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THE THREE-DIMENSIONAL ANHARMONIC OSCILLATOR

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1. The problem of one-dimensional anharmonic oscillator (a.o.) is one of the oldest in quantum mechanics. It has been proved /1/ that a series of the usual perturbation theory diverges for any value of the anharmonic constant. An excellent review of the methods treating the one-dimensional a.o. is given in /2/.

A three-dimensional a.o. is analyzed to a much less extent. In refs. /3/ particular solutions have been found for a generalized a.o.($\alpha r^2 + \beta r^4 + \gamma r^6$) with certain relations between α, β, γ. The treated a.o.($\alpha r^2 + \beta r^4$) does not satisfy them. Pure quartic a.o. βr^4 has been investigated numerically in /4/. One should also mention the exact calculations /5,6/ and quasi-classical treatment /7/.

We feel that there is certain necessity in an approximate simple method which permits one to obtain, in a closed form, the energy levels of an isotropic three-dimensional a.o. with the accuracy 1-2%. We prove that 1st order MPT meets these demands.

The consideration is organized as follows. In Sec.2 using the 1st order MPT we evaluated the energy levels of the three-dimensional a.o. for a broad range of values of the anharmonic constant and angular moments. For $\alpha > 0$ the inaccuracy of the method is not worse than 2%. A very simple formula is obtained for the pure quartic oscillator:

$$\mathcal{E}_{N\ell} = \left(\frac{\hbar^4\beta}{m^2}\,\frac{81}{32}\right)^{\frac{1}{3}} \cdot \left(N + \frac{3}{2}\right)^{\frac{4}{3}} \left[1 - \frac{(\ell + \frac{3}{2})(\ell - \frac{1}{2})}{3\cdot (N + \frac{3}{2})^2}\right]^{\frac{1}{3}},$$

We found that for $\alpha < 0$ and negative energies MPT does not work satisfactorily.

In Sec.3 we apply WKB method to the same problem and find that it gives reasonable description both for large positive energies and at the bottom of the potential well. Yet MPT final expressions are much simpler. There is no need in the solution of the transcendental equation (as in WKB case). Further for $\alpha > 0$ the MPT works equally well both for large and small energies, while WKB expressions are satisfactory only for large energies. We conclude that these methods complement each other.

The results of the present consideration may be applied to nuclear collective Hamiltonians which include large anharmonic terms /8/.

2. The Schroedinger equation for the three-dimensional a.o. is

$$-\frac{\hbar^2}{2m}\Delta\Psi + \alpha r^2\Psi + \beta r^4\Psi = \mathcal{E}\Psi. \tag{2.1}$$

After the transformation to the dimensionless variables ($\rho^2 = \frac{1}{\hbar}\sqrt{2|\alpha|m}\, r^2$) one obtains

$$(-\Delta\rho \pm \rho^2)\Psi + \chi\rho^4\Psi = \epsilon\Psi,$$
$$\mathcal{E} = \frac{1}{2}\hbar\omega_0\cdot\epsilon,\quad \omega_0 = \sqrt{\frac{2|\alpha|}{m}},\quad \chi = \frac{h\beta}{\omega_0|\alpha|m}. \tag{2.2}$$

Upper and lower signs in (2.2) correspond to $\alpha > 0$ and $\alpha < 0$ resp. Following MPT prescription we add and subtract in (2.2) the quantity $\omega^2\rho^2$ (ω will be defined later) and present (2.2) as:

$$(H_0 + H_1)\Psi = \epsilon\Psi,\ H_0 = \omega(-\Delta_{\rho_1} + \rho_1^2),\ \rho_1^2 = \rho^2\cdot\omega,$$
$$H_1 = \rho_1^2(\pm\omega^{-1} - \omega) + \chi\rho_1^4/\omega^2. \tag{2.3}$$

Now treat H_1 in the 1st order of the usual PT:

$$\epsilon_{N\ell} = \langle N\ell|H|N\ell\rangle = (N + \frac{3}{2})(\omega \pm \omega^{-1} + C_{N\ell}\cdot\omega^{-2}). \tag{2.4}$$

Here N is the principal quantum number $N = 2n + \ell$,

$$C_{N\ell} = \frac{3}{2}\chi\cdot(N + \frac{3}{2})d_{N\ell},\quad d_{N\ell} = 1 - \frac{(\ell+\frac{3}{2})(\ell-\frac{1}{2})}{3\cdot(N+\frac{3}{2})^2}.$$

Clearly, exact values of H should not depend upon ω. The MPT prescribes[9] to use as ω one of the real roots of the equation $\partial\epsilon_{N\ell}/\partial\omega = 0$. In our case this leads to:

$$\omega^3 \mp \omega - 2C_{N\ell} = 0. \tag{2.5}$$

Combining (2.4) and (2.5) we get

$$\epsilon_{N\ell} = \frac{1}{2}\cdot(N + \frac{3}{2})(3\omega \pm \omega^{-1}). \tag{2.6}$$

Consider first case $\alpha > 0$. The form of solution depends on the sign of $Q = C_{N\ell}^2 - 1/27$. If $Q > 0$, then:

$$\omega = (C_{N\ell} + \sqrt{Q})^{1/3} + (C_{N\ell} - \sqrt{Q})^{1/3}. \tag{2.7}$$

For $Q < 0$:

$$\omega = \frac{2}{\sqrt{3}}\cos\frac{\phi}{3},\quad \phi = \mathrm{arctg}\sqrt{\frac{1}{27C_{N\ell}^2} - 1}. \tag{2.8}$$

Using (2.6)-(2.8) we calculated the energy levels of the three-dimensional a.o. (see fig.1 for N even and fig.2 for N odd). The numbers on the right of the levels mean N (i.e., they form a broken SU(3) multiplet N). The levels inside a given multiplet differ by the angular momentum ℓ. The values of ℓ are not shown, as the sequence of ℓ inside the multiplet is always the same: the energy of the levels inside a given multiplet decreases as ℓ increases. For example, multiplet N = 8 contains levels (from up to down) with ℓ = 0,2,4,6,8. Consider now particular cases of (2.6).

a) $C_{N\ell} \gg 1$ (strong anharmonicity or large N):

$$\epsilon_{N\ell} = \frac{1}{2}[3\cdot(N + \frac{3}{2})]^{4/3}\cdot\chi^{1/3}\cdot(d_{N\ell})^{1/3}. \tag{2.9}$$

d) $C_{N\ell} \ll 1$(weak anharmonicity and not very large N):

$$\epsilon_{N\ell} = (2N+3)[1 + \frac{3}{8}\chi\cdot(2N+3)\cdot d_{N\ell}]. \tag{2.10}$$

Now compare (2.6) with numerical calculations of [5,6]. They are presented in the first lines of Tables 1 and 2. In the second lines the MPT results are given. The exact results[4] for the pure quartic oscillator are collected in Tables 3 and 4 for even and odd angular momenta, respectively. The MPT gives for this case a particularly simple formula

$$\mathcal{E}_{N\ell} = \frac{\hbar^{4/3}\beta^{1/3}}{m^{2/3}}\cdot(\frac{81}{32})^{1/3}\cdot(N + \frac{3}{2})^{4/3}\cdot(d_{N\ell})^{1/3}.$$

From the inspection of Tables 1-4 we conclude that maximal inaccuracy is about 1%.

For the negative values of α one obtains:

$$\epsilon_{N\ell} = \frac{1}{2}\cdot(N + \frac{3}{2})\cdot(3\omega - \omega^{-1}),\ \omega = (C_{N\ell} + \sqrt{C_{N\ell}^2 + \frac{1}{27}})^{1/3} - (\sqrt{C_{N\ell}^2 + \frac{1}{27}} - C_{N\ell})^{1/3}.$$

For $\chi \gg 1$ the anharmonic term dominates and we arrive at (2.9). The case of weak anharmonicity ($\chi \ll 1$ and N is not very large, so that $C_{N\ell} \ll 1$) is more interesting. One gets:

$$\mathcal{E}_{N\ell} = -\frac{2}{3}V_0\cdot[1 - \frac{(\ell+\frac{3}{2})(\ell-\frac{1}{2})}{3\cdot(N+\frac{3}{2})^2}]^{-1}. \tag{2.11}$$

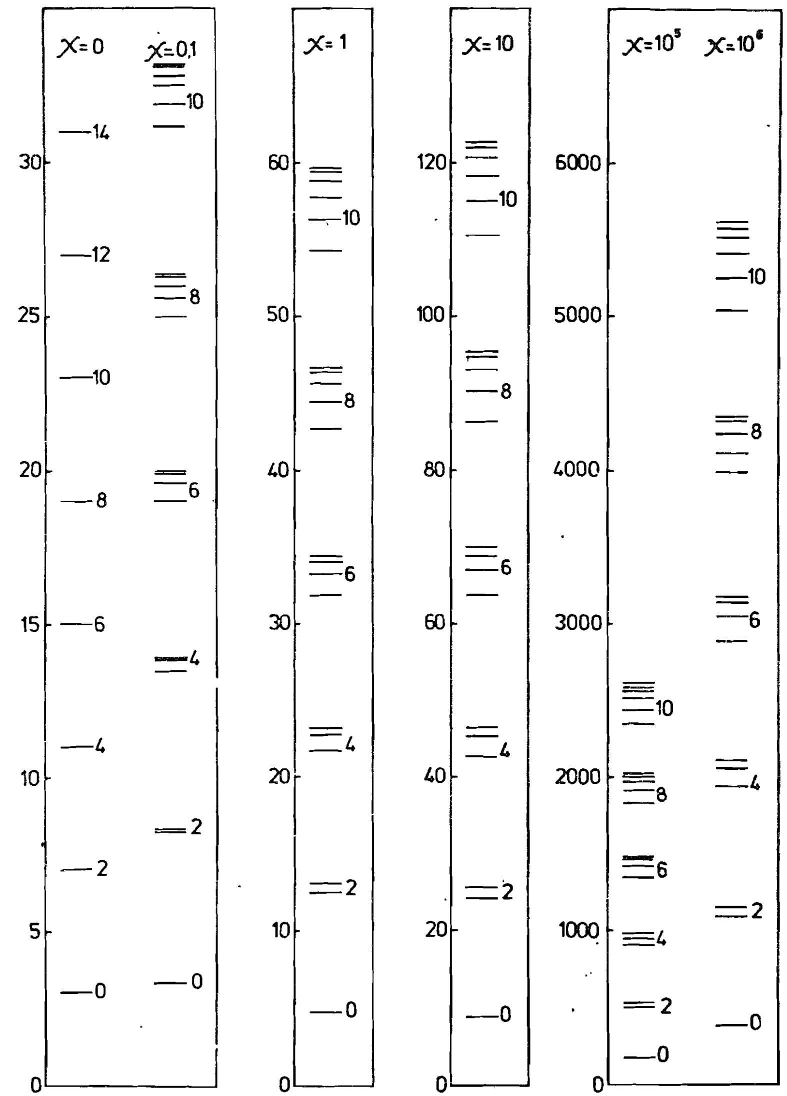

Fig.1. The energy levels of the anharmonic oscillator $(-\Delta + \rho^2 + \chi\rho^4)$ *for* N *even and for different values of the dimensionless constant* $\chi = \hbar\beta\sqrt{\frac{2}{m\alpha^3}}$*.The numbers on the right of levels mean principal quantum numbers* $N = 2n + \ell$*. The energies of levels inside a given multiplet (* N *const) always decrease as angular momentum* ℓ *increases. So, this quantum number is not shown.*

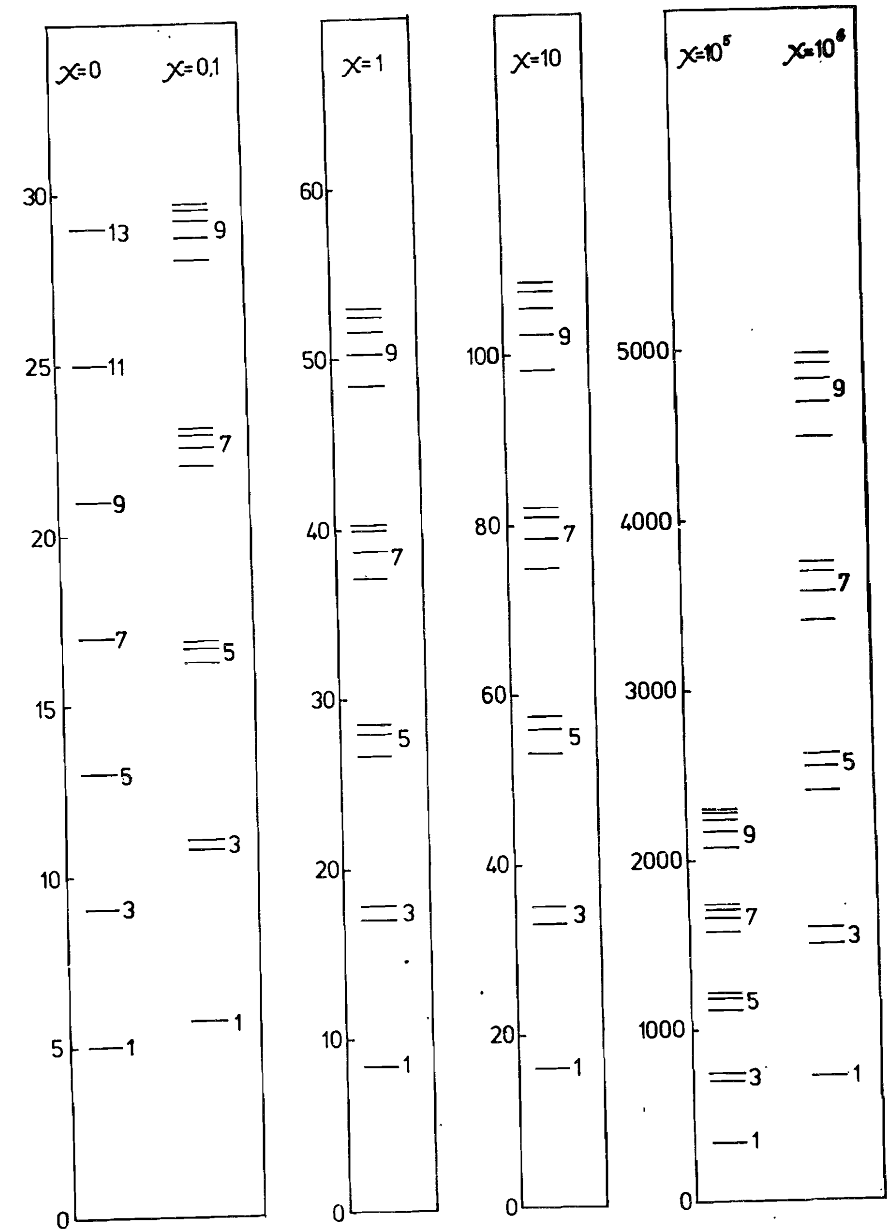

Fig.2. The same as in fig.1 but for N *odd.*

Table 1

Eigenvalues of three-dimensional a.o.
$(H = -\Delta + r^2 + \lambda r^4)$ [5]

$N\ell$ \ λ	0.01	1	100
0,0	3.036 3.037	4.649 4.68	17.83 18.05
4, 0	11.426 11.425	23.298 23.16	99.033 98.22
8, 0	20.21 20.2	46.965 46.62	204.8 202.9
2, 2	7.151 7.151	12.486 12.54	50.61 51.05
6, 2	15.746 15.74	34.208 34.06	147.31 146.45
10, 2	24.712 24.704	59.8 59.4	262.4 260.3
4, 4	11.336 11.337	21.595 21.67	90.1 90.54
8, 4	20.13 20.12	45.72 45.6	198.5 197.8
12, 4	29.27 29.26	73.03 72.64	321.89 319.8

Table 2

Eigenvalues of the three-dimensional a.o.
$(H = -\frac{1}{2}\Delta + \frac{1}{2}\omega^2 r^2 + \lambda r^4)$ [6]

$N\ell$ \ $\omega\lambda$	$\omega = 0$, $\lambda = 1$	$\omega = 1$, $\lambda = 0.5$
100, 0	651.731 645.26	524.604 519.661
100, 50	629.194 627	507.068 505.39
100, 100	565.344 565.53	457.286 457.43
50, 0	263.751 261.16	213.99 212.03
50, 20	257.89 256.5	209.482 208.414
50, 50	229.44 229.6	187.53 187.64
10, 0	35.74 35.4	30.065 29.835
10, 4	34.98 34.8	29.510 29.385
10, 10	31.69 31.78	27.09 27.15
0, 0	2.394 2.424	2.324 2.339

Here $V_0 = \frac{\alpha^2}{4\beta}$ is the absolute value of the potential energy in the minimum which is situated at $r_0^2 = \frac{|\alpha|}{2\beta}$. So, MPT predicts definite concentration of the levels around the value $-\frac{2}{3}V_0$ (when $C_{N\ell} \ll 1$). We shall see in the following section that this result is incorrect.

Table 3

Eigenvalues of the three-dimensional quartic oscillator for even angular momenta [4]
$(H = -\frac{1}{2}\Delta + r^4)$

N \ ℓ	0	2	4	6	8
0	2.394 2.424				
2	7.336 7.291	6.83 6.89			
4	13.38 13.27	13. 12.97	12.16 12.22		
6	20.22 20.04	19.92 19.8	19.23 19.22	18.15 18.22	
8	27.71 27.45	27.45 27.24	26.85 26.76	25.92 25.95	24.69 24.77
10	35.74 35.4	35.51 35.21	34.98 34.79	34.16 34.11	33.06 33.12
20	82.3 81.5	82.15 81.37	81.8 81.1	81.26 80.66	80.5 80.06
30	136.94 135.6	136.83 135.5	136.55 135.3	136.12 135.	135.54 135.5
40	197.78 195.82	197.68 195.75	197.46 195.6	197.1 195.3	196.61 194.9
48	250.18 247.7	250.1 247.6	249.89 247.5	249.58 247.23	249.14 246.9

3. Now apply to the three-dimensional a.o. the quasiclassical method. The Bohr-Sommerfeld condition

$$m \oint f \cdot dr = \sqrt{2m} \int_b^a \left(\mathcal{E} - \alpha x - \beta x^2 - \frac{L^2}{2mx}\right)^{1/2} \frac{dx}{\sqrt{x}} = \hbar \cdot \left(n + \frac{1}{2}\right), \quad (x = r^2). \quad (3.1)$$

leads to the following transcendental equation [7,10]:

$$\frac{\hbar}{g} \cdot \sqrt{\frac{\beta}{2m}}\left(n + \frac{1}{2}\right) = \mathcal{E} \cdot K - \frac{\alpha a}{k^2} \cdot [(k^2 - \gamma^2)K + \gamma^2 \cdot E] - \frac{\beta a^2}{3k^4}.$$

Table 4

Eigenvalues of the three-dimensional quartic oscillator for odd angular momenta

N \ ℓ	1	3	5	7	9
1	44.78				
	4.52				
3	10.1	9.401			
	10.65	9.457			
5	16.6	16.046	15.082		
	16.48	16.028	15.15		
7	23.796	23.331	22.511	21.358	
	23.6	23.22	22.53	21.43	
9	31.578	31.174	30.455	29.436	28.135
	31.3	30.98	30.38	29.48	28.22
11	39.869	39.508	38.865	37.948	36.772
	39.5	39.22	38.7	37.92	36.85
21	89.393	87.149	86.711	86.081	85.261
	86.54	86.35	86.	85.5	84.837
31	142.732	142.541	142.197	141.702	141.056
	141.33	141.18	140.91	140.52	140
41	204.129	203.969	203.681	203.267	202.725
	202.11	202.	201.78	201.44	201.
49	256.916	256.774	256.517	256.147	255.664
	254.38	254.3	254.1	253.78	253.4

$$\times[(3k^4-6\gamma^2k^2+2\gamma^4+k^2\gamma^4)K+2\cdot(3\gamma^2k^2-\gamma^4-k^2\gamma^4)E]-\frac{L^2}{2ma}\cdot\Pi(\gamma^2,k). \qquad (3.2)$$

Here $K(k)$, $E(k)$, $\Pi(\gamma^2,k)$ are complete elliptic integrals of the 1st, 2nd and 3rd kind, resp.; $g=\frac{2}{\sqrt{a-c}}$; $k^2=\frac{a-b}{a-c}$, $\gamma^2=\frac{a-b}{a}$; a,b,c $(a \geq b > 0 > c)$ are roots of the integrand in (3.1). The radius of the particle orbit satisfies the condition: $\sqrt{b} \leq r \leq \sqrt{a}$ (see fig.3). The explicit expressions for a,b,c as well as the value $\mathcal{E}_0$ of the energy at the minimum are presented in the Appendix.

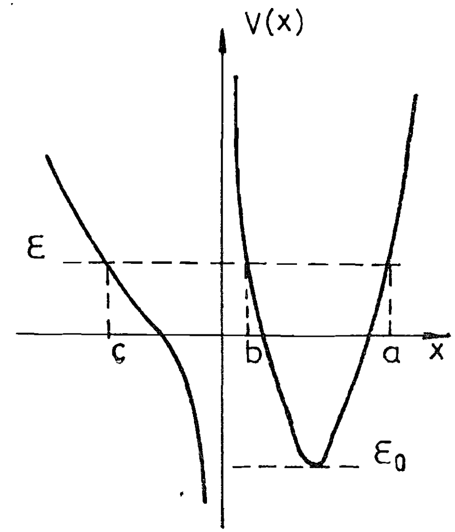

Fig.3. The sum of the potential energy and centrifugal terms for the three-dimensional anharmonic oscillator. The parameters a,b,c and $\mathcal{E}_0$ used in the text are shown.

Consider now some limiting cases of (3.2). For the positive harmonic constant α and for the anharmonicity β tending to zero one has: $a \approx \frac{\mathcal{E}}{2\alpha}+R$, $b \approx \frac{\mathcal{E}}{2\alpha}-R$, $c \approx -\frac{\alpha}{\beta}$, ($R = \sqrt{\frac{\mathcal{E}^2}{4\alpha^2}-\frac{L^2}{2m\alpha}}$). As expected, one obtains the spectrum of a harmonic oscillator: $\mathcal{E} \approx h\omega_0(2n+\ell+\frac{3}{2})$ where we put $L = h(\ell+\frac{1}{2})$ following quasiclassical prescriptions.

Another interesting case is $\mathcal{E} \approx \mathcal{E}_0$ that is the bottom of the potential well. Presenting $\mathcal{E}$ as $\mathcal{E}_0+\mathcal{E}_1$ and developing a,b,c in $\mathcal{E}_1$ we get:

$$a=-[1+\sqrt{\epsilon_0}(\mu+\lambda)]\frac{\alpha}{3\beta},\quad b=-[1+\sqrt{\epsilon_0}(\mu-\lambda)]\frac{\alpha}{3\beta},\quad c=[2\sqrt{\epsilon_0\,\mu}-1]\frac{\alpha}{3\beta},$$

where

$$\mu=1+\frac{1}{2}\frac{\epsilon_1}{\epsilon_0}-\frac{\lambda^2}{6},\quad \lambda=\sqrt{\frac{\epsilon_1\epsilon_0(\frac{1}{2}\epsilon_0-d)}{(\frac{3}{2}\epsilon_0-d)^3}},\quad \epsilon_0=1+\frac{3\mathcal{E}_0\beta}{\alpha^2},\quad \epsilon_1=\frac{3\mathcal{E}_1\beta}{\alpha^2},\quad d=\frac{1}{2}\cdot\frac{27}{4}\frac{\beta^2L^2}{m\alpha^3}.$$

Substituting these Exps. into (3.2) one obtains:

$$\mathcal{E}_1=(2n+1)\hbar\omega_0\cdot(1+\frac{3\mathcal{E}_0\beta}{\alpha^2})^{1/6}. \qquad (3.3)$$

(Free terms as well as the ones proportional to $\sqrt{\epsilon_1}$ are cancelled). The same result is obtained if one develops in the Schroedinger equation the sum of potential energy and centrifugal terms near the minimum. It follows from this that MPT expression (2.1) is incorrect. Finally, for the energies well above the minimum ($\frac{\mathcal{E}}{\mathcal{E}_0} \gg 1$) the following expression is derived: $\mathcal{E} \approx [\frac{3hn}{4K(\frac{1}{\sqrt{2}})}]^{4/3}\frac{\beta^{1/3}}{m^{2/3}}$. On the other hand, the 1st order MPT gives for the same case ($n \gg \ell$); $\mathcal{E} \approx (\frac{3hn}{\pi})^{4/3}\frac{\beta^{1/3}}{m^{2/3}}\frac{1}{2^{5/3}}$. Taking the ratio of these energies we have

$$\frac{\mathcal{E}}{\mathcal{E}'} = \frac{1}{2}\left[\frac{\pi}{K(1/\sqrt{2})}\right]^{4/3} \approx 1.01 .$$

The closed form of quasiclassical results obtained suggests that the corresponding classical equations may be integrated in quadratures. In fact, one easily obtains the particle trajectory. It looks like:

$$r^2(t) = c + (b-c)\{1-k^2\,\mathrm{sn}^2[\sqrt{\frac{2\beta(a-c)}{m}}(t-t_{min}),k]\}^{-1},$$

$$\phi(t) = \phi_{min} + \frac{L}{b}\left(1-\frac{b}{c}\right)\frac{1}{\sqrt{2m\beta(a-c)}}\cdot\Pi\{\mathrm{am}[\sqrt{\frac{2\beta(a-c)}{m}}(t-t_{min})],\frac{k^2c}{b},k\}+\frac{L}{mc}(t-t_{min}).$$

when particle moves from $r_{min}(=\sqrt{b})$ to $r_{max}(=\sqrt{a})$ and

$$r^2(t) = a-(a-b)\,\mathrm{sn}^2[\sqrt{\frac{2\beta(a-c)}{m}}(t-t_{max}),k],$$

$$\phi(t) = \phi_{max} + \frac{L}{a}\frac{1}{\sqrt{2m\beta(a-c)}}\cdot\Pi\{\mathrm{am}[\sqrt{\frac{2\beta(a-c)}{m}}(t-t_{max})],\gamma^2,k\}$$

in the opposite case. Here $\mathrm{sn}(\psi,k)$ is the Jacobian elliptic function; t_{min} and t_{max} are moments of time when particle passes through r_{min} or r_{max}; ϕ_{min} and ϕ_{max} are angular distances of particle at these moments. It follows from this that the period of the radial oscillations (that is the time needed to pass from r_{min} to r_{max} and back) equals $T_r = \sqrt{\frac{2m}{\beta(a-c)}}\,K$. The angular distance covered by the particle for the same time is (in radians): $\Theta_r = \frac{2L}{a}\frac{1}{\sqrt{2m\beta(a-c)}}\cdot\Pi(\frac{\pi}{2},\gamma^2,k)$.

4. We conclude:

1) the MPT final expressions (2.7)-(2.9) and (2.12) are much simpler; there is no need in the solution of the transcendental equation (as in the WKB case);

2) for the positive values of harmonic constant α the MPT works equally well both for large and small energies, while WKB expressions are satisfactory only for large energies;

3) for $\alpha < 0$ the MPT and WKB methods complement each other. In fact, MPT works well for positive energies if $C_{N\ell}$ is sufficiently large. This include the case of large anharmonicity and arbitrary energy and vice versa. On the other hand MPT method breaks for negative energies, while WKB one adequately describes the levels at the bottom of the potential well;

4) for $\alpha = 0$ the MPT gives a particularly simple formula:

$$\mathcal{E}_{N\ell} = \frac{\hbar^{4/3}}{m^{2/3}}\cdot\beta^{1/3}\cdot\left(\frac{81}{32}\right)^{1/3}\cdot\left(N+\frac{3}{2}\right)^{4/3}\cdot\left[1-\frac{(\ell+\frac{3}{2})(\ell-\frac{1}{2})}{3\cdot(N+\frac{3}{2})^2}\right]^{1/3}.$$

Its error is about 1%.

We are indebted to Prof. M.Gmitro for careful reading of this manuscript and some useful comments.

APPENDIX

The following notation is used throughout this Appendix:

$$d=\frac{1}{2}-\frac{27}{4}\frac{\beta^2L^2}{m\alpha^3},\ \epsilon=1+\frac{3\mathcal{E}\beta}{\alpha^2},\ \mathrm{tg}\,\phi=\left[\frac{\epsilon^3}{(\frac{3}{2}\epsilon-d)^2}-1\right]^{1/2};\ D=\frac{1}{2}\sqrt{d^3(d-\frac{1}{2})},$$

$$q=\frac{1}{2}\left[d^2-\frac{9}{4}(d-\frac{3}{8})\right],\ \mathrm{tg}\,\phi_0=\left[\frac{(\frac{9}{16}-d)^3}{q^2}-1\right]^{1/2}.$$

I. The harmonic constant $\alpha > 0$

$$a=[\sqrt{\epsilon}(\cos\frac{\phi}{3}+\sqrt{3}\sin\frac{\phi}{3})-1]\frac{\alpha}{3\beta},\quad b=[\sqrt{\epsilon}(\cos\frac{\phi}{3}-\sqrt{3}\sin\frac{\phi}{3})-1]\frac{\alpha}{3\beta},$$

$$c=-(2\sqrt{\epsilon}\cos\frac{\phi}{3}+1)\frac{\alpha}{3\beta};$$

ϕ changes from 0 to $\pi/2$ as ϵ ranges from its value $\epsilon_0(=1+\frac{3\mathcal{E}_0\beta}{\alpha^2})$ at the minimum to ∞; ϵ_0 will be given below.

II. The harmonic constant $\alpha < 0$

For the ϵ in the interval $(\epsilon_0,\frac{2}{3}d)$ the distances a,b,c are given by:

$$a=-[\sqrt{\epsilon}(\cos\frac{\phi}{3}+\sqrt{3}\sin\frac{\phi}{3})+1]\frac{\alpha}{3\beta},$$

$$b=-[\sqrt{\epsilon}(\cos\frac{\phi}{3}-\sqrt{3}\sin\frac{\phi}{3})+1]\frac{\alpha}{3\beta},\quad c=(2\sqrt{\epsilon}\cos\frac{\phi}{3}-1)\frac{\alpha}{3\beta}.$$

ϕ changes from 0 for $\epsilon=\epsilon_0$ up to $\pi/2$ for $\epsilon=\frac{2}{3}d$. For $\frac{2}{3}d<\epsilon<\infty$ one has:

$$a=-(2\sqrt{\epsilon}\cos\frac{\phi}{3}+1)\frac{\alpha}{3\beta},\ b=[\sqrt{\epsilon}(\cos\frac{\phi}{3}-\sqrt{3}\sin\frac{\phi}{3})-1]\frac{\alpha}{3\beta};\ c=[\sqrt{\epsilon}(\cos\frac{\phi}{3}+\sqrt{3}\sin\frac{\phi}{3})-1]\alpha/3\beta.$$

Here $\phi = \pi/2$ both for $\epsilon = 2/3\,d$ and $\epsilon = \infty$ and takes the minimal value $\operatorname{arctg}\sqrt{4d-1}$ for $\epsilon = d$. ϵ_0 as a function of d is defined as follows:

Range of d	ϵ_0
$-\infty < d \le 0$	$\frac{3}{4} + (q+D)^{1/3} + (q-D)^{1/3}$
$0 \le d \le \frac{3}{8}(3-\sqrt{3})$ $(0 \le \phi_0 \le \frac{\pi}{2})$	$\frac{3}{4} + 2\sqrt{\frac{9}{16} - d}\,\cos\frac{\phi_0}{3}$
$\frac{3}{8}(3-\sqrt{3}) \le d \le \frac{1}{2}$ $(\pi/2 \ge \phi_0 \ge 0)$	$\frac{3}{4} + \sqrt{\frac{9}{16} - d}\,(\cos\frac{\phi_0}{3} + \sqrt{3}\cdot\sin\frac{\phi_0}{3})$
$\frac{1}{2} < d < \frac{9}{16}$	$\frac{3}{4} - (-q-D)^{1/3} - (-q+D)^{1/3}$
$\frac{9}{16} < d < \infty$	$\frac{3}{4} + (D+q)^{1/3} - (D-q)^{1/3}$.

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Трехмерный ангармонический осциллятор

Мы получили явные выражения для уровней энергии трехмерного ангармонического осциллятора $\alpha r^2 + \beta r^4$, используя первый порядок, так называемой, модифицированной теории возмущений /МТВ/ и квазиклассический метод. Расчеты выполнены для широкого спектра значений ангармонической константы β и угловых моментов. Мы показываем, что точность первого порядка МТВ при положительной константе гармоничности α не хуже 1-2%. Результирующие формулы МТВ более просты и не требуют предварительного решения трансцендентного уравнения /как в случае ВКБ метода/. Особенно простые выражения получаются при $\alpha = 0$. Их неточность - около 1%. Оказывается, что МТВ плохо работает при отрицательных энергиях /при этом $\alpha < 0$/, в то время как ВКБ метод дает разумные результаты как при больших энергиях, так и вблизи дна потенциальной ямы. Таким образом, эти методы дополняют друг друга.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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The Three-Dimensional Anharmonic Oscillator

We have obtained, in a closed form, the energy levels of a three-dimensional anharmonic oscillator $\alpha r^2 + \beta r^4$ using 1st order of the so-called modified perturbation theory (MPT) as well as WKB method. The calculations were performed for a broad range of the anharmonic constant β and angular momenta. We prove that inaccuracy of the 1st order MPT for positive harmonic constant is not worse than 1-2%. It is found that for $\alpha < 0$ and negative energies the 1st order MPT is not satisfactory, whereas a quasiclassical treatment gives reasonable results both for large energies and at the bottom of the potential well. We conclude that these methods complement each other. The results of the present consideration may be used to take into account large anharmonic terms occuring in nuclear collective Hamiltonians.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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