ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
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ИССЛЕДОВАНИЙ
ДУБНА

E4 - 8543

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1975

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EVALUATION OF FRESNEL'S CORRECTIONS TO THE EIKONAL APPROXIMATION BY THE METHOD OF SEPARABILIZATION

Submitted to *ЯФ*

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The separabilization method is applied widely to solve the quantum-mechanical problem of 3 and more bodies as it leads ultimately to the solution of one-dimensional equations[1-4].

The content of the method is as follows. Let V be an interaction potential that always may be presented as

$$V = \sum_{i,j} V|i\rangle\langle i|V^{-1}|j\rangle\langle j|V \ .$$

Retaining a final number of forms we obtain a separable potential

$$V^{N} = \sum_{i,j} V|i\rangle\langle i|V^{-1}|j\rangle\langle j|V . \tag{1}$$

Convergence of the approximation (1) is provided by the well-known theorem of the functional analysis that a quite continuous operator V may be approximated according to the norm by the operator V^{N} with any accuracy. In this study we modify the presentation (1), viz, change V by the separable

$$V^{s} = \frac{V|\varphi_{k}^{+}\rangle\langle\varphi_{k'}^{-}|V}{\langle\varphi_{k'}^{-}|V|\varphi_{k}^{+}\rangle} . \tag{2}$$

Schrödinger's equation with the potential (2) is solved in explicit form and for the scattering amplitude we obtain

$$f^{s} = \langle k'|V|\varphi_{k}^{+}\rangle C(k,k'). \tag{3}$$

here $\varphi_\kappa^\pm$ are any approximated solutions of Schrödinger's equation

$$|\Psi_k^+\rangle = |k\rangle + G V |\Psi_k^+\rangle$$
$$\langle\Psi_k^-| = \langle k| + \langle\Psi_k^-| V G. \tag{4}$$

Let us formulate now the **iteration** method of solution. In equation (4) we change V by V^S, and as $\varphi^\pm$ we choose any approximated solutions $\chi^\pm$. Then with the potential (2) the equation (4) is solved in explicit form and $\Psi_0^\pm = \Psi_0^\pm(\chi)$ are found. Thereafter supposing in (2) $\varphi^\pm = \Psi_0^\pm$ the equation is solved again, etc.

Without dwelling on the convergence of this process, we note that it leads at every step for the scattering amplitude to Schwinger's variation principle[5] (2). Really, if

$$\varphi^\pm - \Psi^\pm = \varepsilon^\pm \sim \varepsilon, \tag{5}$$

then $f(\kappa', \kappa) - f^S(\kappa', \kappa) \sim \varepsilon^2$.

We shall use further this property widely.

With decreasing particle energy or increasing scattering angle the role of corrections (called Fresnel ones) connected with deviations from the eikonal approximation increases.

It should be remembered that this approximation corresponds to the solution of the Schrödinger equation with approximated Green's function

$$G_0(\vec{r}) = \frac{m}{ik} e^{ikz}\theta(z)\,\delta(\vec{r}_\perp),$$

here $\vec{\kappa} = (0, 0, k)$ $\quad \vec{r}_\perp \perp \vec{K}$. (6)

The solutions of the equation with such Green's function are known

$$|\varphi_\kappa^+\rangle = \exp\left\{ikz - \frac{im}{k}\int_{-\infty}^{z} V(\vec{r}_\perp, z')\,dz'\right\}$$

$$\langle\varphi_\kappa^-| = \exp\left\{-ikz - \frac{im}{k}\int_{z}^{\infty} V(\vec{r}_\perp, z')\,dz'\right\}. \tag{7}$$

Now functions (7) are inserted into (2) and (3).

As a result the amplitude takes the form

$$f_1(\vec{k},\vec{k}') = C(\vec{k},\vec{k}')\, f_0(\vec{k},\vec{k}')\,; \quad f_0 = \langle \vec{k}' | V | \varphi^{+}_{k} \rangle$$
$$C = f_0(\vec{k},\vec{k}')\Big[f_0(\vec{k},\vec{k}') - \int f_0(\vec{k},\vec{k}-\vec{\eta})\, f_0(\vec{k}-\vec{\eta},\vec{k}')\, g(\vec{k},\vec{\eta})\,\frac{d^3\vec{\eta}}{(2\pi)^3}\Big]. \qquad (8)$$

Here $g = G - G_0$ is the difference between the exact and eikonal Green's functions.

It is clear that amplitude $f_1(\vec{k},\vec{k}')$ in formula (8) differs exactly with consideration of Fresnel's corrections.
From the explicit expression for

$$g(\vec{k},\vec{\eta}) = m\,\frac{\eta^2}{(\vec{k}\vec{\eta} + i\epsilon)(2\vec{k}\vec{\eta} - \vec{\eta}^2 + i\epsilon)} \qquad (9)$$

it follows that a) $C \to 1$ for $E \to \infty$

b) with increase of the scattering angle C increases

c) for the explicit calculation of the correction an off-mass shell amplitude in eikonal approximation is necessary.

For example let us view the particle scattering Gaussian potential

$$V(r) = A\exp(-a^2 r^2).$$

Restricting ourselves to the lowest A/E term in the correction actor C we obtain the result

$$C^{-1} = 1 - I,$$

where

$$I = m\int\frac{d^3\eta}{(2\pi)^3}\,\frac{\vec{\eta}^2}{(\vec{k}\vec{\eta} + i\varepsilon)(2\vec{k}\vec{\eta} - \vec{\eta}^2 + i\varepsilon)}\,\frac{V_{|\vec{\Delta}+\vec{\eta}|}\,V_{\eta}}{V_{\Delta}}. \qquad (10)$$

Here

$$\vec{\Delta} = \vec{k}' - \vec{k}\,,\ \vec{\Delta}\vec{k} \simeq 0\,;\ V_{\Delta} = \langle k' | V | k \rangle = \frac{\pi^{3/2} A}{|a|^3}\, e^{-\Delta^2/(2a)^2}.$$

Calculation of integral leads to the functions of the parabolic cylinder. However in limit cases

a) $\Delta/a \ll 1$ and b) $\Delta/a \gg 1$

the result takes a simple form

$$\text{a)}\quad \bar{I} = -\frac{m}{2\pi}\,\frac{A}{2^3\kappa^2}\left(1+\frac{1}{2}\,\frac{\Delta^2}{2a^2}\right), \qquad (11)$$

$$\text{b)}\quad \bar{I} = -\frac{m}{2\pi}\,\frac{A}{2^3\kappa^2}\,e^{\Delta^2/8a^2}\,\frac{\Delta}{2\sqrt{2}\,a}. \qquad (12)$$

It means that the correction may become essential for large scattering angles.

Calculating Glauber's expression in the framework of the potential theory for the scattering amplitude of a rapid particle on the nucleus the following approximations are made(8).

1. The binding and kinetic energy of nucleus nucleons are neglected. This approximation is often reffered to as adiabatic one $(G \to \tilde{G})$.

2. The eikonal approximation is used $(\tilde{G} \to \tilde{G}_0)$. Here G is exact Green's function.

In the adiabatic approximation Schrödinger's equation may be written in integral form (4). Therefore the results may be applied.

Let us view an example of the elastic πd scattering.

In this case

$$\tilde{G}(r,R) = \delta(\vec{r})\int\frac{d^3p}{(2\pi)^3}\,e^{i\vec{p}\vec{R}}\left[-\frac{\vec{k}^2-\vec{p}^2}{2m}-\frac{(\vec{k}-\vec{p})^2}{2M}+i\varepsilon\right]^{-1}. \qquad (13)$$

Here $\vec{k}, m$ ($\vec{k}', \bar{m}$) are the momentum and mass of the incident (scattered) meson. M is deuteron mass. The difference between $\tilde{G}$ and $\tilde{G}_0$ is

$$g(R,r) = \delta(\vec{r})\frac{m^2}{2\mu}\int\frac{d^3q}{(2\pi)^3}\,e^{i(\vec{k}-\vec{q})\vec{R}}\,g_0(\vec{k},\vec{q})$$

$$g(\vec{k},\vec{q}) = \frac{m^2}{2\mu}\,g_0(\vec{k},\vec{q}),\quad g(\vec{k},\vec{q}) = \frac{m^2}{2\mu}\,\frac{\vec{q}^{\,2}}{(\vec{k}\vec{q}+i\varepsilon)(\vec{k}\vec{q}-\vec{q}^{\,2}+i\varepsilon)}, \qquad (14)$$

where $\mu = \frac{mM}{m+M}$

The expression for the scattering amplitude on fixed centres that includes all Fresnel's correction is as follows

$$f_1(\vec{\kappa},\vec{\kappa}',\vec{\tau}) = C(\vec{\kappa},\vec{\kappa}',\vec{\tau})\, f_0(\vec{\kappa},\vec{\kappa}',\vec{\tau})$$
$$C = f_0(\vec{\kappa},\vec{\kappa}';\vec{\tau})\left[f_0(\vec{\kappa},\vec{\kappa}',\vec{\tau}) - \int\frac{d^3q}{(2\pi)^3} f_0(\vec{\kappa},\vec{\kappa}-\vec{q},\vec{\tau}) f_0(\vec{\kappa}-\vec{q},\kappa';\vec{\tau})\, g(\vec{\kappa},\vec{q})\right]^{-1}. \tag{15}$$

Here f_0 is an amplitude in eikonal (Glauber's) approximation. Averaging then over initial and final wave functions we obtain an expression for πd scattering amplitude. To compare with the results of other studies (7-10) (3) let us calculate expression (15) approximately. Supposing the correction to be small we obtain

$$f_1 = f_0 + \int\frac{d^3q}{(2\pi)^3} f_0(\vec{\kappa},\vec{\kappa}-\vec{q},\vec{\tau}) f_0(\vec{\kappa}-\vec{q},\vec{\kappa}';\vec{\tau})\, g(\vec{\kappa},\vec{q}). \tag{16}$$

In the second term in (16) let is take the amplitude in the following approximation

$$f_0 \simeq e^{i\vec{\Delta}\vec{\tau}/2} f_{0\pi n}(\Delta) + e^{-i\vec{\Delta}\vec{\tau}/2} f_{0\pi p}(\Delta). \tag{17}$$

Note that πN amplitudes are to be taken in the eikonal approximation $(f_{0\pi N})$

As a result we have

$$\begin{aligned} f_1 = f_0 &+ e^{i\vec{\Delta}\vec{\tau}/2}\int g(\vec{q}) f_{0\pi n}(\vec{q}) f_{0\pi n}(-\vec{q}+\vec{\Delta})\frac{d^3q}{(2\pi)^3} + \\ &+ e^{-i\vec{\Delta}\vec{\tau}/2}\int g(\vec{q}) f_{0\pi p}(\vec{q}) f_{0\pi p}(-\vec{q}+\vec{\Delta})\frac{d^3q}{(2\pi)^3} + \\ &+ \int g(\vec{q}) f_{0\pi p}(\vec{q}) f_{0\pi n}(-\vec{q}+\vec{\Delta})\, e^{i(\vec{q}+\vec{\Delta}/2)\vec{\tau}}\frac{d^3q}{(2\pi)^3} + \\ &+ \int g(\vec{q}) f_{0\pi n}(\vec{q}) f_{0\pi p}(-\vec{q}+\vec{\Delta})\, e^{i(\vec{q}-\vec{\Delta}/2)\vec{\tau}}\frac{d^3q}{(2\pi)^3}. \end{aligned} \tag{18}$$

It is seen that the 2nd and 3 rd terms simply renormalize the approximation (17) and correspond with Fresnel's corrections, to the amplitudes of the πN scattering. The 4th and 5th terms are Fresnel's corrections renormalizing the rescattering effects. These corrections have been studied in a number of works[7-10]. By any calculation method an evident difficulty is the question of the off-mass shell amplitude parametrization.

The authors express their sincere gratitude to V.B. Belyaev and B.I. Zaharjev for support and interest in this study and to B.Z. Kopeliovich, L.I. Lapidus, G.I. Likasov, A.V. Tarasov for useful discussions on the present study.

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Received by Publishing Department
on January 20, 1975