mechanics with a nonnegative phase-space distribution function" [34], investigated in details in the works [32-45].

It is necessary to underline, that the "subquantum" functions are not arbitrary in the "quantum mechanics with a probability operator". They must satisfy not only the normalization (43) but the whole complex of requirements, which reflect the dynamical and canonical correspondence between the C- and Q-theories. These requirements include the multitude $HU\{A\}_{exp}$ and, consequently, may be written down only for concrete systems.

10. The main theoretical consequences

In this paragraph we shall consider the main features of the "quantum mechanics with a nonnegative phase-space distibution function" [32-45] because the formulated above "quantum mechanics with a probability operator" is its particular case.

Firstly let us note that a change in the number or explicit form of the "subquantum" functions changes the whole set of the operators (45) and, consequently, all the results of the theory. In this sense by saying the "quantum mechanics with a nonnegative phase-space distribution function" we understand an infinite multutude of theories, each of which corresponds to a fixed set $\{\varphi_k(\xi,t)\}$ satistying the normalization (43).

<u>a) Correspondence rule. Quantum-mechanical formalism</u>. A detailed analysis of the correspondence rule (48) is performed in [30,32,34-37]. Here we give only its main properties, which are independent on the explicit form and the number of "subquantum" functions $\{\varphi_k\}$: the standart commutation relations for coordinate and momentum

$$\hat{q}_j(t)\hat{p}_l(t) - \hat{p}_l(t)\hat{q}_j(t) = i\hbar\delta_{jl}, \quad (49a)$$

self-conjugate operators $\hat{A}(t)$ for real $A(q,p,t)$, nonnegative $\hat{A}(t)$ for $A(q,p,t) \geq 0$, differential operators

$$\hat{A}(t) = \int \alpha_0(\xi,t)A_1(\xi+x,t)d\xi + \int \beta_0(\eta,t)A_2(\eta-i\hbar\nabla_x)d\eta \quad (49b)$$

for classical functions of the multitude

$$M_0 = \{A(q,p,t)\}_0 \supset A_1(q,t) + A_2(p,t). \quad (49c)$$

The functions α_o and β_o in (49b), which turn to be very important in the considered theory, are the following constructions of the "subquantum" functions $\{\varphi_\kappa\}$:

$$\alpha_o(\xi,t) = \sum_\kappa |\varphi_\kappa(\xi,t)|^2, \qquad \beta_o(\eta,t) = \sum_\kappa |\tilde{\varphi}_\kappa(\eta,t)|^2. \quad (50)$$

The totality of the quantum operators $\hat{H}(t) \cup \{\hat{A}(t)\}_{exp}$, the evolution equation (46) and the definition of physical variable values (47) represent the quantum-mechanical formalism of the considered theory. This formalism differs from that of the generally accepted quantum mechanics only by the dependence of the quantum operators on "subquantum" functions.

b) Statistical formalism. Interpretation. The substitution of the quantum operators (in the form (45), or (48)) into the relations (47) gives

$$\langle A\rangle_\psi = \langle A\rangle_F = \int A(q,p,t)\, F(q,p,t)\,dq\,dp, \quad (51a)$$

where F is the quantum distribution function, related with the state $\psi(x,t)$ and the "subquantum" functions $\{\varphi_\kappa(\xi,t)\}$ as follows:

$$F(q,p,t) = (2\pi\hbar)^{-N} \sum_\kappa \left| \int \varphi^*_\kappa(q-\xi,t)\, e^{-\frac{i}{\hbar}\xi p}\, \psi(\xi,t)\,d\xi \right|^2 / (\psi/\psi) \geq 0. \quad (51b)$$

It can be easily proved, that the nonnegative distribution (51b) is normalized to one and, as it has to be, coincides with the quantum average (47) of the probability operator (44).

The differentiation of the distribution (51b) with respect to t, taking into account the equation (46) and the relation (45) between $\hat{H}(t)$ and $H(q,p,t)$, leads to the integral equation

$$\partial_t F(q,p,t) = \mathcal{L}\left[H(q,p,t), \{\varphi_\kappa(\xi,t)\}\right] F(q,p,t). \quad (52)$$

Here $\mathcal{L}[H,\{\varphi_\kappa\}]$ is a linear integral operator in the space of real coordinate-momentum functions, which functionally depends on the classical hamiltonian and the "subquantum" functions (fore more details see [34,43]).

The totality of the classical functions $H(q,p,t) \cup \{A(q,p,t)\}_{exp}$, the evolution equation (52) and the definition of physical variable values (51a) represent the statistical formalism of the theory under consideration. This formalism differs from that of the classical statistics only by the dependence of the evolution on the "subquantum" functions.

The statistical formalism gives the only possible [23,34-38, 43,44] interpretation of the considered theory - the distribution (51b) is the joint coordinate-momentum probability density.

The correspondent integrations of F lead to the probability densities of coordinates

$$\alpha(q,t) = \int F(q,p,t)\,dp = \int \alpha_o(\xi,t)\cdot|\psi(q-\xi,t)|^2 d\xi, \quad (53a)$$

an of momenta

$$\beta(p,t) = \int F(q,p,t)\,dq = \int \beta_o(\eta,t)\cdot|\tilde{\psi}(p-\eta,t)|^2 d\eta, \quad (53b)$$

where α_o and β_o are the "subquantum" constructions (50), „$\sim$" means the transformation (48b). The expressions (53) show, that in the "quantum mechanics with a probability operator" $|\psi|^2$ determines the correspondent probability density, but in a general case does not coincide with it.

c). "Subquantum" uncertainties of coordinates and momenta. Writting down the coordinate uncertainties $\langle(\Delta q_j)^2\rangle$ for a state ψ in the statistical formalism with the help of the probability density (53a) or, that is the same, in the quantum-mechanical formalism as the values (47) of the operators (45) for $A(q,p,t) = (q_j - \langle q_j\rangle)^2$ and minimizing them by varying $\psi \in \mathcal{L}$ one can show [34,44] the following restrictions:

$$\sqrt{\langle(\Delta q_j)^2\rangle} \geq \delta q_j = \sqrt{\int\left[\xi_j - \int \xi_j \alpha_o(\xi,t)d\xi\right]^2 \alpha_o(\xi,t)\,d\xi}. \quad (54a)$$

The analogous problem for the momentum uncertainties gives:

$$\sqrt{\langle(\Delta p_j)^2\rangle} \geq \delta p_j = \sqrt{\int\left[\eta_j - \int \eta_j \beta_o(\eta,t)d\eta\right]^2 \beta_o(\eta,t)\,d\eta}. \quad (54b)$$

Thus, in the theory under consideration in a general case there exist no states with a fixed coordinate, as well, as there exist no states with a fixed momentum.

The "subquantum" coordinate uncertainties δq_j and the "subquantum" momentum uncertainties δp_j lead [34,44] to the generalization of the Heisenberg uncertainty relations:

$$\langle(\Delta q_j)^2\rangle\cdot\langle(\Delta p_\ell)^2\rangle \geq \frac{\hbar^2}{4}\delta_{j\ell} + (\delta q_j)^2\cdot(\delta p_\ell)^2. \quad (55)$$

It should be underlined, that the equality in (54a) (in (54b)) is attained here only for states with an eigenvector of operator $\hat{q}_j(t)$ (of operator $\hat{p}_j(t)$).

d) Limiting cases of the "subquantum" functions. The uncertainty relations (54) indicate such limiting cases of the considered theory, when its coincidence with the generally accepted quantum mechanics is maximum. In fact, to allow the quantum states with a fixed coordinate and the quantum states with a fixed momentum it is necessary and sufficient that the constructions (50), which determine the "subquantum" uncertainties δq_j and δp_j in accordance with (54a) and (54b), would have the properties:

$$\alpha_o(\xi,t) = \delta(\xi), \qquad \beta_o(\eta,t) = \delta(\eta). \tag{56}$$

It is important to note that the set $lim\{\varphi_\kappa\}$ of "subquantum" functions, which provides the properties (56), is not unique. The successions of the sets $\{\varphi_\kappa\} \longrightarrow lim\{\varphi_\kappa\}$ are given and investigated in [40].

If some set $lim\{\varphi_\kappa\}$ is chosen, that is the properties (56) of the "subquantum" constructions (50) have place, then:

1. The "subquantum" uncertainties δq_j and δp_j are equal to zero and the restrictions (54) vanish.

2. The operators (45) for physical variables of the multitude M_o (see (49b) and (49c)) coincide with those of the generally accepted quantum mechanics.

3. The evolution equation (46) with hamiltonians from M_o takes the form of the usual Schrödinger equation.

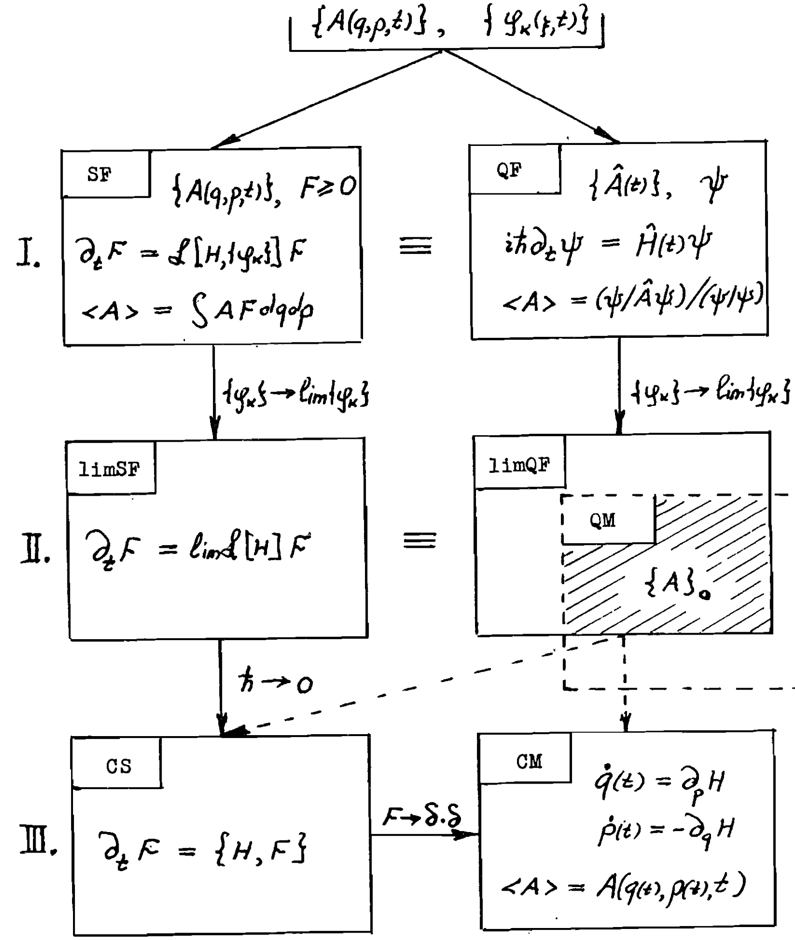
4. The relations (53) define the standard interpretation of a state vector ψ.

5. The restrictions (55) turn to be the usual Heisenberg uncertainty relations.

Thus, the considered theory with any set $lim\{\varphi_\kappa\}$ of "subquantum" functions coincides with the generally accepted quantum mechanics as far as the physical variables of the multitude M_o(49c) are concerned.

e). Limit transitions. Correspondence scheme. Different limiting cases of the considered theory have been investigated in [34 - 44]. The main results are given in the following simplified diagram, named here as "correspondence scheme". The scheme contains three different levels of theories, separated by the concrete content of the relations (54) and (55).

Level I includes the set of theories, written down in the form of quantum-mechanical (QF), or statistical (SF) formalism, each of which is the result of the quantization of the same initial C-theory with the help of some fixed normalized set of "subquantum" functions and has the mentioned above properties.



The limit transition $\{\varphi_\kappa\} \longrightarrow lim\{\varphi_\kappa\}$, leading to a fulfilment of (56), separates from I the subset of theories, which form level II. A theory of level II may be written down either in the quantum-mechanical (limQF), or in the statistical (limSF) formalism. The quantum-mechanical formalism of any theory of level II is close to the generally accepted quantum mechanics (QM). This allows to include the last into the scheme as the dotted parallelogram, partially covering the parallelogram limQF.

The successive limit transition $\hbar \to 0$ in limSF of any theory of level II leads to the classical statistics (CS), because the evolution equation (52) transforms to the classical Liouville equation when $\{\varphi_\kappa\} \longrightarrow \lim\{\varphi_\kappa\}$ and $\hbar \to 0$ [34]. Finally, the choice of the probability density in the form $F(q,p,t) = \delta(q-q(t))\cdot\delta(p-p(t))$ (that is possible only in the theory of level III) brings over to the classical mechanics (CM) with the Hamilton equations for $q(t)$ and $p(t)$.

Classical theories of level III can also be obtained from limQF by the methods (dotted lines in the scheme) of the generally accepted quantum mechanics.

f) Uncertainty problem. In the considered theory, as in any theory which deals with probabilities, any value $<A>$ of a physical variable A must be characterized by some uncertainty $<(\Delta A)^2>$, which has the physical sense of dispersion, i.e. $<(\Delta A)^2>$ is some mean value of the square deviation of A from $<A>$.

In the "quantum mechanics with a nonnegative phase-space distribution function" the mathematical image of a variable A is the quantum operator $\hat{A}(t)$ and, in the same time, the classical function $A(q,p,t)$, related by the probability operator. So, it is logical to write down for the quantum-mechanical formalism -

$$<(\Delta A)^2>^{(Q)} = (\psi/(\hat{A}(t)-<A>_\psi)^2\psi)/(\psi/\psi), \tag{57a}$$

and for the statistical formalism -

$$<(\Delta A)^2>^{(S)} = \int (A(q,p,t)-<A>_F)^2 F(q,p,t)\,dq\,dp\,. \tag{57b}$$

Problems arise from the fact, that the expressions (57a) and (57b) for some A are not equivalent. This follows directly from the correspondence rule (14) with a probability operator, for which, in a general case,

$$O_{\hat F}(A(q,p,t))\cdot O_{\hat F}(A(q,p,t)) \neq O_{\hat F}(A^2(q,p,t))\,. \tag{58}$$

The unequality (58) represents the main feature of the correspondence rules named "non-Neumann". Their consequences have been investigated in details in the work [32].

The choice of definition (57a) gives: the maximal certainty of a value $<A>$ (minimal, or equal to zero dispersion $<(\Delta A)^2>^{(Q)}$) is achieved in the states ψ_α satisfying the eigenvalue problem:

$$\hat{A}(t)\psi_\alpha = O_{\hat F}(A(q,p,t))\psi_\alpha = \alpha\psi_\alpha\,, \qquad \alpha = \alpha^*. \tag{59a}$$

This result for $A = H$ is in accordance with the sense of the stationary states, defined by the equation (46). But the choice of (57a) leads to a partial deterioration of the "consistent probability interpretation" of the theory.

The choice of (57b) conserves the "consistent probability interpretation". The states ψ_d of the maximal certainty of a value $<A>$ in this case satisfy the equation (for more details see [32]):

$$\{O_{\hat F}(A^2(q,p,t)) - 2<A>_F\cdot O_{\hat F}(A(q,p,t))\}\psi_d = \\ = \{d^2 - (<A>_F)^2\}\psi_d\,, \qquad d \geq 0. \tag{59b}$$

But now the dispersion $<(\Delta A)^2>^{(S)}$ in the states ψ_d in a general case is not zero (even is not minimal), i.e. the eigenvalue problem and the stationary states cease to play a fundamental role.

It is quite possible, that the consistent resolution of the uncertainty problem demands that some new definition of $<(\Delta A)^2>$ shoud be proposed. This definition may coincide neither with (57a), nor with (57b). But in any case such a definition must be closely related with the "subquantum" functions, the physical sense of which is not yet clarified.

11. Some concrete applications

In this work we restrict ourselves by a consideration of the simplest physical systems representing, from the point of view of the classical theory, a particle of mass μ in central potential fields. The initial C-theory is then the classical mechanics of a point particle in the three-dimentional space (i.e. $q = \vec{r} = (r_1, r_2, r_3)$ and $p = \vec{p} = (p_1, p_2, p_3)$ with the hamiltonian

$$H(\vec{r},\vec{p}) = \frac{1}{2\mu}\vec{p}^{\,2} + V(|\vec{r}|)\,. \tag{60a}$$

To the set $\{A\}_{exp}$ we include the coordinate, the momentum, the orbital mechanical moment, the kinetic, the potential and the whole energies of the particle, i.e.:

$$\{A(q,p,t)\}_{exp} = (\vec{r}, \vec{p}, \vec{L} = [\vec{r}\times\vec{p}], T = \frac{1}{2\mu}\vec{p}^{\,2}, V = V(|\vec{r}|), E = H). \tag{60b}$$

For the quantization based on a probability operator we choose the Bose algebra $\mathcal{A}_B(3)$ in the concrete representation (42a). Then the coordinate-momentum probability operator takes the form (44).

The "subquantum" functions $\{\varphi_\kappa(\vec{\xi},t)\}$, determining the probability operator, must satisfy the normalization requirement

(43). The probability operator itself must satisfy the dynamical correspondence equations (17) for any variable (60b) (that gives 12 integral equations for $\{\varphi_\kappa\}$) and the canonical correspondence equations (18) for any pair of variable (60b) (that makes 72 equations for $\{\varphi_\kappa\}$). Besides, the set $\{\varphi_\kappa\}$ must provide the invariance of the constructed Q-theory with respect to time translations and to space rotations, because these invariances take place for the considered systems.

After a rather long calculation one may show that all the mentioned above requirements are fulfiled if

$$\varphi_\kappa(\vec{\xi},t) = \varphi_\kappa(|\vec{\xi}|) = \varphi^*_\kappa(|\vec{\xi}|), \qquad \sum_\kappa \int \varphi^2_\kappa(|\vec{\xi}|)d\vec{\xi} = 1. \quad (61)$$

In the "nonrelativistical quantum mechanics with the probability operator (44)", where the "subquantum" functions are defined as (61), the stationary isotropic normalized constructions

$$\alpha_o(|\vec{\xi}|) = \sum_\kappa |\varphi_\kappa(|\vec{\xi}|)|^2, \qquad \beta_o(|\vec{\eta}|) = \sum_\kappa |\tilde{\varphi}_\kappa(|\vec{\eta}|)|^2, \quad (62a)$$

determine, in accordance with (54), the "subquantum" uncertainties of coordinate and momentum:

$$\delta r_j = \delta r = \sqrt{\tfrac{1}{3}\int \xi^2\alpha_o(|\vec{\xi}|)d\vec{\xi}}, \qquad \delta p_j = \delta p = \sqrt{\tfrac{1}{3}\int \eta^2\beta_o(|\vec{\eta}|)d\vec{\eta}}. \quad (62b)$$

Putting the probability operator (44) with the "subquantum" functions (61) into the correspondence rule (14) we have the following quantum operators for the classical functions (60b):

$$\hat{\vec{r}} = \vec{x}, \qquad \hat{\vec{p}} = -i\hbar\nabla_{\vec{x}}, \qquad \hat{\vec{L}} = -i\hbar[\vec{x}\times\nabla_{\vec{x}}], \quad (63a)$$

$$\hat{H} = \hat{T} + \hat{V} = -\frac{\hbar^2}{2\mu}\nabla^2_{\vec{x}} + \frac{3}{2\mu}(\delta p)^2 + \int V(|\vec{x}+\vec{\xi}|)\alpha_o(|\vec{\xi}|)d\vec{\xi}. \quad (63b)$$

The operators (63) act in the vector space $\mathcal{L}$ of states $\psi(\vec{x})$ ($\vec{x}\in R_{\vec{x}}$, isomorphical to $R_{\vec{r}}$).

The evolution equation (46) of the "quantum mechanics with a probability operator" may be reduced, as in the generally accepted quantum mechanics, to the eigenvalue problem of the operator (63b)

$$\hat{H}\psi_{(n)}(\vec{x}) = \langle E\rangle_{(n)}\psi_{(n)}(\vec{x}), \quad (64)$$

which defines the stationary states and the correspondent energy spectrum.

a) A free particle. For $V(|\vec{r}|) \equiv 0$ the equation (64) defines the set of the stationary states

$$\psi_{\vec{P}}(\vec{x}) = \tau^{-1/2} e^{\frac{i}{\hbar}\vec{P}\vec{x}}, \qquad \tau = \int d\vec{x}, \quad (65a)$$

with the energy and the momentum spectrums:

$$\langle E\rangle_{\vec{P}} = \frac{\vec{P}^2}{2\mu} + \frac{3(\delta p)^2}{2\mu}, \qquad \langle\vec{p}\rangle_{\vec{P}} = \vec{P}. \quad (65b)$$

Hence, a free particle has here a "subquantum" energy $E_o = 3(\delta p)^2/2\mu$, that is achieved when $\vec{P} = 0$.

b) An harmonic oscillator. When $V(|\vec{r}|) = \mu\omega^2 r^2/2$, the operator (63b) takes the form:

$$\hat{H} = -\frac{\hbar^2}{2\mu}\nabla^2_{\vec{x}} + \frac{\mu\omega^2\vec{x}^2}{2} + \varepsilon, \qquad \varepsilon = \frac{3(\delta p)^2}{2\mu} + \frac{\mu\omega^2(\delta q)^2}{2}, \quad (66a)$$

which differs from the usual energy operator by the constant ε. That is why, the equation (64) defines the well-known eigenfunctions $\psi_{n_1n_2n_3}(x_1,x_2,x_3)$ of an oscillator with the energy eigenvalues

$$\langle E\rangle_{n_1n_2n_3} = \hbar\omega(n_1+n_2+n_3+\tfrac{3}{2}) + \varepsilon, \qquad n_j = 0,1,\ldots \quad (66b)$$

The "subquantum" energy ε does not effect on the level-differences and is not therefore experimentally observable.

c) An electron in the Coulomb field. When $V(|\vec{r}|) = -Ze^2/|\vec{r}|$, from (63b) we have

$$\hat{H} = -\frac{\hbar^2}{2\mu}\nabla^2_{\vec{x}} + \frac{3(\delta p)^2}{2\mu} - Ze^2 v(x), \quad (67a)$$

where $x = |\vec{x}|$ and $v(x)$ is defined by the construction (62a):

$$v(x) = \frac{4\pi}{x}\int_0^x \xi^2\alpha_o(\xi)d\xi + 4\pi\int_x^\infty \xi\alpha_o(\xi)d\xi. \quad (67b)$$

The eigenvalue problem (64) in this case can be approximately solved prosupposing that "subquantum"uncertainty δr is small. Then, as it follows from (62b), $\alpha_o(|\vec{\xi}|)$ differs essentially from zero only for small $\vec{\xi}$. This allows to evaluate the function (67b): $v(x\to\infty)\to x^{-1}$, $v(0) \approx (\delta r)^{-1}$. Therefore, under small δr the function $v(x)$ can be presented in the form of the sum, containing the undisturbed part x^{-1} and the perturbance $v(x) - x^{-1}$. In such approach the problem (64) defines for 0-order approximation the stationary states $\psi_{nlm}(\vec{x}) = R_{nl}(x)\cdot Y_{lm}(\theta,\varphi)$, known from the correspondent problem of the generally accepted quantum mechanics, and the spectrum of eigenvalues (for more details see [44,45]):

$$\langle E\rangle_{nlm} = -\frac{Ze^2}{2an^2} + \frac{3(\delta p)^2}{2\mu} + \frac{Ze^2}{2an^2}\varepsilon_{nl}\,, \qquad (67c)$$

$$\varepsilon_{nl} = 2\pi\left(\frac{2Z}{n}\right)^{2(l+1)}\frac{C^{2l+1}_{n+l}}{(2l+3)!}\int_0^\infty\left(\frac{\xi}{a}\right)^{2(l+1)}\xi^2 J_0(\xi)\,d\xi\,. \qquad (67d)$$

Here $a = \hbar^2/\mu e^2$ is the Bohr radius, C^m_k are the binomial coefficients, $n = 1,2,\ldots$; $l = 0,1,\ldots,n-1$; $m = -l,\ldots,l$.

The first term in (67a) is the n^2-fold degenerate energy level of a hydron-like atom in the generally accepted quantum mechanics. The second term represents the "subquantum" energy of a free electron (see (65b)) and does not influence the difference between the levels. At last, the third term eliminates the l-degeneration, that changes the level-differences. In particular, we have:

$$\Delta E_{ns,np} = \langle E\rangle_{n00} - \langle E\rangle_{n1m} = \frac{2e^2Z^4}{an^3}\cdot\left(\frac{\delta r}{a}\right)^2. \qquad (67e)$$

The shift (67e) resembles that of Lamb by its dependence on Z and n. This allows to estimate the "subquantum" coordinate uncertainty: $\delta r \approx 4.26\cdot 10^{-12}\ cm$, if (67e) coincides with the Lamb shift, $\delta r < 10^{-14}\ cm$, if the shift (67e) reflects some effect which is beyond the accuracy of the modern experiments.

12. Conclusion

The investigations, the results of which are set forth in the present paper, make it possible to conclude as follows:

1. The elimination of the "incompleteness of the probability interpretation" in quantum theory is possible and may be achieved only by the introducing to the theory of a joint coordinate-momentum probability operator.

2. The probability operator makes it possible to formulate a new method of quantization, that allows to construct a quantum theory, when the initial classical theory is written down in the Hamilton form. The probability operator limitates the arbitrariness in the correspondence rule and by this way gives some restriction to the "incompleteness of the mathematical formalism" of the quantum theory.

3. The quantization, based on a probability operator, leads to a quantum theory which allows to calculate the probabilities to find a system in any infinitesimal volume of the phase-space. That actually corresponds to the point of view of Einstein, de-Broglie, Schrödinger and others, who considered the coordinate and momentum as simultaneously existing physical realities. At the same time, this does not contradict to the Heisenberg uncertainty relation.

4. The quantization, based on a probability operator, leads to the appearence in the quantum theory of some "subquantum" notions, which have no analogues in the generally accepted quantum and classical theories. The "subquantum" notions may influence on the results, which allow an experimental verification. By this reason one may hope on the possibility of some experimental verification of the quantization with a probability operator.

5. The consistent probability interpretation of the quantum theory with a probability operator and the existence of the "subquantum" notions in its mathematical formalism show that such a theory does not pretend to the complete description of the physical reality. That is why it is free from the problems like the Einstein-Podolsky-Rosen paradox.

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Квантование гамильтоновых теорий на основе оператора вероятности

Рассматривается метод квантования с линейным отображением классических функций $A(q,p,t)$ координат, импульсов и времени на квантовые операторы $\hat{A}(t)$ алгебры $\mathcal{A}$ в пространстве квантовых состояний ψ. При этом

$$\hat{A}(t) = \int A(q,p,t)\,\hat{F}(q,p,t)\,dq\,dp \in \mathcal{A},$$

а оператор вероятности $\hat{F}$ обладает свойствами

$$\int \hat{F}(q,p,t)\,dq\,dp = \hat{I}, \quad F_{\psi}(q,p,t) = \langle\psi|\hat{F}(q,p,t)|\psi\rangle \geq 0,$$

и удовлетворяет системе уравнений, отражающих принципы динамического и канонического соответствий между классической и квантовой теориями. Квантование с оператором вероятности приводит к квантовой теории с неотрицательным совместным координатно-импульсным распределением F_{ψ} для любого состояния ψ. Обсуждаются основные следствия квантовой механики с оператором вероятности в сравнении с общепринятыми классической и квантовой теориями. Показано, что квантование с оператором вероятности приводит к появлению в теории новых понятий, называемых в работе "субквантовыми". Следовательно квантовая теория с оператором вероятности не претендует на полное описание физической реальности в терминах классических переменных и по этой причине не содержит проблем типа парадокса Эйнштейна-Подольского-Розена. Приведены результаты ряда конкретных задач: свободная частица, гармонический осциллятор, электрон в кулоновском поле. Эти результаты позволяют надеяться на возможность экспериментальной проверки правомерности квантования на основе оператора вероятности.

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Hamiltonian Theories' Quantization Based on a Probability Operator

The quantization method with a linear reflection of classical coordinate-momentum-time functions $A(q,p,t)$ at quantum operators $\hat{A}(t)$ of an algebra $\mathcal{A}$ in a space of quantum states ψ, based on a probability operator $\hat{F}(q,p,t) \in \mathcal{A}$, is considered. For such a quantization

$$\hat{A}(t) = \int A(q,p,t)\,\hat{F}(q,p,t)\,dq\,dp \in \mathcal{A},$$

where the probability operator $\hat{F}$ has the properties

$$\int \hat{F}(q,p,t)\,dq\,dp = \hat{1}, \quad F_{\psi}(q,p,t) = \langle\psi|\hat{F}(q,p,t)|\psi\rangle \geq 0,$$

and satisfies a system of equations representing the principles of dynamical and canonical correspondences between the classical and quantum theories. The quantization based on a probability operator leads to a quantum theory with a nonnegative joint coordinate-momentum distribution function F_{ψ} for any state ψ. The main consequences of quantum mechanics with a probability operator are discussed in comparison with the generally accepted quantum and classical theories. It is shown that a probability operator leads to an appearance of some new notions called "subquantum". Hence the quantum theory with a probability operator does not pretend to any complete description of physical reality in terms of classical variables and by this reason contains no problems like Einstein-Podolsky-Rosen paradox. The results of some concrete problems are given: a free particle, a harmonic oscillator, an electron in the Coulomb field. These results give hope on the possibility of an experimental verification of the quantization based on a probability operator.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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