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GENERAL SUPERSTABLE CLASSICAL GASES. Van Hove Theorem

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1. Introduction

Methods based on the ideas of superstability estimates are coming to play more and more important role both in the studies of the ultraviolet and infrared stability problems in the constructive field theory[1,2].

The original area where the superstability estimates have been discovered in the fundamental paper by Ruelle[3] and for the first so fruitfully applied is the theory of classical gases (see also recent paper on the quantum gases[4]). The fundamental papers of Ruelle deal with the existence questions for the infinite volume free energy and the infinite volume Gibbs measures. However, during the study of these fundamental questions the effects coming from the nontrivial boundary conditions were not taken into account. It is our aim to provide a detailed study of the possible role played by nontrivial boundary conditions.

We will study general multibody superstable and regular interactions from the point of view of the analysis of the corresponding Dobrushin-Lanford-Ruelle equations[5]. Here in the first paper on these topics, we prove the general version of the van Hove theorem including independence of the (reasonable) large class of boundary conditions. In the forthcoming paper[6] we will deal with the general existence and uniqueness questions for the corresponding infinite volume Gibbs measures. The question about independence of the free energy of the boundary condition had arised in our paper[7] where we formulated new criterion for the uniqueness of the infinite volume Gibbs measures for the class of neutral systems of particles interacting via two-body interaction of the positive type. Applications of the results obtained here to the problem considered in [7] will appear in another publication.

Before starting the work let us explain briefly the main idea of what the superstability estimates are. Let φ_Λ denote the configuration of some physical system like spin system, classical lattice of continual gas, euclidean field, etc. enclosed in the finite volume $\Lambda\subset R^d$. Let $\mathcal{H}_\Lambda(\varphi_\Lambda)$ be the Hamiltonian function of the system and let $\Omega(\Lambda)$ be the corresponding space of configurations of the system under consideration. We say that a system is thermodynamically well behaved when we have the equality:

$$\mathrm{Tr}_{\Omega(\Lambda)}\, e^{-\mathcal{H}_\Lambda(\varphi_\Lambda)} = e^{\alpha(\Lambda)|\Lambda|} \qquad (1.1)$$

with $\sup_\Lambda|\alpha(\Lambda)|<\infty$. The main problem is to obtain an identity like (1.1) and this is a place where the essence of the superstability estimates comes in . Let $\Lambda=\Lambda_1\cup\Lambda_2$ with $\Lambda_1\cap\Lambda_2=\phi$. Then, obviously we can write:

$$\mathcal{H}_\Lambda(\varphi_\Lambda)=\mathcal{H}_{\Lambda_1}(\varphi_{\Lambda_1})+\mathcal{H}_{\Lambda_2}(\varphi_2)+\mathcal{H}^{(\Lambda)}_{\Lambda_1,\Lambda_2}(\varphi_{\Lambda_1},\varphi_{\Lambda_2}), \qquad (1.2)$$

where the term $\mathcal{H}^{(\Lambda)}_{\Lambda_1,\Lambda_2}$ expresses the energy of interactions in between the regions Λ_1 and Λ_2. In formula (1.1) the trace is taken over the space $\Omega(\Lambda)$. In many cases we have an obvious factorisation property $\Omega(\Lambda)=\Omega(\Lambda_1)\otimes\Omega(\Lambda_2)$. Assume now that we are able to find some subset $R^{(\Lambda)}(\Lambda_1,\Lambda_2)$ of $\Omega(\Lambda)$ such that

$$\frac{\mathrm{Tr}_{R^{(\Lambda)}(\Lambda_1,\Lambda_2)}\, e^{-\mathcal{H}_\Lambda(\varphi_\Lambda)}}{\mathrm{Tr}_{\Omega(\Lambda)}\, e^{-\mathcal{H}_\Lambda(\varphi_\Lambda)}} > \frac{1}{2} \qquad (1.3)$$

and we can find a bound

$$\inf_{\varphi_{\Lambda_1}\otimes\varphi_{\Lambda_2}\in R^{(\Lambda)}(\Lambda_1,\Lambda_2)} \mathcal{H}^{(\Lambda)}_{\Lambda_1,\Lambda_2}(\varphi_{\Lambda_1},\varphi_{\Lambda_2}) \ge \Psi^{(\Lambda)}(\Lambda_1,\Lambda_2), \qquad (1.4)$$

where $\Psi^{(\Lambda)}(\Lambda_1,\Lambda_2)$ fulfills some requirements to achieve the existence of the thermodynamic limit for $\alpha(\Lambda)$ explained briefly below (see formula 1.6). Using (1.3) and (1.4) we easily conclude that:

$$\mathrm{Tr}_{\Omega(\Lambda)}\, e^{-\mathcal{H}_\Lambda(\varphi_\Lambda)} \le e^{\ln 2}\, e^{-\Psi^{(\Lambda)}(\Lambda_1,\Lambda_2)} \cdot \mathrm{Tr}_{\Omega(\Lambda_1)}\, e^{-\mathcal{H}_{\Lambda_1}(\varphi_{\Lambda_1})}\, \mathrm{Tr}_{\Omega(\Lambda_2)}\, e^{-\mathcal{H}_\Lambda(\varphi_{\Lambda_2})}. \qquad (1.5)$$

This leads to some extended notion of subadditivity for $\alpha(\Lambda)$:

$$\alpha(\Lambda)\le\frac{2}{|\Lambda|}-\frac{\Psi^{(\Lambda)}(\Lambda_1,\Lambda_2)}{|\Lambda|}+\frac{|\Lambda_1|}{|\Lambda|}\alpha(\Lambda_1)+\frac{|\Lambda_2|}{|\Lambda|}\alpha(\Lambda_2)$$

from which (eventually after iterations) we are often able to conclude the existence of the unique thermodynamic limit for $\alpha(\Lambda)$ whenever $\Lambda\uparrow R^d$ in a suitable sense.

Thus, the core of the whole method consists in finding an app-

ropriate decomposition of the configurational space $\Omega(\Lambda)=R^{(\Lambda)}\cup(\Omega(\Lambda)-R^{(\Lambda)})$ on which (1.3) and (1.4) are valid.

Finally, we comment on the organisation of this paper. In the next section we collect some basic definitions and formulate some preliminary technical results which we will need in the following. Also the precise formulation of the main result of this paper is contained in section 2. Section 3 is entirely devoted to the proof of the main result named as general van Hove theorem. In the appendix we elaborate the proof of the so-called superstability estimates working exclusively in the language of the configurational space and the Poisson integration.

2. Introductory Definitions and Results

2.1. Configuration space /8/

Let Σ be some Borel subset of some real, finite-dimensional space R^d. The set Σ will be called the space of charges. On the Borel σ-algebra of sets of Σ there is given a regular measure μ such that $\mu(\Sigma)<\infty$.

In the sets $(R^d\otimes\Sigma)^{\otimes k}$ let us introduce the following equivalence relation $\sim$. We will say that $\omega,\omega'\in(R^d\otimes\Sigma)^{\otimes k}$ are equivalent iff they differ only by the permutation of the elements composing them. For a given $\Lambda\subset R^d$ let $\omega(\Lambda)$ be the restriction of ω to the set Λ. The subset Ω of $2^{(R^d\otimes\Sigma)}$ having the property that $\mathrm{card}|\omega(\Lambda)|=|\omega(\Lambda)|<\infty$ (counting with the multiplicities) for every bounded $\Lambda\subset R^d$ is called the configurational space of the system. The subset $\Omega(\Lambda)$ of Ω is defined by: $\omega\in\Omega(\Lambda)\Leftrightarrow\omega=\omega(\Lambda)$. Then, for every regular $\Lambda\subset R^d$ we have natural decomposition $\Omega=\Omega(\Lambda)\otimes\Omega(\Lambda^c)$ which corresponds to the notation: $\omega=\omega(\Lambda)\vee\omega(\Lambda^c)$ where we denoted $\omega\vee\omega'$ as a union of the elements composing ω and ω'. We have also $\Omega(\Lambda)=\bigcup_{n=0}^{\infty}\Omega^n(\Lambda)$ where $\Omega^n(\Lambda)=\{\omega\in\Omega(\Lambda)\,|\,|\omega|=n\}$. Moreover, the set $\Omega^k(\Lambda)$ can be identified with $(R^d\otimes\Sigma)^{\otimes k}/\sim$. Therefore, it is possible to transform measureble and topological structures of $(R^d\otimes\Sigma)^{\otimes k}/\sim$ into the set $\Omega^k(\Lambda)$ and then also into $\Omega(\Lambda)$. The corresponding σ-algebras are denoted by $\mathcal{F}(\Lambda)$. We remark that the measure and the topological structures coincide. By $\mathcal{F}$ we denote the σ-algebra $\mathcal{F}(R^d)$.

Let a number $\delta>0$ be given. Then, we define δ-lattice $Z_\delta=\{x\in R^d\,|\,x_i=n_i\frac{\delta}{2},\ i=1,\dots,d\}$ where n_i are integers, δ-cube:

$$\square_\delta(n)=\{x\in R^d\,|\,n_i\delta-\frac{\delta}{2}\le x_i<n_i\delta+\frac{\delta}{2}\}$$

and δ-covers:

$$C_\delta(R^d) = \bigcup_{\underline{r}} \square_\delta(\underline{r}),$$
$$C_\delta(\Lambda) = \bigcup_{\square_\delta(\underline{r})\cap\Lambda\neq\phi} \square_\delta(\underline{r}).$$

A subset Λ of R^d will be called the regular δ-polygon iff Λ is connected, one connected and $\Lambda = \bigcup_{\underline{r}} \square_\delta(\underline{r})$. For a given $\omega \in \Omega$ we denote by $\omega_\delta(\underline{r})$ the restriction of ω to $\square_\delta(\underline{r})$ and by $n_\delta(\omega,\underline{r})$ the cardinality of $\omega_\delta(\underline{r})$.

2.2. <u>Free systems</u>/8/

Let us consider a system of cylindrical sets $C_\Lambda^n = \{\omega\in\Omega \mid \omega(\Lambda) = n\}$ where n runs over integers and Λ over bounded regular subsets of R^d. This system of sets with fixed Λ generates then some σ-algebra of sets $\mathcal{F}'(\Lambda)$. On the generators C_Λ^n of this σ-algebra we then define a measure

$$\bar{\lambda}_{0,\Lambda}(C_\Lambda^n) = \frac{1}{n!}\,\lambda(\Lambda)^n\Big(\int_\Sigma \mu(d\alpha)\Big)^n, \tag{2.2.1}$$

where λ is some Borel and regular measure on R^d. The system $\{\mathcal{F}'(\Lambda), \bar{\lambda}_{0,\Lambda}\}$ defines the projective family of measure spaces whose projective limit can be defined on some $\{\Omega, \mathcal{F}', \bar{\lambda}_0\}$. It is easy to see that one can identify $\mathcal{F}'(\Lambda)$ with $\mathcal{F}(\Lambda)$. The measure $\bar{\lambda}_{0,\Lambda}$ has the remarkable property: if $\Delta\subset\Lambda$ then $\bar{\lambda}_{0,\Lambda} = \bar{\lambda}_{0,\Delta}\otimes\bar{\lambda}_{0,\Lambda-\Delta}$. The most popular choice of λ is the Lebesque measure multiplied by some positive constant z called chemical activity. This case we denote as $\pi^z_{0,\Lambda}$. The different choice of λ leads to the description of the noninteracting system in some external field. The system $\{\Omega, \Omega(\Lambda), \mathcal{F}(\Lambda), \bar{\lambda}_{0,\Lambda}\}$ will be called the free system. For the completeness of our exposition let us note

$$\bar{\lambda}_{0,\Lambda}(\Omega(\Lambda)) = \exp\Big(\lambda(\Lambda)\big(\int_\Sigma d\mu(\alpha)\big)\Big)$$

which follows easily via the mentioned identifications. In this paper we choose $\bar{\lambda}_{0,\Lambda}$ as equal to $\pi^z_{0,\Lambda}$. However, our results are valid for the more general choice of λ as well.

2.3. <u>Interactions</u>

Any measurable function $\mathcal{E}: \Omega\to(-\infty,+\infty)$ will be called interaction and value of $\mathcal{E}$ at the given $\omega\in\Omega$ will be called the energy of the configuration ω. For the statistical mechanics the most interesting interactions are those which are stable.

Def. 2.3.1.

a) An interaction $\mathcal{E}$ is stable $\Longleftrightarrow$

$$\underset{B\in R}{\exists} : \underset{\omega\in\Omega}{\forall} \quad \mathcal{E}(\omega) \geq -|\omega|\cdot B \tag{2.3.1}$$

b) An interaction $\mathcal{E}$ is superstable

$$\underset{\delta}{\forall} : \underset{\substack{A_\delta>0\\ B_\delta\in R}}{\exists} \quad \underset{\omega\in\Omega}{\forall} \quad \mathcal{E}(\omega) \geq \sum_{\underline{r}\in C_\delta(R^d)} A_\delta\, n_\delta^2(\omega,\underline{r}) + \sum_{\underline{r}\in C_\delta(R^d)} B_\delta\, n_\delta(\omega,\underline{r}) \tag{2.3.2}$$

Remark 2.3.1.

The constant A_δ is in general dependent on δ. However, when $\mathcal{E}$ is superstable on some scale δ then it is superstable on any scale $\delta'>0$. For example, taking $\delta=1$ and $\delta'=k^{-n}$ where $k,n\in N$. Then, assuming superstability of the given interaction $\mathcal{E}$ with the constants A_1 and B_1, we obtain:

$$\mathcal{E}(\omega) \geq k^{-nd}A_1 \sum_{\underline{r}\in C_\delta(R^d)} n_\delta^2(\omega,\underline{r}) + \sum_{\underline{r}\in C_\delta(R^d)} B_1 n_\delta(\omega,\underline{r}) \tag{2.3.3}$$

Def. 2.3.2.

A given interaction $\mathcal{E}$ is regular iff there exists a continuous, positive and decreasing function $\Psi: (0,\infty)\to[0,\infty)$ such that

$$\int_0^\infty r^{d-1}\,\Psi(r)\,dr < \infty \tag{2.3.4}$$

and for any two configurations $\omega_1,\omega_2\in\Omega$ their interaction energy $\mathcal{E}(\omega_1,\omega_2)$ defined by:

$$\mathcal{E}(\omega_1,\omega_2) \equiv \mathcal{E}(\omega_1\vee\omega_2) - \mathcal{E}(\omega_1) - \mathcal{E}(\omega_2)$$

can be estimated as follows:

$$|\mathcal{E}(\omega_1,\omega_2)| \leq \frac{1}{2}\sum_{\underline{r}\in Z_\delta}\sum_{\underline{s}\in Z_\delta} \Psi(|\underline{r}-\underline{s}|)\big(n_\delta^2(\omega_1,\underline{r}) + n_\delta^2(\omega_2,\underline{s})\big) \tag{2.3.5}$$

Remark 2.3.2.

Having in mind decomposition $\Omega = \bigcup_{k=0}^\infty \Omega^k$ we can often write:

$$\mathcal{E}(\omega) = \sum_{k=0}^\infty \sum_{\omega_k\subset\omega,\,|\omega_k|=k} \mathcal{E}_k(\omega_k).$$

Then the functions $\mathcal{E}_k$ are called k-particle interactions. The k-particle interaction $\mathcal{E}_k$ can be defined on the space $(R^d\otimes\Sigma)^{\otimes k}/\sim$ and then can be naturally extended to the space $(R^d\otimes\Sigma)^{\otimes k}$. These extensions are called k-particle potentials. It is an interesting and largely open (up to the author's present knowledge) to describe a space of regular and superstable k-potentials with $k>2$. For chargeless systems and $k=2$ some criteria for the (lower) regularity and superstability are known /9/.

Remark 2.3.3.

For the chargeless systems the concept of the lower regularity should be sufficient for the purposes of the present paper. However, it is because of our possible applications that we use here the more restrictive notion of the regularity.

Remark 2.3.4.

In the case when the function Ψ from the definitinn 1.3.2. has asymptotic like $r^{-d-\epsilon}$ with some $\epsilon>0$ as $r\uparrow\infty$, then we say that the interactinn $\mathcal{E}$ is strongly regular. The strong regularity will be needed for the proof of the main result of the present paper (see 1.7).

Definition 2.3.3.

Define the (measurable) action of the translational group $E(d)$ on Ω by:

$$T_a\omega = \omega_a = ((x_1+a,\alpha_1),\ldots,(x_n+a,\alpha_n),\ldots)$$

The interaction $\mathcal{E}$ is translationally invariant iff:

$$\mathcal{E}(\omega_a) = \mathcal{E}(\omega)$$

Similarly we can define the other symmetries of the given interaction $\mathcal{E}$.

2.4. Finite volume equilibrium Gibbs measures and superstability estimates

The finite volume equilibrium Gibbs measure ν_Λ corresponding to the interaction $\mathcal{E}$ is defined on $\{\Omega(\Lambda),\mathcal{F}(\Lambda)\}$ by the formula

$$\nu_\Lambda(d\omega) = (Z_\Lambda(z))^{-1}\exp(-\mathcal{E}(\omega))\,\pi^z_{0,\Lambda}(d\omega). \tag{2.4.1}$$

For at least stable interactions $\mathcal{E}$ this formula has well defined sense. Here $Z_\Lambda(z)$ is the finite volume partition function

$$Z_\Lambda(z) = \int_{\Omega(\Lambda)}\pi^z_{0,\Lambda}(d\omega)e^{-\mathcal{E}(\omega)} = e^{P_\Lambda(z)|\Lambda|}, \tag{2.4.2}$$

where $P_\Lambda(z)$ is the free energy density of the finite volume system. For any regular $\Delta\subset\Lambda$ we can define a finite volume measure $\nu^{(\Lambda)}_\Delta(d\omega(\Delta))$ as a projection of ν_Λ onto $\{\Omega(\Delta),\mathcal{F}(\Delta)\}$ by the formula:

$$\nu^{(\Lambda)}_\Delta(d\omega(\Delta)) = (Z_\Lambda(z))^{-1}\int_{\Omega(\Lambda-\Delta)}\pi^z_{0,\Lambda}(d\omega)e^{-\mathcal{E}(\omega(\Delta)\vee\omega(\Lambda-\Delta))} \tag{2.4.3}$$

The measure $\nu^{(\Lambda)}_\Delta$ is then absolutely continuous with respect to the measure $\pi^z_{0,\Delta}$ with the corresponding Radon-Nikodym derivative:

$$g^{(\Lambda)}_\Delta(\omega(\Delta)) = \frac{d\nu^{(\Lambda)}_\Delta}{d\pi^z_{0,\Delta}}(\omega(\Delta)) = Z^{-1}\int_{\Omega(\Lambda-\Delta)}\pi^z_{0,\Lambda-\Delta}(d\omega(\Lambda-\Delta))e^{-\mathcal{E}(\omega(\Delta)\vee\omega(\Lambda-\Delta))} \tag{2.4.4}$$

The basic technical results used in this and the following papers are the estimates called after Ruelle as superstability estimates.

Theorem 2.4.1.

Let the interaction $\mathcal{E}$ be superstable and regular. Then, there exist constants $\gamma>0$, $\delta\in R$ such that for every regular $\Delta\subset\Lambda$ and $\omega\in\Omega(\Lambda)$ we have:

$$\rho^{(\Lambda)}_\Delta(\omega(\Delta)) \le \exp-\Big(\sum_{r\in\mathcal{C}_\delta(\Delta)}\gamma n^2_\delta(\omega(\Delta),r)+\rho\sum_{r\in\mathcal{C}_\delta(\Delta)}n_\delta(\omega(\Delta),r)\Big). \tag{2.4.5}$$

The function $\rho^{(\Lambda)}_\Delta(\omega(\Delta))$ measures the probability of finding a finite volume system in a state $\omega(\Delta)$ whotever the state $\omega(\Lambda-\Delta)$ is. Again the constants γ and ρ are δ-dependent. It is not hard to see that $\gamma\downarrow 0$ as $\delta\downarrow 0$. The obvious corollaries of this theorem are the following ones.

Corollary 2.4.2.

Let the interaction $\mathcal{E}$ be superstable and regular. Define the set:

$$\Omega_N(\Lambda,\Delta) = \{\omega\in\Omega(\Delta)\,|\,\sum_{r\in\mathcal{C}_\delta(\Delta)}n^2_\delta(\omega,r)\ge N^2|\Delta|\} \tag{2.4.6}$$

with N integer.

There exist constants $\gamma'>0$ and $\rho'\in R$ such that

$$\nu_\Lambda(\{\Omega_N(\Lambda,\Delta)\}) \le \exp(-(\gamma' N^2+\rho')|\Delta|). \tag{2.4.7}$$

Corollary 2.4.3.

If additionally to the hypothesis of Theorem 2.4.1 we assume that the interaction $\mathcal{E}$ is translationally invariant, then the constants γ,γ',ρ and ρ' can be chosen independent of Λ. We elaborate the proofs of these results in the appendix to this paper.

2.5. Infinite volume Gibbs measures

Of special interest is weak (sub) limit $\lim_{\Lambda\uparrow R^d}\nu_\Lambda = \nu_\infty$ of the finite volume Gibbs measure ν_Λ. It is an obvious consequence of Cor. 2.4.2 and Cor. 2.4.3 that the infinite volume measure ν_∞ (whenever exists) is then supported on the set

$$R = \bigcup_{N=1}^{\infty} R_N \qquad \text{where}$$

$$R_N = \{\omega \in \Omega \mid \sum_{1 \le |s| \le l} n^2(\omega, r) \le N(2l+1)^d\} \tag{2.5.1}$$

This enables us to define the conditioned by $\omega \in R$ finite volume Gibbs measures. For a given bounded $\Lambda \subset R^d$ let $\Lambda_n(\Lambda)$ be a sequence of bounded subsets of $R^d - \Lambda$ which tends to $R^d - \Lambda$ monotonically and by inclusion. If the interaction $\mathcal{E}$ is superstable and regular then for any $\tilde{\omega} \in \Omega(\Lambda)$ we can define:

$$\mathcal{E}(\tilde{\omega}, \omega(\Lambda^c)) \equiv \lim_{n \uparrow \infty} \mathcal{E}(\tilde{\omega}, \omega(\Lambda_n))$$

as a measurable function on the set R . After Dobrushin (see also /3/) we then define a conditioned (by ω) Gibbs measure in the finite volume Λ on the space $\{\Omega(\Lambda), \mathcal{F}(\Lambda)\}$ by:

$$\nu_\Lambda(-|\omega) = (Z_\Lambda(\omega))^{-1} e^{-\mathcal{E}(-)} e^{-\mathcal{E}(-|\omega(\Lambda^c))} \pi^z_{0,\Lambda}(-),$$

$$Z_\Lambda(\omega) = \int_{\Omega(\Lambda)} \pi^z_{0,\Lambda}(d\eta) e^{-\mathcal{E}(\eta)} e^{-\mathcal{E}(\eta|\omega(\Lambda^c))} \tag{2.5.2}$$

The set $\mathcal{G}(\mathcal{E}, \bar{\lambda}_0)$ of probabilistic Borel measures on $\{\Omega, \mathcal{F}\}$ such that their conditioned at the σ -algebras $\mathcal{F}(\Lambda^c)$ variants where Λ runs over the bounded and regular subsets of R^d are given exactly by formula (2.5.2) is called the set of infinite volume Gibbs (ground canonical) measures corresponding to the interaction $\mathcal{E}$ and free density $\bar{\lambda}_0$. The subset $\mathcal{G}^T(\mathcal{E}, \bar{\lambda}_0)$ of $\mathcal{G}(\mathcal{E}, \bar{\lambda}_0)$ composed of the Gibbs measures which are supported on the set R is called the set of tempored infinite volume Gibbs measures.

2.6. Boundary conditions

The full set $\mathcal{G}^T(\mathcal{E}, \bar{\lambda}_0)$ can be obtained by taking the thermodynamic limits of certain convex superpositions of the conditioned finite volume Gibbs measures $\nu_\Lambda(-|\cdot)$.

In particular, the limits $\lim_{\Lambda} \nu_\Lambda(-|\omega)$ (whenever they exist) belong to the set $\mathcal{G}(\mathcal{E}, \bar{\lambda}_0)$. This corresponds in our terminology to the pure boundary case of which a subcase $\omega = \phi$ is called the empty boundary condition case.

Let λ^{bc} be a Borel probability measure on $\{\Omega, \mathcal{F}\}$ such that:

i) the projections $\lambda^{bc}_\Lambda(d\omega(\Lambda))$ on $\{\Omega(\Lambda), \mathcal{F}(\Lambda)\}$ are absolutely continuous with respect to $\pi^z_{0,\Lambda}$;

ii) the corresponding Radon-Nikodym derivatives g^λ_Λ obey the superstability estimate of the Theorem 2.4.1.

Any measure λ^{bc} with the properties i) and ii) realizes regular, tempered boundary condition in the following manner:

$$p_\Lambda(\lambda^{bc}) = -\frac{1}{|\Lambda|} \int_{\Omega(\Lambda^c)} \lambda^{bc}(d\omega) \ln Z_\Lambda(\omega) \tag{2.6.1}$$

and

$$\rho^{(\Lambda)}_\Delta(\eta(\Delta)|\lambda^{bc}) = \int_{\Omega(\Lambda^c)} \lambda^{bc}(d\omega) \rho^{(\Lambda)}_\Delta(\eta(\Delta)|\omega(\Lambda^c)). \tag{2.6.2}$$

The detailed study of the thermodynamic limits $\lim_{\Lambda \uparrow R^d} \rho^{(\Lambda)}_\Delta$ will be presented in the forthcoming paper. In this paper we will concentrate on the free energy density $p_\Lambda(\lambda^{bc})$.

2.7. Main result

The result we prove in this paper is formulated in the following theorem.

Van Hove Theorem

Let $\mathcal{E}$ be superstable, strongly, regular and translationally invariant interaction. Let λ^{bc} realise the regular and tempered boundary condition. Let $\{\Lambda_n\}$ be a sequence of δ -polygonal bounded regions in R^d and such that $\Lambda_n \uparrow R^d$ in the sense of van Hove. Then, the unique thermodynamic limit:

$$\lim_{n \uparrow \infty} -\frac{1}{|\Lambda_n|} p_{\Lambda_n}(\lambda^{bc})$$

exists and is equal to lim

$$\lim_{n \uparrow \infty} -\frac{1}{|\Lambda_n|} p_{\Lambda_n}(z)$$

3. Proof of the Van Hove Theorem

3.1. Introductory remarks

Let a regular δ -polygonial region Λ be given. Assume that $\Lambda = \Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2 = \phi$, where Δ_i are again some δ -polygonal sets. Then, we define inductively the following sets. Let

$$V^{(\Lambda)}_1(\Delta_1, \Delta_2) = \max\{\sum_{s \in C_\delta(\Delta_2)} \Psi(|r - s|) ; r \in C_\delta(\Delta_1)\} \tag{3.1.1}$$

then a set $\bar{\sigma}_1(\Delta_1, \Delta_2)$ is defined by

$$\bar{\sigma}_1(\Delta_1, \Delta_2) = \{r \in \Delta_1 \mid \sum_{s \in C_\delta(\Delta_2)} \Psi(|r - s|) = V^{(\Lambda)}_1(\Delta_1, \Delta_2)\}. \tag{3.1.2}$$

Having defined $V^{(\Lambda)}_n$ and $\bar{\sigma}_n$ we define

$$V^{(\Lambda)}_{n+1}(\Delta_1, \Delta_2) \equiv \max\{\sum_{s \in C_\delta(\Delta_2)} \Psi(|r - s|), r \in \Delta_1 - \bar{\sigma}_n\} \tag{3.1.3}$$

and then

$$\overline{\sigma}_{n+1}(\Delta_1,\Delta_2)=\{r\in\Delta_1 \mid \sum_{s\in C_\delta(\Delta_2)} \Psi(|r-s|)\geq V^{(\Lambda)}_{n+1}(\Delta_1\Delta_2)\}. \quad (3.1.4)$$

From this definition it follows:

i) $V_1^{(\Lambda)}(\Delta_1,\Delta_2) > V_2^{(\Lambda)}(\Delta_1,\Delta_2) >$.

ii) $\overline{\sigma}_1(\Delta_1,\Delta_2) \subseteq \overline{\sigma}_2(\Delta_1,\Delta_2) \subseteq \dots \subseteq \Delta_1$

iii) $|\overline{\sigma}_n(\Delta_1,\Delta_2)| \geq n$.

iv) for bounded Δ_1 induction is finite and ends when $\overline{\sigma}_n=\Delta_1$.

Similarly we can define $V_n^{(\Lambda)}(\Delta_2,\Delta_1)$ and $\overline{\sigma}_n(\Delta_2,\Delta_1)$. The sets $\overline{\sigma}_n$ then induce definitions of some nonzero v_Λ-measure subsets of $\Omega(\Lambda)$:

$$\Omega^{(\Lambda)}_{N,n}(\Delta_1,\Delta_2)=\{\omega\in\Omega(\Lambda) \mid \sum_{r\in\overline{\sigma}_n(\Delta_1\Delta_2)} n^2(\omega(\Delta,r)\leq N^2|\overline{\sigma}_n(\Delta_1,\Delta_2)|\} \quad (3.1.5)$$

and similarly for $\Omega^{(\Lambda)}_{N,n}(\Delta_2,\Delta_1)$.

Finally, we define

$$\Omega^{(\Lambda)}_{N}(\Delta_1;\Delta_2)=(\bigcap_n \Omega^{(\Lambda)}_{N,n})\cap(\bigcap_n \Omega^{(\Lambda)}_{N,n}). \quad (3.1.6)$$

The above definitions result in the following Lemma.

Lemma 3.1.1.

Let the interaction $\mathcal{E}$ be regular. Then the following estimates are valid.

1) for any $\omega\in\Omega^{(\Lambda)}_{N}(\Delta_1,\Delta_2)$ we have:

$$\mathcal{E}(\omega(\Delta_1),\omega(\Delta_2))\leq \frac{N^2}{2}\sum_{r\in C_\delta(\Delta_1)}\sum_{s\in C_\delta(\Delta_2)}\Psi(|r-s|) \quad (3.1.7)$$

$$+\frac{1}{2}\sum_{r\in C_\delta(\Delta_1)}\sum_{s\in C_\delta(\Delta_2)}\Psi(|r-s|)\,n^2_\delta(\omega(\Delta_2),s).$$

2) for any $\omega\in\Omega^{(\Lambda)}_{N}(\Delta_1;\Delta_2)$ we have:

$$\mathcal{E}(\omega(\Delta_1),\omega(\Delta_2))\leq N^2\sum_{r\in C_\delta(\Delta_1)}\sum_{s\in C_\delta(\Delta_2)}\Psi(|r-s|). \quad (3.1.8)$$

Proof:

ad 1)

$$\frac{1}{2}\sum_{r\in C_\delta(\Delta_1)}\sum_{s\in C_\delta(\Delta_2)}\Psi(|r-s|)\,n^2_\delta(\omega(\Delta_1),r)$$

$$\leq \frac{1}{2}\sum_{k=1} V_k^{(\Lambda)}(\Delta_1,\Delta_2)\sum_{r\in\overline{\sigma}_k} n^2_\delta(\omega(\Delta_1),r)+\frac{1}{2}|\overline{\sigma}_1|V_1^{(\Lambda)}(\Delta_1,\Delta_2)$$

$$\leq \frac{N^2}{2}(V_k^{(\Lambda)}(\Delta_1,\Delta_2)|\overline{\sigma}_1|+\frac{1}{2}\sum_k V_k^{(\Lambda)}(\Delta_1,\Delta_2)|\overline{\sigma}_k|)$$

$$\leq \frac{N^2}{2}\sum_{s\in C_\delta(\Delta_2)}\Psi(|r-s|).$$

The case of 2 then follows immediately. The following Lemma will play a crucial role in the following.

Lemma 3.3.2.

Let $\mathcal{E}$ be superstable and regular interaction defined on $\Omega(\Lambda)$. Let $\Lambda=\Delta_1\cup\Delta_2$ be as above. Then, there exists an integer N independent of Δ_1 and Δ_2 such that:

$$v_\Lambda(\{\Omega^{(\Lambda)}(\Delta_1;\Delta_2)\})>\frac{3}{4}$$

Proof:

From the superstability estimates (Th. 2.4.1) it follows:

$$v_\Lambda(\{\Omega^{(\Lambda)}(\Delta_1,\Delta_2)\})$$

$$\leq\sum_j v_\Lambda(\{\Omega^{(\Lambda)}_{N,j}(\Delta_1,\Delta_2)\})$$

$$\leq\sum_j \exp(-(\gamma' N^2-\delta')_j)<\infty$$

Moreover, the last sum tends to zero when $N\to\infty$.

q.e.d.

3.2. Empty boundary condition case

We start with the proof of the existence of infinite volume free energy in the empty boundary case. This is essentially proved already in the Ruelle paper /3/.

Theorem 3.2.1.

Let $\mathcal{E}$ be a superstable, translationally invariant and regular interaction. Let $\{\Lambda_n\}$ be a sequence of regular δ-polygonal subsets of R^d such that $\Lambda_n\uparrow R^d$ in the sense of van Hove.

Then, the unique thermodynamic limit exists

$$p_\infty(z)=\lim_{n\uparrow\infty}-\frac{1}{|\Lambda_n|}\ln Z_{\Lambda_n}(z).$$

Proof:

Let us decompose $\Lambda_n = \Delta_1 \cup \Delta_2$ into two regular, disjoint δ -polygonal regions. Choose the integer N such that Lemma 3.1.1. holds. Then we have:

$$Z_\Lambda(z) = \int_{\Omega_N^{(\Lambda)}(\Delta_1;\Delta_2)} \pi^z_{o,\Lambda}(\omega(\Lambda)) e^{-\mathcal{E}(\omega(\Lambda))} + \int_{\Omega(\Lambda)-\Omega_N^{(\Lambda)}(\Delta_1;\Delta_2)} \pi^z_{o,\Lambda}(d\omega(\Delta)) e^{-\mathcal{E}(\omega(\Lambda))}$$

$$\leq e^{N^2 \sum_{r\in\Delta_1} \sum_{s\in\Delta_2} \Psi(|r-s|)} \int_{\Omega_N^{(\Lambda)}(\Delta_1,\Delta_2)} \pi^z_{o,\Lambda}(d\omega(\Lambda)) e^{-\mathcal{E}(\omega(\Delta_1))} e^{-\mathcal{E}(\omega(\Delta_2))} + \frac{1}{2} Z_\Lambda \tag{3.2.1}$$

This leads to the upper bound of the form:

$$Z_\Lambda(z) \leq 2 Z_{\Delta_1} \cdot Z_{\Delta_2} \exp\Big(N^2 \sum_{r\in\Delta_1} \sum_{s\in\Delta_2} \Psi(|r-s|)\Big). \tag{3.2.2}$$

To obtain a lower bound on Z_Λ we proceed similarly:

$$Z_\Lambda(z) \geq \int_{\Omega_N^{(\Lambda)}(\Delta_1;\Delta_2)} \pi^z_{o,\Lambda}(d\omega(\Lambda)) e^{-\mathcal{E}(\omega(\Lambda))}$$

$$\geq e^{-N^2 \sum_{r\in\Delta_1} \sum_{s\in\Delta_2} \Psi(|r-s|)} \cdot \int_{\Omega_N^{(\Lambda)}(\Delta_1,\Delta_2)} \cdot \pi^z_{o,\Lambda}(d\omega(\Lambda)) e^{-\mathcal{E}(\omega(\Delta_1))} e^{-\mathcal{E}(\omega(\Delta_2))} \tag{3.2.3}$$

$$\geq e^{-N^2 \sum_{r\in\Delta_1} \sum_{s\in\Delta_2} \Psi(|r-s|)} \cdot Z_{\Delta_1} \cdot Z_{\Delta_2}$$

$$- \int_{(\Omega_N^{\Lambda}(\Delta_1;\Delta_2))^c} \pi^z_{o,\Lambda}(d\omega(\Lambda)) e^{-\mathcal{E}(\omega(\Delta_1))} e^{-\mathcal{E}(\omega(\Delta_2))} e^{-N^2 \sum_{r\in\Delta_1} \sum_{s\in\Delta_2} \Psi(|r-s|)}$$

(from the Lemma 1)

$$\geq Z_{\Delta_1} \cdot Z_{\Delta_2} e^{-N^2 \sum_{r\in\Delta_1} \sum_{s\in\Delta_2} \Psi(|r-s|)} - \frac{1}{2} Z_{\Delta_1} \cdot Z_{\Delta_2} e^{-N^2 \sum_{r\in\Delta_1} \sum_{s\in\Delta_2} \Psi(|r-s|)}$$

from which we get lower bound of the form:

$$Z_\Lambda \geq 2^{-1} Z_{\Delta_1} \cdot Z_{\Delta_2} e^{-N^2 \sum_{r\in\Delta_1} \sum_{s\in\Delta_2} \Psi(|r-s|)}. \tag{3.2.4}$$

Now we iterate the above procedures. Let now Λ be decomposed into n disjoint, regular δ -polygonal regions Δ_i such that $\bigcup_1^n \Delta_i = \Lambda$. Then by a simple induction we get:

$$\prod_{i=1}^n 2^{-1} Z_{\Delta_i} e^{-N^2 \sum_{r\in\Delta_i} \sum_{s\notin\Delta_i} \Psi(|r-s|)} \leq Z_\Lambda \leq \prod_{i=1}^n 2 Z_{\Delta_i} e^{\frac{N^2}{2} \sum_{r\in\Delta_i} \sum_{s\notin\Delta_i} \Psi(|r-s|)} \tag{3.2.5}$$

From this estimate the proof then follows immediately.

q.e.d.

Remark 3.2.2.

It seems to be possible to extend this result to the situation, where $\{\Lambda_n\}$ is an arbitrary sequence of bounded subsets such that $\partial\Lambda_n$ are integrable and moreover $\Lambda_n \uparrow R^d$ in the sense of van Hove. The idea is the following. For any n let δ_n be a δ_n-cover of Λ_n. Let then $\Lambda^-_{\delta,n} = \bigcup_\Omega \square_{\delta,\Omega}$ where $\square_{\delta,n} \subset \Lambda_n$ and $\Lambda^+_{\delta,n} = \bigcup_\Omega \square_\delta(\Omega)$ where at least $\square_\delta(\Omega) \cap \Lambda_n \neq \emptyset$. Then, we have two sequences of the regular, δ_n -polygonal regions $\{\Lambda^-_{\delta,n}\}$ and $\{\Lambda^+_{\delta,n}\}$ which both tend to R^d in the sense of van Hove. In applying the arguments used to prove Th. 3.1. we need some control of the N which now became to be n-dependent (see remark 1.3.1). It is plausilible to choose such δ_n that the proof can be applied to both sequences $\{\Lambda^-_{n,\delta_n}\}$ and $\{\Lambda^+_{n,\delta_n}\}$. Then the "three-sequence Lemma" will give a proof for the sequence $\{\Lambda_n\}$, but we have not checked the details yet.

3.3. Pure boundary condition case

Now we proceed to the proof of van Hove theorem in the case of pure boundary condition. For this we choose $\omega \in \Omega(a)$ where the set $\Omega(a)$ defined below is of measure one with respect to any Gibbs measure which fulfills superstability estimates. For a given regular, δ -polygonal bounded region Λ we define its B -boundary as:

$$\partial_B(\Lambda) = \Big\{x \in \Lambda \;\Big|\; \sum_{r\in C_\delta((\Lambda\cup\partial_B(\Lambda))^c)} \Psi(|r-x|) < 2B\Big\}.$$

This definition does not determine B -boundary uniquely. However, when we have sequence $\{\Lambda_n\}$of the regular, δ -polygonal, bounded subsets, then we can define sequence of B -boundaries $\{\partial_B(\Lambda_n)\}$ in such a way that $\lim_{n\uparrow\infty} |\partial_B(\Lambda_n)|/|\Lambda_n| = 0$ and $\{\partial_B(\Lambda_n)\}$ are δ -polygonals. The proof goes essentially as in the empty boundary case. We are looking for the appropriate lower and upper bounds for $Z_\Lambda(\omega)$.

Upper bound

Choose $B < A$ where A is the superstability constant.

$$Z_\Lambda(\omega) = e^{\mathcal{E}(\omega(\partial_B(\Lambda)))} e^{-\mathcal{E}(\omega(\partial_B(\Lambda)))} Z_\Lambda(\omega)$$

$$= e^{\mathcal{E}(\omega(\partial_B(\Lambda)))} \int_{\Omega(\Lambda)} \bar{\lambda}_{0,\Lambda}(d\eta)\, e^{-\mathcal{E}(\eta \vee \omega(\partial_B(\Lambda)))} e^{-\mathcal{E}(\eta, \omega(\partial_B(\Lambda)))} \cdot e^{-\mathcal{E}(\eta, \omega(\Lambda^c))}. \tag{3.3.1}$$

Let $\Lambda' = \Lambda \cup \partial_B(\Lambda)$. Then on $\{\Omega(\Lambda'), \mathcal{F}(\Lambda')\}$ we can define a new measure $\nu_\Lambda^{+,B}$ (indexed by $\omega(\partial_B(\Lambda))$) by:

$$\nu_\Lambda^{+,B}(d\eta(\Lambda')) = (Z_\Lambda^+(\omega))^{-1}\, \bar{\lambda}_{0,\Lambda} \otimes \delta_\omega(\eta(\partial_B(\Lambda)))\, e^{-\mathcal{E}_\omega^+(\eta(\Lambda'))}, \tag{3.3.2}$$

where δ_ω is the δ-measure concentrated at $\eta(\partial_B(\Lambda)) = \omega(\partial_B(\Lambda))$, i.e., for any $\mathcal{F}(\partial_B(\Lambda))$ measurable and integrable function F we have:

$$\int_{\Omega(\partial_B(\Lambda))} \delta_\omega(\eta(\partial_B(\Lambda)))F(\eta) \equiv F\,\omega(\partial_B(\Lambda)); \tag{3.3.3}$$

$\mathcal{E}_\omega^+$ is the new interaction on $\Omega(\Lambda')$ (now Λ, B and ω dependent) defined by:

$$\mathcal{E}_\omega^+(\eta(\Lambda')) = \mathcal{E}(\eta(\Lambda')) - \mathcal{E}(\eta(\Lambda), \eta(\partial_B(\Lambda))) + \mathcal{E}(\eta(\Lambda), \omega(\Lambda^c))$$

$$= \mathcal{E}(\eta(\Lambda')) + \mathcal{E}(\eta(\Lambda), \omega(\Lambda'^c)) \tag{3.3.4}$$

and

$$Z_\Lambda^+(\omega) = \int_{\Omega(\Lambda')} \bar{\lambda}_{0,\Lambda}(d\eta(\Lambda)) \otimes \delta_\omega(\eta(\partial_B(\Lambda)))\, e^{-\mathcal{E}_\omega^+(\eta(\Lambda'))}. \tag{3.3.5}$$

It is easy to observe that the new interaction $\mathcal{E}_\omega^+$ is superstable and regular on $\Omega(\Lambda')$. For example, let us check the superstability of $\mathcal{E}_\omega^+$:

$$\mathcal{E}(\eta(\Lambda')) \geq A \sum_{r \in C_\delta(\Lambda')} n_\delta^2(\eta(\Lambda'), r) + B \sum_{r \in C_\delta(\Lambda')} n_\delta(\eta(\Lambda'), r)$$

$$- \frac{1}{2} \sum_{r \in C_\delta(\Lambda)} \sum_{s \in C_\delta(\Lambda'^c)} \Psi(|r - s|)\, n_\delta^2(\eta(\Lambda), r)$$

$$- \frac{1}{2} \sum_{r \in C_\delta(\Lambda)} \sum_{s \in C_\delta(\Lambda')} \Psi(|r - s|)\, n_\delta^2(s, \omega(\Lambda'^c))$$

$$\geq (A - B) \sum_{r \in C_\delta(\Lambda')} n_\delta^2(\eta(\Lambda'), r) + \sum_{r \in C_\delta(\Lambda')} \mathrm{const}(\omega(\Lambda^c))\, n_\delta(\eta(\Lambda'), r), \tag{3.3.6}$$

where we have used to assume regularity of $\mathcal{E}$ and the definition of $\partial_B(\Lambda)$.

We conclude that to the measure $\nu_\Lambda^{+,B}$ the superstability estimates of the Theorem 1.1.1 with some constants δ^+ and ρ^+ apply. For the given Λ' let us define $\Omega_N(\Lambda', \Lambda'^c)$ as in §1.3. Then, for sufficiently large N we have:

$$e^{-\mathcal{E}(\omega(\partial_B(\Lambda)))} Z_{\Lambda'}(\omega)$$

$$\leq \Big(\int_{\Omega_N(\Lambda'; \Lambda'^c)} + \int_{\Omega_N^c(\Lambda'; \Lambda'^c)}\Big) \bar{\lambda}_{0,\Lambda}(d\eta(\Lambda)) \otimes \delta_\omega(\eta(\Lambda' - \Lambda))\, e^{-\mathcal{E}_\omega^+(\eta(\Lambda'))}$$

$$\leq \int_{\Omega_N(\Lambda'; \Lambda'^c)} (\bar{\lambda}_{0,\Lambda}(d\eta(\Lambda)) \otimes \delta_\omega(\Lambda' - \Lambda))\, e^{-\mathcal{E}(\eta(\Lambda'))} e^{-\mathcal{E}(\eta(\Lambda), \omega(\Lambda'^c))} + \frac{1}{2} Z_\Lambda^+(\omega). \tag{3.3.7}$$

But on $\Omega_N(\Lambda'; \Lambda'^c)$ we have

$$|\mathcal{E}(\eta(\Lambda) | \omega(\Lambda'^c))| \leq \frac{N^2}{2} \sum_{r \in C_\delta(\Lambda)} \sum_{s \in C_\delta(\Lambda'^c)} \Big(\Psi(|r - s|) + \frac{1}{2} \sum_{r \in C_\delta(\Lambda)} \sum_{s} \Psi(|r - s|)\, n_\delta^2(\omega(\Lambda'^c), s)\Big). \tag{3.3.8}$$

Therefore, using regularity of $\mathcal{E}$ we obtain

$$Z_\Lambda(\omega) \leq e^{\mathcal{E}(\omega(\partial_B(\Lambda)))} e^{\frac{N^2}{2} \sum_{r \in C_\delta(\Lambda)} \sum_{s \in C_\delta(\Lambda'^c)} \Psi(|r - s|)} \tag{3.3.9}$$

$$\cdot e^{\frac{1}{2} \sum_{r \in C_\delta(\Lambda)} \sum_{s \in C_\delta(\Lambda'^c)} \Psi(|r - s|)\, n_\delta^2(\omega(\Lambda'^c), s)} Z_\Lambda \cdot Z'_\omega$$

$$+ \frac{1}{2} Z_\Lambda^+$$

From which it follows that:

$$Z_\Lambda(\omega) \leq 2\, e^{\mathcal{E}(\omega(\partial_B(\Lambda)))}\, e^{\frac{N^2}{2}\sum_{r\in C_\delta(\Lambda)}\sum_{s\in C_\delta(\Lambda^c)} \Psi(|r-s|)}$$

$$e^{\frac{1}{2}\sum_{r\in C_\delta(\Lambda)}\sum_{s\in C_\delta(\Lambda^c)} \Psi(|r-s|)\, n^2(\omega(\Lambda^c), s)} \cdot Z_\Lambda \cdot Z'_\omega\,, \tag{3.3.10}$$

where

$$Z'_\omega = e^{-\mathcal{E}(\omega(\partial_B(\Lambda)))}\, e^{\frac{N^2}{2}\sum_{r\in C_\delta(\Lambda)}\sum_{s\in C_\delta(\Lambda^c)} \Psi'(|r-s|)}\,. \tag{3.3.11}$$

Lower Bound

Now let $\Lambda = \Lambda' \cup \partial_B(\Lambda')$. Let χ be a characteristic function of some event in $\Omega(\partial_B(\Lambda'))$ to be defined below. Then, we have:

$$Z_\Lambda(\omega) \geq \int_{\Omega(\Lambda)} \bar\lambda_{0,\Lambda'}(d\eta(\Lambda')) \otimes \bar\lambda_{0,\Lambda-\Lambda'}(d\eta(\Lambda-\Lambda'))\, e^{-\mathcal{E}(\eta(\Lambda))}\, e^{-\mathcal{E}(\eta,\omega(\Lambda^c))} \cdot \chi$$

$$\geq \int_{\Omega(\Lambda)} \bar\lambda_{0,\Lambda}(d\eta(\Lambda')) \cdot \chi\, e^{-\mathcal{E}(\eta(\Lambda'))}\, e^{-\mathcal{E}(\eta(\Lambda'),\omega(\partial_B(\Lambda')))}$$

$$e^{-\mathcal{E}(\eta(\Lambda-\Lambda'))}\, e^{-\mathcal{E}(\eta(\Lambda),\omega(\Lambda^c))}\,. \tag{3.3.12}$$

Choose now

$$\chi(\eta(\Lambda-\Lambda')) = \{\eta \in \Omega(\Lambda-\Lambda') \mid \forall_{r\in C_\delta(\Lambda-\Lambda')}\; n_\delta^2(\eta, r) \leq N_-\} \equiv \chi_{N_-}\,.$$

Then, we have

$$Z_\Lambda(\omega) \geq \int_{\Omega(\Lambda)} \bar\lambda^*_{0,\Lambda}(d\eta(\Lambda))\, e^{-\mathcal{E}^-_\omega(\eta(\Lambda))} \equiv Z^-_\Lambda(\omega) \cdot \tag{3.3.13}$$

where a new measure

$$\bar\lambda^*_{0,\Lambda}(d\eta(\Lambda)) = \bar\lambda_{0,\Lambda}(d\eta(\Lambda)) \cdot \chi_{N_-} \tag{3.3.14}$$

is defined on $\{\Omega(\Lambda), \mathcal{F}(\Lambda)\}$ and a new interaction :

$$\mathcal{E}^-_\omega(\eta(\Lambda)) = \left(\mathcal{E}(\eta(\Lambda)) + \mathcal{E}(\eta(\Lambda),\omega(\Lambda^c))\right) \cdot \chi_{N_-}(d\eta(\Lambda-\Lambda'))\,. \tag{3.3.15}$$

The new interaction $\mathcal{E}^-_\omega$ is again superstable and regular. Therefore, the procedure analogical as in § 3.2 can be applied. This leads to the following lower bound:

$$Z_{\Lambda'\cup\partial_B(\Lambda')}(\omega) \geq$$

$$\frac{1}{2}\, e^{-N^2\sum_{r\in C_\delta(\Lambda)}\sum_{s\in C_\delta(\Lambda^c)} \Psi(|r-s|)}$$

$$e^{-N^2\sum_{r\in C_\delta(\Lambda)}\sum_{s\in C_\delta(\Lambda^c)} \Psi(|r-s|)} \cdot Z^-_{\partial_B(\Lambda')} \cdot Z_\Lambda$$

$$\cdot\, e^{-\frac{N^2}{2}\sum_{r\in C_\delta(\Lambda)}\sum_{s\in C_\delta(\Lambda-\Lambda')} \Psi(|r-s|)}\,;\; N_- = N\,, \tag{3.3.16}$$

where

$$Z^-_{\partial_B(\Lambda')}(\omega) = \int_{\Omega(\partial_B(\Lambda'))} \bar\lambda_{0,\partial_B(\Lambda')}(d\omega)\, \chi_{N_-}(\Lambda-\Lambda')\, e^{-\mathcal{E}(\omega(\partial_B(\Lambda)))}\,. \tag{3.3.17}$$

To finish the proof we note two simple observations.

Lemma 3.3.1. Let the measure λ on $\{\Omega, \mathcal{F}\}$ be tempered and regular in the sense of § 2.6.
Then, there exists $a>0$ such that the set

$\Omega(a) = \{\omega \in \Omega \mid \forall_{r\in Z_\delta} : n^2(\omega, r) \leq a \log r$
for suff. large $a\}$ has λ measure equal 1.

Proof: This is a simple application of the Tchebyshev inequality combined with the superstability estimates of Th. 2.4.1. See also /10/.

Lemma 3.3.2. Whenever $\{\Lambda_n\}$ sequence of regular, δ-polygonal regions tends to R^d in the sense of van Hove, then:

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|}\left(\sum_{r\in C_\delta(\Lambda_n)}\sum_{s\in C_\delta(\Lambda_n^c)} \Psi(|r-s|)\right) = 0\,. \tag{3.3.18}$$

If moreover $\Psi(r) \sim r^{-d-\epsilon}$ for some $\epsilon>0$ then also:

$$\forall_{\alpha>0}\; \lim_{n\uparrow\infty} \frac{1}{|\Lambda_n|}\sum_{r\in C_\delta(\Lambda_n)}\left(\sum_{s\in C_\delta(\Lambda_n^c)} \log|s|^\alpha\, \Psi(|r-s|)\right) = 0 \tag{3.3.19}$$

From this two observations, the upper bound (3.3.10) and (3.3.16) and the Theorem 3.1.1., there easily follows

$$\limsup_{n\to\infty} p_{\Lambda_n}(\omega) \leq \lim_{n\to\infty} p_{\Lambda_n}(z) \leq \liminf_{n\to\infty} p_{\Lambda_n}(\omega) \tag{3.3.20}$$

whenever Λ_n tends to R^d in the sense of Van Hove and $\omega \in \Omega(a)$.

3.4. General regular and tempered boundary condition

For a given $\omega \in \Omega(a)$ we proceed as in the case of pure boundary condition and we obtain for $Z_\Lambda(\omega)$ upper (3.3.10) and lower (3.3.16) bounds. After taking the logarithms of all and division by $|\Lambda_n|$ we integrate with the given regular and tempered measure λ^{bc}. This leads to the upper bound

$$
\begin{aligned}
&\limsup_{n\to\infty} P_{\Lambda_n}(\lambda^{bc}) \\
&\leq \limsup_{n\to\infty} \frac{1}{|\Lambda_n|} P_{\Lambda_n}(z) \\
&+ N^2 \limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \int \lambda^{bc}(d\omega(\Lambda^c)) \sum_{\underline{r}\in C_\delta(\Lambda_n)} \sum_{\underline{s}\in C_\delta(\Lambda_n'^c)} \Psi(|\underline{r}-\underline{s}|) \\
&+ \frac{1}{2} \limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \int \lambda^{bc}(d\omega(\Lambda^c)) \sum_{\underline{r}\in C_\delta(\Lambda_n)} \sum_{\underline{s}\in C_\delta(\Lambda_n^c)} \Psi(|\underline{r}-\underline{s}|)\, n^2(\underline{s},\omega(\Lambda^c)) \\
&+ \limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \int \lambda^{bc}(d\omega(\Lambda^c))\, \mathcal{E}(\omega(\mathcal{C}_B(\Lambda_n))) \\
&+ \limsup_{n\to\infty} N^2 \frac{1}{|\Lambda_n|} \sum_{\underline{r}\in C_\delta(\Lambda_n\cup\partial_B(\Lambda_n))} \sum_{\underline{s}\in C_\delta((\Lambda_n+\partial_B(\Lambda_n))^c)} \Psi(|\underline{r}-\underline{s}|) \\
&\leq P_n(z),
\end{aligned}
\tag{3.4.1}
$$

when taking account of the definition of λ^{bc}, Lemma 3.3.2 and Lemma 3.3.2.

q.e.d.

Appendix

The text below should be compared with that of the original Rulle papers /3/, especially of §2. All the notation is that of (§2, /3/) but the difference is that we will work on the configurational space Ω instead of /3/ where the spaces $\bigcup_n \Lambda^{\otimes n}$ are used.

Definition A.1.

For $\mathring{\omega} \in \Omega(\Lambda)$ we define the finite volume correlation function:

$$\rho^{(\Lambda)}(\mathring{\omega}) = (Z_\Lambda)^{-1} \int_{\Omega(\Lambda)} \bar{\lambda}_{0,\Lambda}(d\omega)\, e^{-\mathcal{E}(\omega\vee\mathring{\omega})}. \tag{A.1}$$

Lemma A.1. There exists a number $\xi > 0$ such that:

$$\rho^{(\Lambda)}(\mathring{\omega}) \leq \xi^{|\mathring{\omega}|}. \tag{A.2}$$

Proof:

Let $\omega_1 \in \Omega^1(\Lambda)$ then we define $\mathring{\omega}' = \mathring{\omega} \vee \omega_1$. Assume that for Lemma A.1 is valid. Let us define:

$$\overset{1}{\Omega}_\Lambda(\omega_0) = \{\omega \in \Omega(\Lambda) \mid \forall_{j\geq \ell} \sum_j n^2(\omega\vee\omega_0, r) \leq \psi_j V_j\} \tag{A.3}$$

and

$$\overset{2}{\Omega}_{\Lambda,q}(\omega_0\vee\omega_1) = \{\omega\in\Omega(\Lambda) \mid q \text{ is the maximal integer such that } \sum_{r\in[q]} n^2(\omega_0'\vee\omega, r) \geq \psi_q V_q\} \tag{A.4}$$

Then, we have:

$$[\overset{1}{\Omega}_\Lambda(\omega_0)]^c \subseteq \bigcup_{q=\ell}^{\infty} \overset{2}{\Omega}_{\Lambda,q}(\omega_0\vee\omega_1)$$

for any choice of $\omega_1 \in \Omega^1(\Lambda)$

and consequently:

$$\rho^{(\Lambda)}(\mathring{\omega}') \leq \overset{1}{\rho}{}^{(\Lambda)}(\mathring{\omega}') + \sum_{q=\ell}^{\infty} \rho_q^{(\Lambda)}(\mathring{\omega}'), \tag{A.5}$$

where

$$\overset{1}{\rho}{}^{(\Lambda)}(\mathring{\omega}') = (Z_\Lambda)^{-1} \int_{\overset{1}{\Omega}_\Lambda(\omega_0)} e^{-\mathcal{E}(\omega\vee\mathring{\omega}')} \lambda_{0,\Lambda}(d\omega) \tag{A.6}$$

and

$$\rho_q^{(\Lambda)}(\mathring{\omega}') = (Z_\Lambda)^{-1} \int_{\overset{2}{\Omega}_{\Lambda,q}} e^{-\mathcal{E}(\omega\vee\mathring{\omega}')}\, \bar{\lambda}_{0,\Lambda}(d\omega) . \qquad (A.7)$$

Using Lemma of /3/ and chosing Δ in such a way that $\omega_1 \in \Omega'(\Delta)$ we have:

$$\overset{1}{\rho}^{(\Lambda)}(\mathring{\omega}') = (Z_\Lambda)^{-1} \int_{\overset{1}{\Omega}^{(\Lambda)}(\omega_0)} e^{-\mathcal{E}(\omega\vee\mathring{\omega}\vee\omega')}\, \bar{\lambda}_{0,\Lambda}(d\omega)$$

$$= (Z_\Lambda)^{-1} \int_{\overset{1}{\Omega}^{(\Lambda)}(\omega_0)} e^{-\mathcal{E}(\omega_1)} e^{-\mathcal{E}(\mathring{\omega}\vee\omega)} e^{-\mathcal{E}(\omega_1,\mathring{\omega}\vee\omega)}\, \bar{\lambda}_{0,\Lambda}(d\omega)$$

$$\leq (Z_\Lambda)^{-1} \int_{\overset{1}{\Omega}^{(\Lambda)}(\omega_0)} e^{-\mathcal{E}(\omega_1)} e^{-\mathcal{E}(\mathring{\omega}\vee\omega)} \exp\Big[\frac{1}{2}\sum_{r\in Z^\nu}\Psi(|r|) + \frac{1}{2}\Psi(0)\sum_{r\in[P]} n^2(\omega_0\vee\omega_1,r) + \frac{1}{2}\sum_{r\notin[P]}\Psi(|r|)\, n^2(\omega_0\vee\omega_1,r)\Big]\bar{\lambda}_{0,\Lambda}$$

$$\leq (Z_\Lambda)^{-1} e^{D} \int_{\overset{1}{\Omega}^{(\Lambda)}(\omega_0)} e^{-\mathcal{E}(\omega_1)} e^{-\mathcal{E}(\mathring{\omega}\vee\omega)}\, \bar{\lambda}_{0,\Lambda}(d\omega)$$

$$\leq (Z_\Lambda)^{-1} e^{D} \int_{\Omega(\Delta)} \bar{\lambda}_{0,\Delta}(d\omega)\, e^{-\mathcal{E}(\omega_1)} \int_{\Omega(\Lambda)} e^{-\mathcal{E}(\mathring{\omega}\vee\omega)}\, \bar{\lambda}_{0,\Lambda}(d\omega)$$

$$\leq (Z_\Lambda)^{-1} e^{const}\, \rho^{(\Lambda)}(\mathring{\omega}\vee\omega).$$

Now we proceed to estimate $\overset{2}{\rho}_q^{(\Lambda)}$. Let us introduce the notation: $\Lambda_q = \Lambda \cap [q+1]$; $\Lambda - \Lambda_q = \Lambda_q^c$ and cosequently let us decompose:

$$\mathring{\omega}'\vee\omega = \mathring{\omega}'(\Lambda_q) \vee (\mathring{\omega}'\vee\omega)(\Lambda_q^c) \equiv \omega'\vee\omega'' .$$

Let denote also $N_q = |\mathring{\omega}'(\Lambda_q)| \geq 1$ if we assume that $\omega_1 \in [q]$. Then we have

$$\overset{2}{\rho}_q^{(\Lambda)}(\mathring{\omega}') = (Z_\Lambda)^{-1} \int_{\overset{2}{\Omega}_q(\mathring{\omega}')} e^{-\mathcal{E}(\omega')} e^{-\mathcal{E}(\omega'')} e^{-\mathcal{E}(\omega',\omega'')}\, \bar{\lambda}_{0,\Lambda}$$

$$\leq e^{-C\psi_{q+1}V_{q+1}} (Z_\Lambda)^{-1} \int_{\overset{2}{\Omega}_q(\mathring{\omega}')} e^{-\mathcal{E}(\omega'')}\, \bar{\lambda}_{0,\Lambda_q}(d\omega(\Lambda_q)) \otimes \bar{\lambda}_{0,\Lambda_q^c}$$

$$\leq e^{-C\psi_{q+1}V_{q+1}} (Z_\Lambda)^{-1} \Big(\int_{\Omega(\Lambda_q)} \bar{\lambda}_{0,\Lambda_q}(d\omega(\Lambda_q))\Big) \int_{\Omega(\Lambda)} e^{-\mathcal{E}(\omega'')}\, \bar{\lambda}_{0,\Lambda}(d\omega)$$

$$\leq e^{-C\psi_{q+1}V_{q+1}}\, e^{o(|V_{q+1}|)} \cdot \rho^{(\Lambda)}(\mathring{\omega}' - \mathring{\omega}'(\Lambda_q))$$

$$\leq e^{-C\psi_{q+1}V_{q+1}}\, e^{o(|V_{q+1}|)} \cdot \xi^{|\mathring{\omega}' - \mathring{\omega}'(\Lambda_q)|} .$$

Taking into account that $\psi_q \to \infty$ as $q \to \infty$ we finally obtain $\overset{2}{\rho}^{(\Lambda)}(\omega_0\vee\omega_1) \leq E\, \xi^{|\omega_0|}$ for a suitable ξ .

Therefore,

$$\rho^{(\Lambda)}(\omega_0\vee\omega_1) \leq (1+E)\,\xi^{|\omega_0|} \leq \xi^{|\omega_0|+1}$$

if we choose $1+E<\xi$.

q.e.d.

As the reader has presumably remarked, all the things are working as in the language of $\cup\Lambda^n$ space. In a similar manner we are then able to prove:

Lemma A.2.

There exists $\gamma > 0$ and $\rho \in R^d$ such that (uniformly in Λ if ξ is translationally invariant)

$$\rho_{\Lambda}(\hat{\omega}) \le \exp \sum_{\underset{\sim}{r} \in \mathbb{Z}_1} \left(-\gamma\, n^2(\hat{\omega}, r) + \rho\, n(\hat{\omega}, r)\right)$$

Lemma A.2 immediately leads to the proof of Theorem 2.4.1.

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Вниманию организаций и лиц, заинтересованных в получении публикаций Объединенного института ядерных исследований

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Гелерак Р. E4-85-969

Общие сверхустойчивые классические газы. Утверждение Ван-Хова

Рассматриваются общие сверхустойчивые классические непрерывные газы. Базируясь на так называемых оценках сверхустойчивости Рюэля, доказывается, что предельная свободная энергия не зависит от класса граничных условий умеренного роста.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Gielerak R. E4-85-969

General Superstable Classical Gases. Van Hove Theorem

For general multibody, superstable and regular interactions, it is proven that the infinite volume free energy does not depend on the regular, tempered boundary conditions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985