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AN APPROXIMATE TWO-SOLITON SOLUTION OF A GENERALIZED SCHRÖDINGER EQUATION IN NUCLEAR HYDRODYNAMICS

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1. INTRODUCTION

An important task of permanent interest in all many-body theories is the discussion of strong (or in principle strong) solvable models, in order to get results which are qualitatively distinguished from those obtained via perturbational methods and which are independent of computer calculations. Examples investigated in detail are the soliton solutions of some nonlinear equations. Their importance has been increased during the last decade in many domains of theoretical physics and applications.

For nuclear physics a well-known example of an analytically solvable model is the time-dependent Hartree theory in one spatial dimension with an attractive (δ-function) contact interaction. In this case the Hartree equations result in a system of coupled nonlinear Schrödinger equations

$$\left(i\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} + g\sum_{b=1}^{N} |\Psi_b(x,t)|^2\right)\Psi_a(x,t) = 0\,, \tag{1}$$

which possess soliton solutions in each component Ψ_a, investigated by Nogami and Warke [1]. Such solutions can simulate an isolated nucleus or nucleus-nucleus scattering. Some models such as the full TDHF-equations are included and possible extensions of the models and methods are considered by the same authors [2]. The scattering problem has also been discussed in [3], see also the review article [4], by use of the two-soliton solution of the nonlinear Schrödinger equation with cubic nonlinearity (S3) [5,6].

The next extension which allows an analytical treatment partly is the inclusion of three-body contact interaction. It means the assumption for the potential being of the form

$$V = -\alpha\delta(x - x') + \beta\delta(x - x')\delta(x - x'')\,, \tag{2}$$

which may be considered as the next step in the hierarchy of the Skyrme-forces [7]. The Hartree approximation of the Schrödinger equation for a one-particle wave function and (n+1)- and (2n+1)-body interactions have been investigated in [8].

For the form (2) of the interaction the coupled Hartree equations have no exact analytical solutions. But the hydrodynamic approximation results in an extended nonlinear Schrödinger equation with fifth order of nonlinearity (S5) [8,9].

$$i\hbar\frac{\partial u}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial x^2}+\alpha|u|^2u-\beta|u|^4u=0, \qquad (3)$$

or, after a convenient choice of time and length units,

$$iu_t+u_{xx}+2|u|^2u-3\delta^2|u|^4u=0 \qquad (4)$$

with $\delta^2=2\beta/3\alpha$. (the lower indices t and x denote the partial derivative). For an attractive two-body and a repulsive three-body interaction is $\alpha>0$ and $\beta>0$, so that δ^2 describes the relation of three- to two-body forces. A precise statement about the relative magnitude of realistic two- and three-body forces depends on the model, but all considerations result in the conclusion the latter are small (e.g., /4,8,10/). The eq.(3) is mentioned by a lot of authors /2,8,9,11,12/. For nuclear physics the consequences of this equation have extensively been discussed by Kartavenko /9/.

Equation (4) possesses an exact one-soliton solution (section 2). An exact two-soliton solution of this equation does not exist but we want to discuss in this letter the pattern of a nucleus-nucleus collision by use of an approximate two-soliton solution. The construction of the N-soliton solution of the eq. S3 via Bäcklund transformations is referred to in section 3. In section 4 the ansatz is proposed for a two-soliton solution of the S5 eq. via Bäcklund transformations. Further the consequences of this solution as a model for (elastic) nucleus-nucleus scattering are considered.

2. THE ONE-SOLITON SOLUTION OF S5

The extended nonlinear Schrödinger equation (4) possesses exact one-soliton solutions having (in the c.m. system, i.e., the zero is chosen by $x_0=0$, $k_1=k_2=k$ and $v_1=-v_2=v$) the analytical expression

$$u^{(1)}_{1/2}=\frac{\sqrt{2}\,k\exp(i(\pm vx+(k^2-v^2)t))}{\sqrt{D\,\mathrm{ch}(2k(x\mp 2vt))+1}} \qquad (5)$$

with $D=\sqrt{1-4k^2\delta^2}$, $0\le D\le 1$. A constant phase ϕ_0 is dropped. The upper index (1) stands for one-soliton solution, and the lower indices stand for the motion in +x or -x direction respectively. Firstly, solution (5) was remarked in /2/ and found by many authors /8,9,11/.

Solutions $u^{(1)}_{1/2}$ have the properties:

1. $u^{(1)}_{1/2}$ for $\delta\to 0\,(D\to 1)$ pass over to the well-known one-soliton solution of the cubic Schrödinger equation

$$q^{(1)}_{1/2}=\frac{k\exp(i(\pm vx+(k^2-v^2)t))}{\mathrm{ch}(k(x\mp 2vt))}.$$

2. The asymptotic behaviour with respect to the space coordinate is $|u^{(1)}|\to 0$ and $|u^{(1)}_x|\to 0$ if $x\to\pm\infty$.

3. For the asymptotic case $t\to\pm\infty$ the linear superposition $(u^{(1)}_1+u^{(1)}_2)$ is solution of eq. (4) with an error $\exp(-2|t|)$ as in the S3 case.

The one-soliton solutions are nonperturbative. They are envelope-solitons.

It can be proved by the well-known methods from the soliton theory that a U-V pair introduced by Zakharov and Shabat /5/ does not exist. So we have to deal with the so-called quasisolitons, as has been pointed out by Machankov in the review article /12/.

3. THE CONSTRUCTION OF THE N-SOLITON SOLUTION OF EQ. S3

In the literature a lot of methods are offered for the construction of N-soliton solutions. Here is the inverse scattering theory of Zakharov and Shabat /5/ or the so-called direct method of Hirota /13/ ought to be mentioned. The approach of Kay and Moses /14/ is used in /1/ for the construction of solutions of the coupled S3 eqs. (1). The most fashionable method to express a N-soliton solution by the one-soliton solution via Bäcklund transformations for the whole AKNS-class was published by Meinl and Neugebauer /15/. As an example we present, follwing /15/, the two-soliton solution of eq. S3: $iq_t+q_{xx}+2|q|^2q=0$, namely:

$$q^{(2)}=-2i\,\frac{\begin{vmatrix}1 & \frac{1}{\alpha_1} & \lambda_1 & \lambda_1^2\\ 1 & -\alpha_1^* & \lambda_1^* & \lambda_1^{*2}\\ 1 & \frac{1}{\alpha_2} & \lambda_2 & \lambda_2^2\\ 1 & -\alpha_2^* & \lambda_2^* & \lambda_2^{*2}\end{vmatrix}}{\begin{vmatrix}1 & \frac{1}{\alpha_1} & \lambda_1 & \frac{\lambda_1}{\alpha_1}\\ 1 & -\alpha_1^* & \lambda_1^* & -\lambda_1^*\alpha_1^*\\ 1 & \frac{1}{\alpha_2} & \lambda_2 & \frac{\lambda_2}{\alpha_2}\\ 1 & -\alpha_2^* & \lambda_2^* & -\lambda_2^*\alpha_2^*\end{vmatrix}} \qquad /6/$$

$$\alpha_j = \frac{k_j}{q_j^*}\left(1 \pm \sqrt{1 - \frac{|q_j|^2}{k_j^2}}\right), \quad q_j \equiv q^{(1)}_{1/2}, \quad \lambda_j = -\frac{v_j}{2} + i\frac{k_j}{2}. \tag{7}$$

Thereby $q_j^{(1)}$ $(j = 1, 2)$ is the one-soliton solution of direction j; k_j and v_j are the corresponding amplitudes and velocities, respectively; the λ_j are the spectral parameters of the appropriate Riemann-Hilbert problem; star denotes the complex conjugate. The function $q^{(2)}$ is the well-known exact two-soliton solution [5,6], built from the two different one-soliton solutions $q_j^{(1)}$.

4. THE APPROXIMATE TWO-SOLITON SOLUTION OF THE S5 EQ.

In sec. 3 it was remarked that the S5 eq. is not completely integrable in the sense of the strong soliton theory in contradiction to the S3 eq. For the reason of smallness of the parameter δ^2 the S5 eq. belongs to the class of nearly integrable nonlinear systems. The physical and computational problems which are connected with many nearly integrable soliton equations has been treated by Machankov [12]. Thereby the approximation of zeroth order is a completely integrable (soliton) equation and the investigation concerns the perturbation of it. If the perturbation can be parametrized by a small parameter the solution is "easy" by approximate methods. But for nonlinear equations perturbation methods are not unique, and the solutions are not necessarily consistent with the original equation. Because of the nonexistence of standardized methods in this letter the use of the Bäcklund transformation, which in the case of the "closed relatives" S3 gives the exact solution, is offered for construction of an approximate two-soliton solution of the S5 eq. Retaining in the mind that the S5 eq. has two essential properties, (i) existence of exact one-soliton solutions, (ii) for the small parameter $\delta^2 \to 0$ the S5 eq. passes over in the S3 eq., for which N-soliton solutions (inclusive a construction principle, see sec. 3) are known. So, we are able to choose an ansatz for the two-soliton solution of the S5 eq. by application of the Bäcklund transformation of the S3 eq. to the one-soliton solution of the S5 eq. In order to realize this ansatz one has to replace in (7) the q_i by the $u_i^{(1)}$ from the S5 eq. This gives the following expression for the new auxiliary functions α_i (with $z_i = k_i(x - 2v_i t)$)

$$\alpha_i = \frac{k_i}{u_i^{(1)*}}\left(1 \pm \sqrt{1 - \frac{2}{D\,\mathrm{ch}(2z_i) + 1}}\right).$$

It does not exist for $D\,\mathrm{ch}(2z_i) < 1$, a small vicinity of the peaks of the soliton at $z_i \approx 0$. Outside this area the resulting

solution from (6): $u^{(2)}$ is in a very good agreement with the S5 eq. which is proved by simple computer calculations. The following change in the auxiliary functions, then denoted by β_i, may avoid these difficulties at $z_i \approx 0$:

$$\beta_i = \frac{k_i}{u_i^{(1)*}}\left(1 \pm \sqrt{1 - \frac{1 + D}{D\,\mathrm{ch}(2z_i) + 1}}\right),$$

or

$$\beta_i = \frac{k_i}{u_i^{(1)*}}\left(1 + \frac{\sqrt{2D}\,\mathrm{sh}(z_i)}{\sqrt{D\,\mathrm{ch}(2z_i) + 1}}\right).$$

Since $D \le 1$ this substitution makes the solution $u^{(2)}$ well defined and gives for all z_i an allowed procedure for construction of an approximate solution. Practically, the function $u^{(2)}$ is not changed outside the "tops" about $z_i \approx 0$. It is proved that all other reasonable variations of the functions α_i give results being in a worse agreement with the S5 eq. Unfortunately the analytical expression by substituting the function $u^{(2)}$ in the S5 eq. becomes too large, but a simple numerical proof leads to the following properties. Defining the error $R(\delta^2)$ and the quality Q of the solution of the S5 eq. (besides the question about stability) as

$$iu_t^{(2)} + u_{xx}^{(2)} + 2|u^{(2)}|^2 u^{(2)} - 3\delta^2 |u^{(2)}|^4 u^{(2)} = 0 + R(\delta^2) \tag{8}$$

and

$$Q = \frac{|R(\delta^2)|}{|u^{(2)}|}, \qquad 0 \le Q < \infty \tag{9}$$

(for $\delta \to 0$ or for an exact solution $Q = 0$ is valid.) it is found that: 1. For all z_i the relation $Q \ll 1$ is worth, with exception of $D\,\mathrm{ch}(2z_i) < 1$, where $Q \approx 1$ is valid, and 2. $Q \ll 1$ is worth for all reasonable values of k_i, v_i, δ^2. The latter statement is a hint that $u^{(2)}$ is a stable solution. For all values of $(x - 2v_i t)$ the quality Q of $u^{(2)}$ is much better (i.m. Q are smaller) than the Q-values from other combinations of $u_i^{(1)}$ with soliton-like asymptotics.

The mean result considering two-soliton solutions as models for a (elastic) scattering process is the time delay, which, following [3], is defined by the asymptotic behaviour

$$\begin{aligned} t \to -\infty: &\quad u^{(2)} = u_1^{(1)}(x, t) + u_2^{(2)}(x, t) \\ t \to +\infty: &\quad u^{(2)} = u_1^{(1)}(x, t - \Delta t) + u_2^{(1)}(x, t - \Delta t) \end{aligned} \tag{10}$$

with

$$\Delta t = -\frac{1}{2vk}\ln\frac{v^2+k^2}{D^2v^2}. \tag{11}$$

For $D = 1$ the time delay (or the phase of the elastic scattering) passes over in the result of the S3 treatment /3,6/.

5. CONCLUSIONS

The solution $u^{(2)}$ of the S5 eq. (4), constructed via Bäcklund transformations, has all properties of an approximate two-soliton and is able to describe elastic nucleus-nucleus collisions (without energy-transfer) in the hydrodynamic approximation.

This example shows that it would be reasonable to start a full perturbation procedure for numerical construction of a "scattering solution" of S5 with the use of a Bäcklund transformation in the first step.

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Функе Х. E4-86-40

Приближенное двухсолитонное решение обобщенного уравнения Шредингера в ядерной гидродинамике

Предлагается процедура построения приближенного двухсолитонного решения нелинейного уравнения Шредингера с нелинейностью $\alpha|u|^2 u - \beta|u|^4 u$ с помощью преобразования Бэклунда, известного из теории кубического уравнения Шредингера. Показано, что полученное таким образом решение имеет минимальную погрешность и может быть использовано при описании упругого ядро-ядерного рассеяния в гидродинамическом приближении.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1986

Funke H. E4-86-40

An Approximate Two-Soliton Solution of a Generalized Schrödinger Equation in Nuclear Hydrodynamics

The construction of an approximate two-soliton solution of a nonlinear Schrödinger equation with nonlinearity proportional to $\alpha|u|^2 u - \beta|u|^4 u$ via Bäcklund transformation known from the cubic Schrödinger equation is proposed. It is shown that this solution has a minimal error and may be used for description of elastic nucleus-nucleus scattering in the hydrodynamic limit.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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