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COMPLETELY INTEGRABLE CASE IN THE THREE-PARTICLE PROBLEMS WITH HOMOGENEOUS POTENTIALS

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Introduction

In this paper we intend to generalise and develop the results obtained in ref.[1] for the problem of three classical particles with arbitrary masses $m_{1,2,3}$, that move along a fixed straight line OX. Consider the systems with the total energy

$$E=(1/2)\sum_{k=1}^{3} m_k \dot{x}_k^2 + V(x_1,x_2,x_3) \ , \tag{1}$$

where the potential $V(x_1,x_2,x_3)=V(x_1-x_2,x_2-x_3,x_3-x_1)$ is the homogeneous coordinate function

$$V(sx_1,sx_2,sx_3)=s^n V(x_1,x_2,x_3) \ . \tag{2}$$

In the center of mass system we apply a modification of the hyperspherical Delves coordinates[2,3] $\rho\in[0,\infty)$, $\varphi\in[0,2\pi)$. In them $x_k=c_k(s_1s_2s_3)^{-1/2}\rho_k\cos(\varphi-\varphi_{ij,k})$, where $c_k=\cos\Psi_k$; $s_k=\sin\Psi_k$, $\Psi_k=\mathrm{arctg}(m_k/\mu)$, $\Psi_k\in[0,2\pi]$; $\mu=\sqrt{m_1m_2m_3/(m_1+m_2+m_3)}$ is the reduced mass, $\varphi_{ij,k}$ are the angles of pair collisions and the Hamiltonian is

$$H=(1/2\mu)(p_\rho^2+\rho^{-2}p_\varphi^2)+g\alpha(\varphi)\rho^n \ . \tag{3}$$

Here g is the dimensional constant and the form of the dimensionless function $\alpha(\varphi)=\alpha(\varphi+2\pi)$ is defined by the potential. For instance, in the two-particle interaction with $V_{ij}\sim|x_i-x_j|^n$ from $V=V_{12}+V_{23}+V_{31}$ we get $\alpha(\varphi)=\sum_{k=1}^{3}\alpha_k|\sin(\varphi-\varphi_{ij,k})|^n$, where α_k are some constants; for the three-particle potential $V=V(I)$, where $I=\mu\rho^2$ is the system moment of inertia, we have $V=g\rho^n$ (at $n\neq0$) or $V=g\ln(\rho/\rho_0)$ (at $n=0$) and $\alpha=1$, etc.

1. The trajectory equation for the three-particle system

In the coordinates we have chosen, one can easily reduce the Hamilton system of equations to one differential equation of the second order for the system trajectory given by the function $\rho(\varphi)=\mathcal{R}\xi(\varphi)$, where $\mathcal{R}$ is the length scale. By analogy with the reduction in the problem with $V_{ij}\sim|x_i-x_j|^{-1}$ [4] we can get

$$(\xi'/\xi)'-(1+\xi'^2/\xi^2)[\omega-(\alpha'/2\alpha)(\xi'/\xi)-\varepsilon/\alpha\xi^{2\omega-2}](1-\varepsilon/\alpha\xi^{2\omega-2})^{-1}=0 \ , \tag{4}$$

where $\varepsilon = E/\mathcal{R}^n g$ is the dimensionless total energy, $\omega = 1+n/2$ and the prime denotes a derivative with respect to φ.

The law of motion is obtained by quadrature

$$t - t_0 = \int_{\varphi_0}^{\varphi} d\varphi [(\mu/2)(\rho'^2 + \rho^2)(E - g\alpha\rho^n)^{-1}]^{1/2} .$$

Suffice it to solve the problem at $\varepsilon = 0$ and $\varepsilon = 1$ since the following proposition holds:

Proposition 1: Transformations $x \to sx$ and $t \to s^{\omega} t$ transform a solution of the problem with the total energy $E \neq 0$ into the solution with the total energy $s^n E$.

Proof is based on the properties of the total energy (1) and the Hamilton equations at the described scale transformations and is easily carried out by using (2).

2. Completely integrable case

At $\varepsilon = 0$ the classical three-particle problem with the Newton or Coulomb two-particle interaction can be considered as a completely integrable one in a generalised sense[1]. Let us show that this takes place for potentials (2) at any n.

Proposition 2: At $\varepsilon = 0$ eq.(4) is reduced to the first order differential equation.

Indeed, at $\varepsilon = 0$ substitutions $\vartheta = \rho'/\rho$ and $\gamma = \alpha'/2\alpha$ transform eq.(4) into the first order Abel equation[5]

$$\vartheta' - (1+\vartheta^2)(\omega - \gamma\vartheta) = 0 . \tag{5}$$

At the initial conditions ζ_0, φ_0 the trajectory $\zeta(\varphi, \varphi_0)$ is expressed through the solution $\vartheta(\varphi, \varphi_0)$ of eq. (5) in the form

$$\zeta(\varphi, \varphi_0) = \zeta_0 \exp\Big[\int_{\varphi_0}^{\varphi} d\varphi\, \vartheta(\varphi, \varphi_0)\Big] . \tag{6}$$

Proposition 3: At $\varepsilon = 0$ there exists an additional first integral of the system, that does not depend explicitly on time

$$\varphi_0 = \Phi(\rho p_\rho p_\varphi^{-1}, \varphi) . \tag{7}$$

Indeed, the function $\varphi_0 = \Phi(\vartheta, \varphi)$ inverse in φ_0 for the solution $\vartheta = \vartheta(\varphi, \varphi_0)$ of eq.(5) satisfies the equation $\partial_\varphi \Phi + (1+\vartheta^2)(\omega - \gamma\vartheta)\partial_\vartheta \Phi = 0$ used for calculating Poisson bracket

$$[\Phi, H] = p_\varphi^{-1}(2\gamma\rho p_\rho p_\varphi^{-1} - n) H .$$

It is seen that at $H = E = 0$ we have $[\Phi, H] = 0$, i.e. by the Birkhoff terminology[6] Φ is the conditional first integral of the problem. Its value may be due to the energy distribution between particles.

Proposition 4: At $\varepsilon = 0$ the variables in the Hamilton-Jacobi equation

$$(1/2\mu)[(\partial_\rho W)^2 + \rho^{-2}(\partial_\varphi W)^2] + g\alpha\rho^n = E \tag{8}$$

are separated.

Indeed at $E = 0$ the ansatz

$$W(\rho, \varphi) = \sqrt{2\mu|g|}\, \rho^{\omega} f(\varphi) \tag{9}$$

transforms eq.(8) into that on $f(\varphi)$:

$$f'^2 + \omega^2 f^2 = \pm\alpha(\varphi) , \tag{10}$$

where $\pm = \operatorname{sign} g$. One can easily verify that the relations $\vartheta = f(\omega^{-2}\alpha - f^2)^{-1/2}$ and $f = \omega^{-1}\vartheta\alpha^{1/2}(1+\vartheta^2)^{-1/2}$ define the relation between the solutions of eqs. (5) and (10).

By using the solution $f(\varphi, \varphi_0)$ of eq.(10), one can obtain the trajectory $\zeta(\varphi, \varphi_0)$ from the relation $\partial_{\varphi_0} W = \mathrm{const} = \sqrt{2\mu|g|}\, \mathcal{R}^{\omega} \zeta_0^{\omega} \partial_{\varphi_0} f(\varphi, \varphi_0)$ in the form

$$\zeta(\varphi, \varphi_0) = \zeta_0 [\partial_{\varphi_0} f(\varphi, \varphi_0) / \partial_{\varphi_0} f(\varphi_0, \varphi_0)]^{-1/\omega} . \tag{11}$$

Comparing of (11) with (7) provides one more relation between ϑ and f

$$\partial_{\varphi_0} f(\varphi, \varphi_0) = \partial_{\varphi_0} f(\varphi_0, \varphi_0) \exp\Big[-\omega \int_{\varphi_0}^{\varphi} d\varphi\, \vartheta(\varphi, \varphi_0)\Big] .$$

The above separation of variables in the Hamilton-Jacobi equation is unusual. It is related with factorization of the action itself rather than with factorization of the exponent of W.

Proposition 5: Invariant manifold of solutions with $\varepsilon = 0$ is diffeomorphic to a two-dimensional torus.

Indeed, substitution of $\vartheta = tg\,\chi$ transforms (5) into an equation on torus[7,8] $\mathbb{T}^{(2)}_{\chi\varphi} = \mathbb{S}^{(1)}_{\chi} \times \mathbb{S}^{(1)}_{\varphi}$ $(\mathbb{S}^{(1)} = \mathbb{R}^{(1)}/\mathrm{mod}(2\pi))$:

$$\chi' = \omega - \gamma(\varphi)\,tg\,\chi \,. \qquad (12)$$

3. Form of the function $\alpha(\varphi)$ and its analytical continuation

If V depends on the distances $|x_i - x_j|$, the function $\alpha(\varphi)$ can be continued to a holomophic function on a certain Riemann surface on the basis of the values of $\alpha(\varphi)$ in an interval,where $x_i - x_j$ does not change its sign. Since the uniqueness points $x_i - x_j = 0$ are not the points of branching of the function $|x_i - x_j| = \sqrt{(x_i - x_j)^2}$, we get the Riemann surface between the sheets of which there are no transitions when going around these points. Therefore, we can limit our consideration to only one sheet. For simplicity we assume it to be diffeomorphic the $\mathbb{C}^{(1)}_{\varphi}$ plane and an analytically continued function $\alpha(\varphi)$ to be moremorphic on it

$$\alpha(\varphi) = a(\varphi)/b(\varphi)\,, \qquad (13)$$

where $a(\varphi) = a(\varphi + 2\pi)$ and $b(\varphi) = b(\varphi + 2\pi)$ are an entire functions on $\mathbb{C}^{(1)}_{\varphi}$. These requirements are fulfilled in the most interesting physical problems with $n \in \mathbb{Z}$.

Under the assumptions accepted the fixed singular points of eq. (12) are $\{\chi_K = K\pi\,;\, \varphi = \varphi_0\,, \varphi_\infty\}$ and $\{\chi_K = (K+1/2)\pi\,;\, \varphi = \varphi_0'\}$, where $K \in \mathbb{Z}$; φ_0 and φ_0' are zeroes of α and α', respectively, and φ_∞ are the poles of $\alpha(\varphi)$. Introduce also the notation $\Delta(\varphi_1^s, \varphi_2^s)$ for an open interval on $\mathbb{R}^{(1)}_{\varphi}$ limited by the coordinates φ_1^s and φ_2^s of two singular points with adjacent projections onto $\mathbb{R}^{(1)}_{\varphi}$. On $\Delta(\varphi_1^s, \varphi_2^s)$ $\alpha(\varphi)$ has a uniquely defined real-valued inverse function $\varphi = \varphi(\alpha)$. Since $\gamma(\varphi)$ is independent of sign $\alpha(\varphi)$, further in studying the solutions of eq.(12) without loss of generality we shall assume that $\alpha(\varphi) > 0$ for $\varphi \in \Delta(\varphi_1^s, \varphi_2^s)$.

4. Trajectory on the torus $\mathbb{T}^{(2)}_{\chi\varphi}$ and the form of an additional integral et $\varepsilon = 0$.

Proposition 6: Under the assumption (13) on the form of the function $\alpha(\varphi)$, one can describe the system trajectories on $\mathbb{T}^{(2)}_{\chi\varphi}$ in the parametric form

$$\begin{pmatrix} \chi(\Delta\tau;\chi_0,\varphi_0) \\ \varphi(\Delta\tau;\chi_0,\varphi_0) \end{pmatrix} = exp\left[\Delta\tau\, \vec{X}_\omega(\chi_0,\varphi_0)\right] \begin{pmatrix} \chi_0 \\ \varphi_0 \end{pmatrix}, \qquad (14)$$

where $\{\chi_0, \varphi_0, \tau_0\}$ is the initial condition and $\vec{X}_\omega(\chi,\varphi)$ is an autonomous analytical vector field

$$\vec{X}_\omega(\chi,\varphi) = \vec{X}_0 + \omega\vec{Y} = [2\omega ab\cos\chi - (a'b - b'a)\sin\chi]\partial_\chi + 2ab\cos\chi\,\partial_\varphi\,. \qquad (15)$$

The Teyler series (14) has a nonzero convergence radius in $\Delta\tau = \tau - \tau_0$ in the vicinity of any initial condition $\{\chi_0, \varphi_0, \tau_0\} \in \mathbb{C}^{(3)}_{\chi\varphi\tau}$.

The proof is based on a direct application of the Cauchy theorem and its generalizations[6-10] to an analytical system of equations

$$\frac{d}{d\tau}\begin{pmatrix} \chi \\ \varphi \end{pmatrix} = \vec{X}_\omega(\chi,\varphi)\begin{pmatrix} \chi \\ \varphi \end{pmatrix}, \qquad (16)$$

that is used for studying eq.(12).

Corollary: Analytical dependence of $\vec{X}_\omega$ on ω leads to a holomorphic nature of solutions with respect to ω in the vicinity of point $\omega = 0$. Then, according to the Poincare[6,10] at small ω one can search for solutions with respect to perturbation theory.

The case $\omega = 0$, i.e.,the problem with potentials (2) at $n = -2$ is exactly solved at any $\alpha(\varphi)$. The solution is defined by the integral $I_0(\chi,\varphi)$ in the form

$$I_0(\chi,\varphi) = \sqrt{\alpha(\varphi)}\,\sin\chi = const\,. \qquad (17)$$

This allows one to construct an original perturbation theory for the potentials of class (2) keeping unchanged the initial function $\alpha(\varphi)$ (that may depend on n) and considering ω as an independent parameter in eqs. (4), (5), (10), (12) or (16). In this approach we shall get an exactly solvable zero approximation with $\omega = 0$ minimally changing the initial problem. In particular, the properties of two-particle interactions at a fixed moment of inertia $I = \mu\rho^2$ are conserved. Then, Hamiltonian (3) is written down in the form

$H = H_0 + \Delta H_\omega$, where $\Delta H_\omega = \omega g \alpha (I/\mu)^{-1} \ln(I/\mu) \sum_{k=0}^{\infty} [\omega \ln(I/\mu)]^k/(k+1)!$, i.e., logarithmic terms arise in it, that have first been discovered by Fock[11] in the expansion of the wave function in the quantum three-particle problem.

Definition: Series in powers ω obtained in the above described way are called the ω-expansions of solutions of the three-particle problem.

Proposition 7: Using the variable I_0 from (17) one can write down the ω-expansion of the integral $I_\omega(\chi,\varphi)$ from (12) in the form

$$I_\omega(\chi,\varphi) = \sum_{k=0}^{\infty} (-\omega)^k \int_{\tilde{\varphi}}^{\varphi} d\varphi_k \cdots \int_{\tilde{\varphi}}^{\varphi} d\varphi_1 \Big[\prod_{\ell=1}^{k} \sqrt{\alpha(\varphi_\ell) - I_0^2}\, \partial_{I_0} \Big] I_0 \,. \tag{18}$$

This series absolutely converges in the interval $[\tilde{\varphi}, \tilde{\tilde{\varphi}}] \subset \Delta(\varphi_1^s, \varphi_2^s)$ at $0 \le I_0^2 < m$ if

$$|\omega| \Delta\varphi \sqrt{M/m} \le (4/\sqrt{3}) \Big[\operatorname{arctg} \sqrt{3(m - \sqrt{m - I_0^2})/(m + \sqrt{m - I_0^2})} - \pi/3 \Big] + \arccos(|I_0|/\sqrt{m}) \,, \tag{19}$$

where $0 < m = \min\limits_{\varphi \in [\tilde{\varphi}, \tilde{\tilde{\varphi}}]} \alpha(\varphi)$ and $M = \max\limits_{\varphi \in [\tilde{\varphi}, \tilde{\tilde{\varphi}}]} \alpha(\varphi) < \infty$.

Indeed, the solution of the equation

$$\partial_\varphi J_\omega + \omega \sqrt{\alpha(\varphi) - I_0^2}\, \partial_{I_0} J_\omega = 0$$

for $J_\omega(I_0,\varphi) = I_\omega(\chi,\varphi)$ under the initial condition $J_{\omega=0} \equiv I_0$ is easily obtained as a formal series (18). To estimate its convergence, we should like to note that for the integrand Π_k in (18) we have $|\Pi_k| \le M^{k/2} |Q_k|$. Under the condition $0 \le I_0^2 < m$ one can easily obtain that

$$|Q_k| = \Big| \Big[\prod_{\ell=1}^{k} \sqrt{1 - I_0^2/\alpha(\varphi_\ell)}\, \partial_{I_0} \Big] I_0 \Big| \le m^{(1-k)/2} \big\{ [(2 - \sqrt{1 - x^2})\partial_x]^k x \big\} \,,$$

where $0 \le x = |I_0|/\sqrt{m} < 1$. Then, the substitution $x = \sin\psi(y)$, where the function $\psi(y)$ holomorphic at zero is determined from $y = (4/\sqrt{3}) \operatorname{arctg}[\sqrt{3}\, \operatorname{tg}(\psi/2)] - \psi$, leads to

$$|I_\omega| \le \sqrt{m} \sum_{k=0}^{\infty} (1/k!) (|\omega| \Delta\varphi \sqrt{M/m}\, \partial_y)^k \sin\psi(y) = \sqrt{m} \sin\psi(y + |\omega| \Delta\varphi \sqrt{M/m}) \le \sqrt{m} \,.$$

The convergence radius of the last series is defined by the critical point $\psi_c = \pi/2$, nearest to $\psi = 0$, of the function $y(\psi)$ which provides (19).

Let us construct also a nonperturbative representation of the integral $I_\omega(\chi,\varphi)$ of eq.(12) as the Fourier series. For this purpose the following proposition is useful:

Proposition 8: Determination of the integral $I_\omega(\chi,\varphi)$ is reduced to the solution of the degenerate Beltrami equation

$$\exp[i\omega\varphi(z\bar{z})]\partial_z F + \exp[-i\omega\varphi(z\bar{z})]\partial_{\bar{z}} F = 0 \tag{20}$$

on the function $F(z,\bar{z})$ mapping the plane $\mathbb{C}_z^{(1)}$ into a certain line in $\mathbb{C}_F^{(1)}$. In eq. (20) $\varphi = \varphi(z\bar{z}) = \overline{\varphi(z\bar{z})}$, where $\varphi(\alpha)$ is the inverse function for $\alpha(\varphi)$.

The proof is based on the change of variables

$$z = \sqrt{\alpha(\varphi)} \exp[i(\omega\varphi - \chi)] \,; \quad \alpha(\varphi) = z\bar{z} \,, \quad \chi = \omega\varphi + (1/2i)\ln(\bar{z}/z) \,, \tag{21}$$

under which the region $\{\chi,\varphi\} \in [0,2\pi) \times \Delta(\varphi_1^s, \varphi_2^s)$ is mapped onto the segment $0 \le \alpha(\varphi_0) = \alpha_0 < |z| < \infty$ of the plane $\mathbb{C}_z^{(1)}$. The change of variables transforms eq. (12) into $\exp(i\omega\varphi)d\bar{z} - \exp(-i\omega\varphi)dz = 0$ whose integral $F(z,\bar{z}) = \mathrm{const}$ satisfies obviously eq. (20). Due to the real-valued nature of $\varphi(z\bar{z})$ eq.(20) leads to the zero Jacobian $J = |\partial_z F|^2 - |\partial_{\bar{z}} F|^2 = 0$ of the mapping $F(z,\bar{z})$ $\mathbb{C}_z^{(1)} \to \mathbb{C}_F^{(1)}$.

Corollary: As a result of degeneracy, the real and **imaginary** parts of the function $F(z,\bar{z})$ satisfy separately eq. (20).

Eq.(20) is reduced to the integral Tricomi equation[12,13]

$$\hat{U} G(z,\bar{z}) \equiv \iint (\zeta - z)^{-2} \exp\{i\omega[\varphi(\zeta\bar{\zeta}) + \varphi(z\bar{z})]\} G(\zeta,\bar{\zeta})\, d\zeta \wedge d\bar{\zeta}/2\pi i = -G(z,\bar{z}) \,.$$

As a result of degeneracy, the operator $\hat{U}$ is unitary in $L^{(2)}(\mathbb{C}_z^{(4)})$ and has no compressing properties. Therefore, by the method described in refs.[12,13] one cannot find its eigenfunction $G(z,\bar{z})$ that corresponds to the eigenvalue-1 and allows one to express $F(z,\bar{z})$ through

$$F(z,\bar{z}) = \iint (\zeta - z)^{-1} \exp[i\omega\varphi(\zeta\bar{\zeta})] G(\zeta,\bar{\zeta})\, d\zeta \wedge d\bar{\zeta}/2\pi i \,.$$

We shall present another way of solving eq.(20).

Proposition 9: The integral $I_\omega(\chi,\varphi)$ is defined by the Fourier series

$$I_\omega(\chi,\varphi)=(\sqrt{\alpha}/2)\,\mathrm{Im}\Big\{\sum_{\kappa=0}^{\infty} I_\kappa(\varphi)\exp[i(2\kappa-1)(\chi-\omega\varphi)]\Big\}, \qquad (22)$$

where $I_0=1$ and for $\kappa\geq 0$ we have in $\Delta(\varphi_1^s,\varphi_2^s)$:

$$I_\kappa=(-\alpha)^{-\kappa}\int_{\varphi_0}^{\varphi} d\varphi_\kappa\, e^{2i\omega\varphi_\kappa}\alpha_\kappa^{2\kappa-2}\frac{d}{d\varphi_\kappa}\Big(\alpha_\kappa^{-2\kappa+3}\int_{\varphi_0}^{\varphi_\kappa} d\varphi_{\kappa-1}\cdots\int_{\varphi_0}^{\varphi_2} d\varphi_1 e^{2i\omega\varphi_1}\frac{d}{d\varphi_1}\alpha_1\Big) \qquad (23)$$

with $\alpha_\kappa=\alpha(\varphi_\kappa)$. Series (22) converges absolutely and uniformly in ω and φ at least at $\mathrm{Im}\chi>\ln\sqrt{3}$. It defines the function $I_\omega(\chi,\varphi)$ that can be continued analytically to all the points on $\mathbb{R}^{(1)}_\chi$ apart from singular points χ_{sing} of the solutions of eq.(12).

To prove this we substitute into (20) the series

$$F(z,\bar z)=(1/2)z\sum_{\kappa=-\infty}^{\infty}(\bar z/z)^\kappa F_\kappa(z\bar z). \qquad (24)$$

The recurrence relations for the coefficients $F_\kappa(z\bar z)=F_\kappa(\alpha)$ give at $\kappa\geq 1$:

$$F_\kappa=(-\alpha)^{-\kappa}\int_{\alpha_0}^{\alpha} d\alpha_\kappa e^{2i\omega\varphi_\kappa}\alpha_\kappa^{2\kappa-2}\frac{d}{d\alpha_\kappa}\Big[\alpha_\kappa^{-2\kappa+3}\int_{\alpha_0}^{\alpha_\kappa} d\alpha_{\kappa-1}\cdots\int_{\alpha_0}^{\alpha_2} d\alpha_1 e^{2i\omega\varphi_1}\frac{d}{d\alpha_1}(\alpha_1 F_0)\Big], \qquad (25a)$$

$$F_{-\kappa}=-(-\alpha)^{-\kappa-1}\int_{\alpha_0}^{\alpha} d\alpha_\kappa e^{-2i\omega\varphi_\kappa}\alpha_\kappa^{2\kappa}\frac{d}{d\alpha_\kappa}\Big(\alpha_\kappa^{-2\kappa+1}\int_{\alpha_0}^{\alpha_\kappa} d\alpha_{\kappa-1}\cdots\int_{\alpha_0}^{\alpha_2} d\alpha_1 e^{-2i\omega\varphi_1}\alpha_1^2\frac{dF_0}{d\alpha_1}\Big), \qquad (25b)$$

where $\varphi_\kappa=\varphi(\alpha_\kappa)$ is the same branch of the function $\varphi(\alpha)$ at all κ .

It is seen from (25b) that series (24) can be truncated from below by the choice of $F_0(\alpha)$. For $F_0=1$ we get $F_{-\kappa}=0$ for all $\kappa\in\mathbb{Z}_+$, and after returning to the variable φ from (25a) we get (23). Passing to the variables $\{\chi,\varphi\}$ according to (21) (taking into account corollary following from proposition 9), from (24) we find (22).

From (25a) we get the estimate

$$|F_\kappa|\leq \kappa^{-1}\alpha^{-\kappa}\int_{\alpha_0}^{\alpha} d(\alpha_\kappa)^\kappa |R_\kappa|,$$

where R_κ are expressed through $R_1=1$ with the help of the relations

$$R_{\kappa+1}=\exp(i\omega\varphi)R_\kappa-(2-1/\kappa)\alpha^{-\kappa}\int_{\alpha_0}^{\alpha} d(\alpha_\kappa)^\kappa\exp(i\omega\varphi_\kappa)R_\kappa .$$

Hence we get the estimate $\max|R_\kappa|\leq(3-1/\kappa)\max|R_{\kappa-1}|$ that, allowing for $R_1=1$ provides $|R_\kappa|<3^{\kappa-1}$. Consequently $|F_\kappa|<3^{\kappa-1}\kappa^{-1}$, wherefrom at $\mathrm{Im}\chi>\ln\sqrt{3}$ we find for series (22) the estimate

$$|I_\omega(\chi,\varphi)|\leq(\sqrt{\alpha}/6)\exp(\mathrm{Im}\chi)\{3+|\ln[1-3\exp(-2\,\mathrm{Im}\chi)]|\}.$$

A possibility for the function $I_\omega(\chi,\varphi)$ in $\mathbb{C}^{(1)}_\chi$ to be continued to nonsingular points of $\mathbb{R}^{(1)}_\chi$ follows from singular points of eq. (12) in $\mathbb{C}^{(1)}_\chi$ being isolated.

Conclusion

Propositions 2-9 show that the case $\varepsilon=0$ in the three-particle problems considered has all qualitative characteristics of the Hamiltonian systems that are completely integrable by the Liouville-Arnold theorem[6,14,16] except for, perhaps, integrability of the first order equations for the trajectory by using a single quadrature. In our opinion, this gives reasons to believe that the case $\varepsilon=0$ is completely integrable in a generalized sense.

Some of these problems, for instance, the problems with $V=V(\rho)$, $V=g\rho^{-2}\alpha(\varphi)$ and $V=g\rho^2\sum_{\kappa=1}\alpha_\kappa\sin^2(\varphi-\varphi_{ij,\kappa})$ are completely integrable by the Liouville-Arnold theorem. Their first-order equations for the trajectory are solved by using only one quadrature. Imposing two conditions $F_{N_+}=0$ and $F_{-N_-}=0$ on the coefficients (25) at some $N_\pm\in\mathbb{Z}_+$, one can obtain the potentials $V_{N_+N_-}$ for which the case $\varepsilon=0$ is solved by using a finite number of quadratures. In the general case, as is seen from (18), (22) and (23), an infinite number of quadratures is needed for solving the problem at $\varepsilon=0$.

At $\varepsilon=1$ the problems with homogeneous potentials V of the general form are not obviously completely integrable even in a generalized sense. This is favoured by unsuccessful attempt to find an additional first integral in the classical three-particle problem in the two-particle Newton interaction. A strict proof of such a "nonintegrability" is related to investigation of the properties of the solutions of eq.(4) and may provide a deeper insight into the problem of integrability of dynamical systems.

It follows from proposition 1 that the problems considered possess the simplest possible nontrivial properties of integrability. They always have one invariant torus (at $\varepsilon=0$) and the remaining part of the phase space either has no invariant tori (if the case

$\varepsilon = 1$ is not completely integrable) or it is filled by the similar invariant tori (if the problem is completely integrable at $\varepsilon = 1$). This qualitative simplicity alongside with the availability of the problems for an analytical investigation make them a convenient and interesting object for studying.

The problems raised in the concluding part of this paper will be discussed in more detail elsewhere.

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Физиев Л.П. E4-86-325

Вполне интегрируемый случай в задачах трех частиц с однородными потенциалами

Рассмотрены уравнения траектории классических задач трех частиц на прямой с потенциалами, которые являются однородными функциями координат. Предлагается рассмотреть случай нулевой полной энергии как полностью интегрируемый в обобщенном смысле, так как в нем понижается до первого порядок дифференциального уравнения траектории, разделяются переменные в уравнении Гамильтона-Якоби, имеются дополнительный первый интеграл и инвариантный тор, несмотря на то, что система не является интегрируемой по теореме Лиувилля-Арнольда. Построены решения: 1/ в параметрическом виде; 2/ в виде сходящихся пертурбативных рядов нового типа; 3/ в виде сходящихся рядов Фурье.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1986

Fiziev P.P. E4 -86-325

Completely Integrable Case in the Three-Particle Porblems with Homogeneous Potentials

The trajectory equations are considered for classical three-particle problems on a straight line with potentials that are homogeneous coordinate functions. It is proposed to consider the case of a zero total energy as a completely integrable one in a generalized sense since in it the order of the differential trajectory equation is lowered to the first one, the variables in the Hamiltonian-Jacobi equation are separated, there are additional first integral and invariant torus in spite of the fact that the system cannot be integrable by the Liouville-Arnold theorem. The solutions are constructed: 1/in a parametric form, 2/in a form of convergent perturbative series of a new type, 3/ as convergent Fourier series.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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