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KRONECKER PRODUCTS WHICH DECOMPOSE INTO TWO IRREDUCIBLE REPRESENTATIONS

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1. Introduction

The aim of this paper is to obtain for the semisimple Lie algebras of types B_n, C_n and D_n a list of pairs of irreducible representations which have the property that the Kronecker product of the elements of a given pair decomposes into a direct sum of two irreducible representations.

Using a procedure due to Okubo /1/ these results prove to be consistent with our previous results in which, after determining the second-degree polynomial identities which can be satisfied by linear representations ϱ of a Lie algebra /2,3/, the representations ϱ (satisfying the identities) have been obtained for the semisimple Lie algebras C_n $(n \geqslant 2)$ and D_n $(n \geqslant 5)$ /4,5/.

2. The procedure of Okubo

Okubo has proved /1/ that there exists a direct connection between the polynomial identities satisfied by finite-dimensional representations of Lie algebras and the Clebsch-Gordan series of the direct products of these representations with other finite-dimensional representations. This connection results in the following way:

For any two finite-dimensional representations ρ_Λ and ρ_Ω (of maximal weights Λ and Ω respectively) of a reductive Lie algebra L acting in the linear spaces V_Λ and V_Ω, an operator A acting on $V_\Lambda \otimes V_\Omega$:

$$A = \sum_{i=1}^{n} \rho_\Lambda(e_i) \otimes \rho_\Omega(e^i) \qquad (2.1)$$

may be defined. In formula (2.1) $\{e_i\}_{i=1}^{n}$ is a basis in the Lie algebra L and $\{e^j\}_{j=1}^{n}$ is another basis in L having the property

$$(e_i, e^j) = \delta_{ij} \qquad (2.2)$$

if we denote by $(\, , \,)$ a nondegenerate bilinear form on L (the existence of which is ensured by the property of L to be reductive).

The operator A commutes with the operators of the direct product of the representations ρ_Λ and ρ_Ω

$$\rho_\Lambda(x) \otimes I_\Omega + I_\Lambda \otimes \rho_\Omega(x) \qquad (2.3)$$

for any $x \in L$. Hence, by Schur's lemma, in any irreducible subspace of $V_\Lambda \otimes V_\Omega$ the operator A is proportional with the unity operator. Thus, if

$$\rho_\Lambda \otimes \rho_\Omega = \bigoplus_{k=1}^{m} \rho_{\Xi_k} \qquad (2.4)$$

and

$$V_\Lambda \otimes V_\Omega = \bigoplus_{k=1}^{m} V_{\Xi_k} \qquad (2.5)$$

we obtain for A the spectral decomposition

$$A = \sum_{k=1}^{m} \lambda_k P_{\Xi_k}, \qquad (2.6)$$

where P_{Ξ_k} is the projector on the subspace V_{Ξ_k}.

The polynomial of minimal degree, which vanishes if its indeterminate is replaced by the operator A, generates the polynomial identities satisfied by the representations ρ_Λ or ρ_Ω /1/: the polynomial identities for ρ_Λ (ρ_Ω) are obtained if in the polynomial satisfied by the operator A in expression (2.1) of A, $\rho_\Omega(e^i)$ ($\rho_\Lambda(e_i)$) are replaced by matrices of these operators in a fixed basis in V_Ω (V_Λ).

Equations (2.4), (2.6) tell us that the minimum degree of the polynomial equation $\mathcal{P}(A) = 0$ satisfied by A is equal to the number of distinct eigenvalues λ_k of A (or, what amounts to the same, to the number of distinct eigenvalues of the Casimir operator) i.e., to the number of distinct terms in the Clebsch-Gordan series (2.4).

3. The procedure developed in /2,3/ and subsequent results

In paper /2/ it has been proved that the polynomial identities of degree k which are satisfied by the generators of a Poisson bracket realization of a Lie algebra L can be obtained by equating to zero the basis vectors of subrepresentations of the symmetric part $(\mathrm{ad}^{\otimes k})_s$ of the direct k-th power of the adjoint representation of L.

The polynomial identities satisfied by linear representations of L are obtained by symmetrization /3/ from the polynomial identities satisfied by the corresponding Poisson bracket realizations.

In paper /3/ we deduced the expressions of the second-degree polynomial identities for the non-exceptional semisimple Lie algebras $A_n (n \geq 3)$, $B_n (n \geq 2)$, $C_n (n \geq 2)$ and $D_n (n \geq 5)$. (These identities are obtained by the vanishing of second-degree tensor operators which transform under subrepresentations of $(\mathrm{ad} \otimes \mathrm{ad})_s$).

In order to be an intrinsic property of the Lie algebra L, a set of polynomial identities has to be invariant under the automorphisms of L, hence under the adjoint group. Intrinsic polynomial identities can be thus provided only by subrepresentations of the extension of the adjoint representation to the symmetric or to the universal enveloping algebras: the procedure developed in /2,3/ allows thus the determination of all intrinsic polynomial identities.

Subsequent papers /4,5,6/ were aimed to extract the information contained in the polynomial identities derived in /3/. In papers /4,5/ it was proved that the tensor operators deduced in /3/ for the algebras $sp(2n,C)$ $(n \geq 2)$ and $so(2n,C)$ $(n \geq 5)$ determine the weights of the linear representations of these algebras for (the states of) which these tensor operators vanish.

The Clebsch-Gordan series of $(\mathrm{ad} \otimes \mathrm{ad})_s$ for the semisimple Lie algebras C_n and D_n are the following:

$$\text{for } C_n (n \geq 2):\ (\mathrm{ad} \otimes \mathrm{ad})_s = (0) \oplus (\Lambda_2) \oplus (4\Lambda_1) \oplus (2\Lambda_2), \qquad (3.1)$$

$$\text{for } D_n (n \geq 5):\ (\mathrm{ad} \otimes \mathrm{ad})_s = (0) \oplus (2\Lambda_1) \oplus (\Lambda_4) \oplus (2\Lambda_2), \qquad (3.2)$$

where by Λ_i, $i=1,\dots,n$ we denoted the fundamental weight system of a Lie algebra of rank n and by (Λ_i) the corresponding representations.

Let us denote a tensor operator associated with representation σ by T_σ. The following results hold /4,5/:

For $sp(2n,C)$:

a) The only representations ρ for (the states of) which the second-degree operator $T_{(\Lambda_2)}$ vanishes are $\rho = (k\Lambda_n)$

b) The only representation ρ for which the second-degree tensor operator $T_{(4\Lambda_1)}$ vanishes is $\rho = (\Lambda_1)$

c) There are no representations ρ of $sp(2n,C)$ for which the tensor operator $T_{(2\Lambda_2)}$ vanishes.

For $so(2n,C)$:

a) The only representations ρ for which the second-degree tensor operator $T_{(2\Lambda_1)}$ vanishes are $\rho = (k\Lambda_{n-1})$ and $\rho = (k\Lambda_n)$

b) The only representations for which the second-degree tensor operator $T_{(\Lambda_4)}$ vanishes are $\rho = (k\Lambda_1)$

c) There exists a representation, $\rho = (\Lambda_n)$, of $so(2n,C)$ for which the tensor operator $T_{(2\Lambda_2)}$ vanishes.

4. Representations the Kronecker product of which has a Clebsch-Gordan series composed of two terms

To determine, using Okubo's procedure, all finite-dimensional representations which satisfy second-degree polynomial identities we have to find (keeping the same notation as in section 3) all the pairs of representations (Λ) and (Ω) such that the Clebsch-Gordan series of their Kronecker product contains only two terms

$$(\Lambda)\otimes(\Omega) = (\Lambda+\Omega)\oplus(\Xi). \tag{4.1}$$

We shall give solutions of this problem for algebras of types B_n C_n and D_n.

The following result due to Dynkin /8/ allows the calculation of a second term (Ξ) in the Clebsch-Gordan series of $(\Lambda)\otimes(\Omega)$.

Let L be a semisimple Lie algebra and let (Λ) and (Ω) be two irreducible representations of L labelled by their maximum weights Λ and Ω.

Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be a minimal chain of simple roots connecting the weights Λ and Ω, i.e., a set of simple roots such that

$$(\Lambda,\alpha_1)\neq 0 \quad , \quad (\alpha_k,\Omega)\neq 0$$
$$(\alpha_i,\alpha_{i+1})\neq 0 \quad (i=1,\dots,k-1) \tag{4.2}$$

and such that no proper subset of $\{\alpha_1,\dots,\alpha_k\}$ having the same properties exists. Then

$$\Xi(\alpha_1,\alpha_2,\dots,\alpha_k) = \Lambda+\Omega-(\alpha_1+\alpha_2+\dots+\alpha_k) \tag{4.3}$$

is the maximum weight for one and only one of the irreducible components $(\Phi_1),(\Phi_2),\dots,(\Phi_\ell)$ in the decomposition of the Kronecker product

$$(\Lambda)\otimes(\Omega) = (\Phi_1)\oplus(\Phi_2)\oplus\dots\oplus(\Phi_\ell).$$

Let us apply Dynkin's theorem to the algebras B_n, C_n and D_n.

i) Algebras of type B_n

Let us denote by (Λ_i), $i=1,\dots,n$ the fundamental representations and consider for algebras of type B_n the Kronecker product $(m\Lambda_1)\otimes(\Lambda_n)$; this product admits as a first term in its Clebsch-Gordan decomposition the representation $(m\Lambda_1+\Lambda_n)$.

We shall deduce a second term by using Dynkin's theorem and then prove calculating the dimensionalities of the representations involved that no other terms exist. The same procedure will be followed for the algebras of types C_n and D_n.

Expressed in terms of the basis vectors ε_i, $i=1,2,\dots,n$ in R^n, the basis composed of simple roots in B_n is

$$\alpha_1=\varepsilon_1-\varepsilon_2,\quad \alpha_2=\varepsilon_2-\varepsilon_3,\quad \dots,\quad \alpha_{n-1}=\varepsilon_{n-1}-\varepsilon_n,\quad \alpha_n=\varepsilon_n \tag{4.4}$$

and the fundamental weights of B_n have the expressions /9/

$$\Lambda_i = \varepsilon_1+\varepsilon_2+\dots+\varepsilon_i \quad (1\leq i<n)$$
$$\Lambda_n = \frac{1}{2}(\varepsilon_1+\varepsilon_2+\dots+\varepsilon_n) = \frac{1}{2}(\alpha_1+2\alpha_2+\dots+n\alpha_n). \tag{4.5}$$

The minimal chain between

$$\Lambda_1 = \varepsilon_1 = \alpha_1+\alpha_2+\dots+\alpha_n \tag{4.6}$$

and Λ_n is $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$. Hence, a second term in the Clebsch-Gordan series of $(m\Lambda_1)\otimes(\Lambda_n)$ has the maximum weight:

$$m\Lambda_1+\Lambda_n-\varepsilon_1 = (m-1)\Lambda_1+\Lambda_n. \tag{4.7}$$

Applying Weyl's dimension formula, we get

$$\dim(m\Lambda_1) = (2n+2m-1)\frac{(2n+m-2)!}{(2n-1)!\,m!}$$
$$\dim(\Lambda_n) = 2^n \tag{4.8}$$
$$\dim(m\Lambda_1+\Lambda_n) = \frac{2^n(2n+m-1)!}{(2n-1)!\,m!}$$

and the equality

$$\dim(m\Lambda_1)\dim(\Lambda_n) = \dim(m\Lambda_1+\Lambda_n)+\dim((m-1)\Lambda_1+\Lambda_n) \tag{4.9}$$

leads to the result

$$(m\Lambda_1)\otimes(\Lambda_n) = (m\Lambda_1+\Lambda_n)\oplus((m-1)\Lambda_1+\Lambda_n) \tag{4.10}$$

which had to be proved.

ii) Algebras of type C_n

Let us consider the Kronecker product

$$(\Lambda_1)\otimes(m\Lambda_n) = (\Lambda_1+m\Lambda_n)\oplus\cdots \tag{4.11}$$

and determine a second term in the Clebsch-Gordan series of the r.h.s.

Expressed in terms of the basis vectors ε_i $(i=1,2,\dots,n)$ in R^n, a basis composed of simple roots of C_n is

$$\alpha_1=\varepsilon_1-\varepsilon_2,\quad \alpha_2=\varepsilon_2-\varepsilon_3,\quad\dots,\quad \alpha_{n-1}=\varepsilon_{n-1}-\varepsilon_n,\quad \alpha_n=2\varepsilon_n \tag{4.12}$$

and the fundamental weights of C_n have the expression /9/:

$$\Lambda_i = \varepsilon_1+\varepsilon_2+\dots+\varepsilon_i \qquad (1\le i\le n). \tag{4.13}$$

The minimal chain between Λ_1 and Λ_n is again $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ and a second term in the Clebsch-Gordan series (4.11) has the maximum weight

$$\begin{aligned}\Lambda_1+m\Lambda_n-(\alpha_1+\alpha_2+\dots+\alpha_n) &= \Lambda_1+m\Lambda_n-(\varepsilon_1+\varepsilon_n)\\ &= (m-1)\Lambda_n+\Lambda_{n-1}.\end{aligned} \tag{4.14}$$

We have $\dim\Lambda_1=2n$; applying again Weyl's formula and denoting by f the factor

$$f = \prod_{1<i<j<n}(j-i)\prod_{1<i<j<n}(2n+2m+2-i-j)\prod_{i=2}^{n-1}2(n+m+1-i) \tag{4.15}$$

we obtain

$$\begin{aligned}\dim(m\Lambda_n) &= f\,\frac{(n-2)!\,(n-1)!\,(2n+2m)!}{(2m+1)!}\\ \dim(\Lambda_{n-1}+(m-1)\Lambda_n) &= f\,\frac{2m\,(n-2)!\,n!\,(2n+2m)!}{(n+2m+1)\,(2m+1)!}\\ \dim(\Lambda_1+m\Lambda_n) &= f\,\frac{(2n+2m+2)\,n!\,(n-2)!\,(2n+2m)!}{(n+2m+1)\,(2m+1)!}\end{aligned} \tag{4.16}$$

whence

$$\dim(\Lambda_1)\dim(m\Lambda_n) = \dim(\Lambda_1+m\Lambda_n)+\dim(\Lambda_{n-1}+(m-1)\Lambda_n) \tag{4.17}$$

follows and thus

$$(\Lambda_1)\otimes(m\Lambda_n) = (\Lambda_1+m\Lambda_n)\oplus(\Lambda_{n-1}+(m-1)\Lambda_n) \tag{4.18}$$

iii) Algebras of type D_n

Denoting again the basis vectors in R^n by ε_i $(i=1,2,\dots n)$ the basis in D_n has the expression

$$\alpha_1=\varepsilon_1-\varepsilon_2,\quad \alpha_2=\varepsilon_2-\varepsilon_3,\quad\dots,\quad \alpha_{n-1}=\varepsilon_{n-1}-\varepsilon_n,\quad \alpha_n=\varepsilon_{n-1}+\varepsilon_n. \tag{4.19}$$

The fundamental weights are /9/

$$\begin{aligned}\Lambda_i &= \varepsilon_1+\varepsilon_2+\dots+\varepsilon_i \qquad (1\le i\le n-2)\\ \Lambda_{n-1} &= \tfrac{1}{2}(\varepsilon_1+\varepsilon_2+\dots+\varepsilon_{n-2}+\varepsilon_{n-1}-\varepsilon_n)\\ \Lambda_n &= \tfrac{1}{2}(\varepsilon_1+\varepsilon_2+\dots+\varepsilon_{n-2}+\varepsilon_{n-1}+\varepsilon_n).\end{aligned} \tag{4.20}$$

The minimal chain between Λ_1 and Λ_{n-1} is $\{\alpha_1,\alpha_2,\dots,\alpha_{n-2},\alpha_{n-1}\}$, the minimal chain between Λ_1 and Λ_n is $\{\alpha_1,\alpha_2,\dots,\alpha_{n-2},\alpha_n\}$. As

$$\alpha_1+\alpha_2+\dots+\alpha_{n-2}+\alpha_{n-1} = \varepsilon_1-\varepsilon_n \tag{4.21}$$

and

$$\alpha_1+\alpha_2+\dots+\alpha_{n-2}+\alpha_n = \varepsilon_1+\varepsilon_n \tag{4.22}$$

we can write the expressions for the following Kronecker products, the validity of which will result from Dynkin's theorem and from a calculation of dimensionalities

$$(m\Lambda_1)\otimes(\Lambda_{n-1}) = (m\Lambda_1+\Lambda_{n-1})\oplus((m-1)\Lambda_1+\Lambda_n) \qquad (4.23)$$

$$(m\Lambda_1)\otimes(\Lambda_n) = (m\Lambda_1+\Lambda_n)\oplus((m-1)\Lambda_1+\Lambda_{n-1}) \qquad (4.24)$$

$$(\Lambda_1)\otimes(m\Lambda_{n-1}) = (\Lambda_1+m\Lambda_{n-1})\oplus((m-1)\Lambda_{n-1}+\Lambda_n) \qquad (4.25)$$

$$(\Lambda_1)\otimes(m\Lambda_n) = (\Lambda_1+m\Lambda_n)\oplus(\Lambda_{n-1}+(m-1)\Lambda_n). \qquad (4.26)$$

The dimensions involved in these calculations are

$$\dim(\Lambda_1) = 2n \;;\quad \dim(\Lambda_{n-1}) = \dim(\Lambda_n) = 2^{n-1} \qquad (4.27)$$

$$\dim(m\Lambda_1) = \frac{(m+2n-3)!\,(m+n-1)}{m!\,(2n-3)!\,(n-1)}$$

$$\dim(m\Lambda_1+\Lambda_{n-1}) = \dim(m\Lambda_1+\Lambda_n) = \frac{2^{n-2}(m+2n-2)!}{m!\,(2n-3)!\,(n-1)}.$$

Also, denoting by $\bar{f}$ the factor

$$\bar{f} = \prod_{1<i<j<n} \frac{m+2n-i-j}{2n-i-j} \qquad (4.28)$$

we have

$$\dim(m\Lambda_{n-1}) = \dim(m\Lambda_n) = \bar{f}\,\frac{(m+2n-3)!}{m!\,(2n-3)!}$$

$$\dim(\Lambda_1+m\Lambda_{n-1}) = \dim(\Lambda_1+m\Lambda_n) = \qquad (4.29)$$

$$= \bar{f}\,\frac{(m+2n-2)!\;n}{m!\,(2n-3)!\,(m+n-1)}$$

$$\dim(\Lambda_{n-1}+(m-1)\Lambda_n) = \dim(\Lambda_n+(m-1)\Lambda_{n-1})$$

$$= \bar{f}\,\frac{(m+2n-3)!\;n}{(m-1)!\,(2n-3)!\,(m+n-1)}.$$

Taking into account Eqs. (4.18) and (4.23)-(4.26), and using Okubo's procedure may lead to a verification of part of the results obtained in /4/ and /5/ and reminded in Section 3. Indeed, the inspection of these equations points out that for semisimple Lie algebras of type C_n only representations ϱ of the types (Λ_1) and $(k\Lambda_n)$ (k = positive integer) can verify second-degree polynomial identities and that, similarly, for algebras of type D_n, only the representations $(k\Lambda_1)$, $(k\Lambda_{n-1})$, $(k\Lambda_n)$ (k = positive integer) can have this property.

This verification gives however no information concerning the representation (Λ) under which transforms the tensor operator $T_{(\Lambda)}$ which vanishes on (the states of) the representation ϱ.

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Кронекеровские произведения, распадающиеся на два неприводимые представления

Целью работы является получение для полупростых алгебр Ли типа B_n, C_n и D_n множества пар неприводимых представлений таких, что Кронекеровское произведение представлений каждой пары распадается в прямую сумму двух неприводимых представлений.

Для проверки используется теорема Дынкина и вычисляются размерности представлений.

Пары представлений, полученные таким образом, следующие: $\{(m\Lambda_1), (\Lambda_n)\}$ для алгебр типа B_n, $\{(\Lambda_1), (m\Lambda_n)\}$ для алгебр типа C_n, $\{(m\Lambda_1), (\Lambda_{n-1})\}$, $\{(m\Lambda_1), (\Lambda_n)\}$, $\{(\Lambda_1), (m\Lambda_{n-1})\}$ и $\{(\Lambda_1), (m\Lambda_n)\}$ для алгебр типа D_n /где Λ_i обозначает наибольший вес фундаментального представления (Λ_i), а m -произвольное положительное число/.

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Kronecker Products which Decompose into Two Irreducible Representations

The aim of the work is to determine, for the semisimple Lie algebras of types B_n, C_n and D_n sets of pairs of irreducible representations having the property that the Kronecker product of the representations of each pair decomposes into a direct sum of two irreducible representations.

The proof uses a theorem due to Dynkin and a calculation of dimensionalities.

The pairs of representations obtained in this way are $\{(m\Lambda_1), (\Lambda_n)\}$ for algebras of type B_n, $\{(\Lambda_1), (m\Lambda_n)\}$ for algebras of type C_n, and $\{(m\Lambda_1), (\Lambda_{n-1})\}$ $\{(m\Lambda_1), (\Lambda_n)\}$, $\{(\Lambda_1), (m\Lambda_{n-1})\}$, $\{(\Lambda_1), (m\Lambda_n)\}$ for algebras of type D_n. /where Λ_i denotes the highest weight of the fundamental representation (Λ_i) and m is an arbitrary positive integer/.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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