ЦБ

объединенный
институт
ядерных
исследований
дубна

E4-86-227

P.P.Fiziev

COMPLETE INTEGRABILITY IN THE CLASSICAL PROBLEM OF THREE PARTICLES ON A STRAIGHT LINE

Submitted to "Physics Letters A"

1986

In ref. /1/ a total reduction is made of the classical problem of three particles with arbitrary masses $m_{1,2,3}$ $(m_1+m_2+m_3=M)$ which move along a fixed straight line and interact with each other by two-particle potentials V_{ij} of the gravitational or electric type. There, a useful tool turned out to be a modification of the Delves hyperspherical coordinates /2-4/: $\rho\in[0,\infty)$, $\varphi\in[0,2\pi]$. Upon eliminating the motion of the center of masses the Hamiltonian in these coordinates reads:

$$H=(1/2\mu)(p_\rho^2+\rho^{-2}p_\varphi^2)+g\alpha(\varphi)\rho^{-1}, \tag{1}$$

where

$$\alpha(\varphi)=\sum_{k=1}^{3}\alpha_k|\sin(\varphi-\varphi_{ij,k})|^{-1}, \tag{2}$$

and

$$\varphi_{23,1}=(1/3)(\Psi_3-\Psi_2)+\pi/6,\ \varphi_{31,2}=(1/3)(\Psi_1-\Psi_3)+5\pi/6,\ \varphi_{12,3}=(1/3)(\Psi_2-\Psi_1)+9\pi/6,$$

are angles of two-body collisions, for the gravitation problem $g=-\mu M\sqrt{s_1s_2s_3}$, $\alpha_k=c_k s_k^{-2}$; for particles with charges $e_{1,2,3}$ we have $g=(e_1e_2e_3)^{2/3}\sqrt{s_1s_2s_3}$, $\alpha_k=(e_ie_j/e_k^2)^{1/3}s_k^{-1}$; the angles $\Psi_k\in[0,2\pi]$ are determined by $\operatorname{tg}\Psi_k=m_k/\mu$; $\Psi_1+\Psi_2+\Psi_3=\pi$ and $s_k=\sin\Psi_k$, $c_k=\cos\Psi_k$. The reduced mass of a three-particle system is given by /2,3/: $\mu=\sqrt{m_1m_2m_3/M}$.

As a result of the reduction, the study of dynamics is reduced to the study of the system trajectory $\rho(\varphi)=\mathcal{R}\xi(\varphi)$, where $\mathcal{R}$ is a length scale. The law of motion is obtained by the quadrature:

$$t-t_0=\pm\int_{\varphi_0}^{\varphi}d\varphi\sqrt{(\mu/2)(\rho'^2+\rho^2)(E-g\alpha/\rho)^{-1}},$$

where $\rho'=d\rho/d\varphi$. For $\xi(\varphi)$ we have the second-order equation

$$(\xi'/\xi)'-(1/2)(1+\xi'^2/\xi^2)[1-(\alpha'/\alpha)(\xi'/\xi)-2\varepsilon\xi/\alpha](1-\varepsilon\xi/\alpha)^{-1}=0, \tag{3}$$

where $\varepsilon=E\mathcal{R}/g$ is the dimensionless total energy of the system. From (3) it is seen that it is sufficient to solve the problem for two values of the dimensionless total energy: $\varepsilon=1$ and $\varepsilon=0$.

In our opinion, the case $\varepsilon = 0$ should be considered completely integrable: equation (3) reduces to a first-order equation for $\vartheta(\varphi) = \xi'/\xi$, in terms of which $\xi(\varphi)$ is expressed as follows:

$$\xi(\varphi) = \xi_0 \exp\left[\int_{\varphi_0}^{\varphi} d\varphi\, \vartheta(\varphi, \varphi_0)\right], \tag{4}$$

where ξ_0 and φ_0 are initial conditions. Setting $\gamma(\varphi) = \alpha'/\alpha$ we arrive at the equation for $\vartheta(\varphi)$:

$$\vartheta' - (1/2)(1+\vartheta^2)[1 - \gamma(\varphi)\vartheta] = 0. \tag{5}$$

At $\varepsilon = 0$ there exists an extra first integral explicitly independent of the time:

$$\varphi_0 = \Phi(\rho p_\rho p_\varphi^{-1}, \varphi). \tag{6}$$

Here $\varphi_0 = \Phi(\vartheta, \varphi)$ is an inverse in ϑ function for the solution $\vartheta = \vartheta(\varphi, \varphi_0)$ to equation (5), and it obeys the equation

$$\partial_\varphi \Phi + (1/2)(1+\vartheta^2)[1 - \gamma(\varphi)\vartheta]\partial_\vartheta \Phi = 0,$$

with the aid of which it is easy to calculate the Poisson bracket:

$$[\Phi, H] = H p_\varphi^{-1}(1 + \gamma \rho p_\rho p_\varphi^{-1})\partial_\vartheta \Phi. \tag{7}$$

It is seen that the system is not completely Liouville-integrable /5-8/. By the Birkhoff terminology Φ is a conditional first integral since $[\Phi, H] = 0$ only at $H = E = 0$.

According to our conception of complete integrability, when $\varepsilon = 0$, there occurs separation of variables in the Hamilton-Jacobi equation of the problem

$$(1/2\mu)[(\partial_\rho W)^2 + \rho^{-2}(\partial_\varphi W)^2] + g\alpha(\varphi)\rho^{-1} = E. \tag{8}$$

This proceeds not in a standard manner, but just via the factorization of the action $W(\rho, \varphi)$. Indeed, setting

$$W(\rho, \varphi) = \sqrt{2\mu|g|\rho}\, f(\varphi) \tag{9}$$

we obtain from (2) and (8) the equation

$$f'^2 + (1/4) f^2 = \pm \alpha(\varphi), \tag{10}$$

where $\pm 1 = -\mathrm{sign}\, g$. Knowing the solution $f(\varphi, \varphi_0)$ to this equation we may find the trajectory from the relation $\partial_{\varphi_0} W = const = \sqrt{2\mu|g|\mathcal{R}\xi_0}\, \partial_{\varphi_0} f(\varphi_0, \varphi_0)$ in the form

$$\xi(\varphi) = \xi_0 [\partial_{\varphi_0} f(\varphi_0, \varphi_0)/\partial_{\varphi_0} f(\varphi, \varphi_0)]^2. \tag{11}$$

Comparison of (4) with (11) leads to the relation

$$\partial_{\varphi_0} f(\varphi, \varphi_0) = \partial_{\varphi_0} f(\varphi_0, \varphi_0) \exp\left[-(1/2)\int_{\varphi_0}^{\varphi} d\varphi\, \vartheta(\varphi, \varphi_0)\right].$$

The peculiarities we have analysed for the case $\varepsilon = 0$ give, as we think, enough grounds for considering it completely integrable. Thus we arrive at a more general conception of complete integrability of dynamic systems. It is based only on the possibility of lowering the order of differential equations down to one. The conventional notion of complete integrability according to Liouville-Arnol'd theorem /5-8/ also implies a concrete procedure of lowering the order of equations of a given problem and leads to their solution in quadratures.

Equations (5) and (10) cannot be solved in quadratures. We shall investigate their solutions and carefully examine the above problems of complete integrability of dynamic system elsewhere. Here we only note that the case $\varepsilon = 0$ at zeroth total momentum of the syste, corresponds to an invariant manifold of the phase flux of the general three-body problem, which represents a submanifold of its bifurcation manifold /6,9/. It can be shown that it is diffeomorphic to a two-dimensional torus.

The author is very grateful to Barbashov B.M., Matveenko A.V., Fizieva Ts.Ya., Donkov A.D., Nikolov P.A., Mescheryakov V.A., for help, remarks and support, and to Belyaev V.B., Smorodinski Ya.A., Melnikov V.K. and Chirikov B.V. for useful discussions.

References

1. P.P.Fiziev, Ts.Ya.Fizieva, communication, JINR, R-2-86-131, Dubna (1986), in Russian.
2. L.M.Delves, Nucl.Phys. 9 (1959) 391; 20 (1960) 275.
3. P.P.Fiziev, Ts.Ya.Fizieva, preprint, JINR, E-2-86-119, Dubna (1986).
4. P.P.Fiziev, Ts.Ya.Fizieva, communication, JINR, R-2-86-130, Dubna (1986), in Russian.

5. V.I.Arnol'd, Mathematical Methods of Classical Mechanics (Springer, N.Y. 1978).
6. V.I.Arnol'd, V.V.Kozlov, A.I.Neistadt, Itogi Nauki i Tehniki, ser.Sovremennye Problemy Matematiki, Vol.3. (Ac.Nauk USSR, M. 1985), in Russian.
7. R.A.Abracham, J.F.Marsden, Foundations of Mechanics (Benjamin, N.Y. - Amsterdam, 1978).
8. G.D.Birkhoff, Dynamical Systems (Am.Math.Soc., N.Y., 1927).
9. S.Smale, Invent. Math. 10 (1970) 305.

Received by Publishing Department
on April 11, 1986.

Физиев П.П. E4-86-227
Полная интегрируемость в классической задаче трех частиц на прямой линии

Рассмотрены уравнения траектории классической задачи трех частиц на прямой линии с двухчастичным гравитационным или электрическим взаимодействием. Предполагается рассматривать случай нулевой полной энергии системы как полностью интегрируемый в обобщенном смысле, так как в нем понижается порядок дифференциального уравнения траектории, имеется дополнительный первый интеграл и разделяются переменные в уравнении Гамильтона - Якоби, несмотря на то, что система не является полностью интегрируемой по Лиувиллю.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1986

Fiziev P.P. E4-86-227
Complete Integrability in the Classical Problem of Three Particles on a Straight Line

We analyse equations of the trajectory of the classical problem of three particles on a straight line with the two-body gravitational or electric interaction. We propose to consider the case of the zeroth total energy of the system completely integrable in a generalized sense: in this case the order of a differential equation of the trajectory is lowered, an extra first integral exists, and variables in the Hamilton - Jacobi equation can be separated despite that the system is not completely Liouville-integrable.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1986