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REPRESENTATIONS WHICH SATISFY IDENTITIES ASSOCIATED WITH TENSOR OPERATORS.

Second-Degree Tensor Operators Transforming under the Representations: (Λ_4), $(2\Lambda_2)$ of so(2n, c) and $(4\Lambda_1)$, $(2\Lambda_2)$ of sp(2n, c)

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1. Introduotion

It is known [1] that if the enveloping algebra of a Lie algebra L acts on (states of) a degenerate representation of this algebra (i.e., a representation for which one or more Dynkin indices vanish) then the generators are not independent but satisfy polynomial relations or syzygies in addition to the commutation relations (cf. also [2,3]).

In previous papers [4,5] we proved that polynomial relations satisfied by the generators of representations of a Lie algebra can be obtained by equating to zero the symmetrized basis vectors of subrepresentations of the symmetric part of direct powers of the adjoint representation.

Such polynomial relations which are obtained by the vanishing of tensor operators T_σ which transform under a subrepresentation σ of $(ad \otimes ad)_s$ say, contain information about the representation ϱ on (the states of) which $T_\sigma = 0$ holds; otherwise stated it is to expected that tensorial syzygies specify the representations on which they are valid.

In previous papers [2,3,6] we started to extract the information contained in second-degree tensorial syzygies $T_\sigma = 0$ associated with subrepresentations σ of (ad $\otimes$ ad $)_s$. In particular, in paper [6] (which, in the following, will be designed by (I)) we considered this problem for second-degree tensor operators $T_{(2\Lambda_1)}$ which transform under the representation $(2\Lambda_1)$ of $so(2n,C)$ and $T_{(\Lambda_2)}$ which transform under the representation (Λ_2) of $sp(2n,C)$. We proved that

i) The representations ϱ of $so(2n,C)$ on (the states of) which $T_{(2\Lambda_1)}$ vanishes are either $\varrho = (k\Lambda_{n-1})$ or $\varrho = (k\Lambda_n)$

ii) The representations ϱ of $sp(2n,C)$ on (the states of) which $T_{(\Lambda_2)}$ vanishes are $\varrho = (k\Lambda_n)$

(We denoted by (Λ_i) ($i = 1, \dots$ rank L) the fundamental representations of the Lie algebra L).

In the present paper we continue to extract information from the relations $T_\sigma = 0$ associated with the other subrepresentations $\sigma \subset$ (ad $\otimes$ ad $)_s$. Taking into aooount that [7]:

for $so(2n,C)$: $(ad\otimes ad)_s = (0)\oplus(2\Lambda_1)\oplus(\Lambda_4)\oplus(2\Lambda_2)$

for $sp(2n,C)$: $(ad\otimes ad)_s = (0)\oplus(\Lambda_2)\oplus(4\Lambda_1)\oplus(2\Lambda_2)$

it remains to determine the representations ϱ (if any) on which the tensor operators $T_{(\Lambda_4)}$ and $T_{(2\Lambda_2)}$ of $so(2n,C)$ and $T_{(4\Lambda_1)}$ and $T_{(2\Lambda_2)}$ of $sp(2n,C)$ vanish.

This will be done in the subsequent sections.

Throughout the present paper we use the notation introduced in (I).

2. Representations for which the tensor operator $T_{(\Lambda_4)}$ of $so(2n,C)$ vanishes

Expressions for second-degree tensor operators $T_{(\Lambda_4)}$ transforming under the subrepresentation $(\Lambda_4)\subset(ad\otimes ad)_s$ of $so(2n,C)$ ($n\geqslant 5$) have been obtained in /5/ in terms of the generators of $so(2n,C)$ defined by the Lie relations

$$[M_{ij},M_{kl}] = \delta_{il}M_{jk}+\delta_{jk}M_{il}-\delta_{ik}M_{jl}-\delta_{jl}M_{ik}. \tag{2.1}$$

The expressions of the components of the tensor operator $T_{(\Lambda_4)}$ are

$$T_{(\Lambda_4)}(p,q;r,s) = \{M_{pq},M_{rs}\}+\{M_{ps},M_{qr}\}+\{M_{pr},M_{sq}\}, \tag{2.2}$$

where $\{\,,\,\}$ designate the anticommutator.

It is easy to prove that the operators of the natural representation (Λ_1) defined by

$$M_{ij} = e_{ij}-e_{ji}$$

with $(e_{ij})_{kl}=\delta_{ik}\delta_{jl}$ satisfy the identity $T_{(\Lambda_4)}=0$.

We shall prove in this section that (as a sort of converse of this statement) we can state

Theorem I. The representations ϱ of the algebra $so(2n,C)$ ($n\geqslant 5$) on (the states of) which $T_{(\Lambda_4)}=0$ are $\varrho=(k\Lambda_1)$.

The highest-weight vector v_ϱ satisfies the equations

$$-C_{kl}v_\varrho = C_{kl}v_\varrho = A_{lk}v_\varrho = 0 \quad \text{for any } l>k>1. \tag{2.3}$$

Proof. As pointed out in (I) in order to prove that the tensor operator $T_{(\Lambda_4)}$ vanishes on (the states of) the representation ϱ it is sufficient to prove that $T_{(\Lambda_4)}v_\varrho=0$ for the highest weight

vector v_ϱ of representation ϱ. We shall use now this property to determine the highest weight of ϱ.

In order to do that we shall consider the component $T_{(\Lambda_4)}(2i,2i-1;2j,2j-1)$ ($i,j\leq n$) and express it in terms of the Cartan-Weyl basis defined in (I). We get

$$T_{(\Lambda_4)}(2i,2i-1;2j,2j-1) = -2A_{ii}A_{jj}-\{B_{ij},C_{ij}\}+\{A_{ij},A_{ji}\}. \tag{2.4}$$

Reminding that in the Cartan-Weyl basis A_{ii} are generators of the Cartan subalgebra of $so(2n,C)$

$$A_{ii}v_\varrho = f_i v_\varrho \tag{2.5}$$

and that A_{ij} ($i<j$) and B_{ij} (any i,j) are raising operators

$$A_{ij}v_\varrho = 0 \quad (i<j)\quad ; \quad B_{ij}v_\varrho = 0 \tag{2.6}$$

while A_{ij} ($i>j$) and C_{ij} (any i,j) are lowering operators we get from (2.4)

$$T_{(\Lambda_4)}(2i,2i-1;2j,2j-1) = -2f_j(f_i+1)v_\varrho. \tag{2.7}$$

Assuming $i<j$, taking $i=1$ and asking that

$$T_{(\Lambda_4)}v_\varrho = 0 \tag{2.8}$$

we obtain the following solution for the weight of the representation ϱ

$$f_1\neq 0, \quad f_2=f_3=\dots=f_n=0. \tag{2.9}$$

(The same result is obtained assuming $i>j$; for i=j, equation (2.8) is identically satisfied).

Thus $\varrho=(k\Lambda_1)$.

Let us now deduce from relation (2.8) the information concerning the highest weight vector v_ϱ.

To do that, we shall calculate the explicit expressions of equation (2.8) for different components of the tensor operator $T_{(\Lambda_4)}$. Taking again into account relations (2.5), (2.6), we obtain

$$T_{(\Lambda_4)}(2i-1,2i;2l,2k-1)v_\varrho = (f_i+1)(C_{kl}+A_{lk})v_\varrho = 0 \tag{2.10}$$

and

$$T_{(\Lambda_4)}(2i-1,2i;2k-1,2l-1)v_\varrho = -\sqrt{-1}(f_i+1)(C_{kl}-A_{lk})v_\varrho = 0 \tag{2.11}$$

if we assume, in both relations, that $i<k<l$.

Equations (2.10), (2.11) prove condition (2.3).

Consider now the following equalities valid for $k<l<i$

$$T_{(\Lambda_4)}(2i-1,2i;2k,2l-1)v_\xi = \tag{2.12}$$
$$= [2f_i(C_{kl}-A_{lk}) - \{A_{il},C_{ki}\} + \{A_{ik},C_{li}\}]v_\xi = 0$$

$$T_{(\Lambda_4)}(2i-1,2i;2k-1,2l-1)v_\xi = \tag{2.13}$$
$$= \frac{\sqrt{-1}}{2}[2f_i(C_{kl}-A_{lk}) + \{A_{il},C_{ki}\} - \{A_{ik},C_{li}\}]v_\xi = 0.$$

Subtraction of (2.13) from (2.12) gives

$$\{A_{ik},C_{ki}\}v_\xi = \{A_{il},C_{ki}\}v_\xi . \tag{2.14}$$

If $k\neq 1$ both members vanish as a consequence of (2.3). Let $k=1$. From (2.14) and (2.3) we get $C_{li}A_{i1}v_\xi = A_{il}C_{i1}v_\xi$ whence, using (2.14)

$$-[A_{il},C_{li}]v_\xi = [A_{il},C_{1i}]v_\xi \tag{2.15}$$

which, using the Lie relations of $so(2n,C)$ (cf. (I)) gives

$$-C_{1l}v_\xi = C_{l1}v_\xi$$

which is satisfied due to the antisymmetry of the generators C_{il} of $so(2n,C)$. This is a confirmation for the fact that the only generators of $so(2n,C)$ which applied to v_ξ lead to a non-vanishing result are

A_{j1} and $C_{1j}=-C_{j1}$ for any j.

3. Representations for which the tensor operator $T_{(4\Lambda_1)}$ of $sp(2n,C)$ vanishes

The expression of the second-degree tensor operators $T_{(4\Lambda_1)}$ which transform under the representation $(4\Lambda_1)\subset(ad\otimes ad)_s$ of $sp(2n,C)$ $(n\geq 2)$ has been obtained in /5/ in terms of the generators $S_{ij}=S_{ij}$ $(i,j=1,2,\dots,2n)$ which satisfy the Lie relations

$$[S_{ij},S_{kl}] = g_{kj}S_{il} - g_{il}S_{kj} - g_{ik}S_{jl} + g_{lj}S_{ki}, \tag{3.1}$$

where $g_{ij} = \delta_{i,j+n} - \delta_{i+n,j}$ $(i,j=1,2,\dots,2n)$.

In terms of these generators the components of the tensor operator $T_{(4\Lambda_1)}$ have the expression

$$T_{(4\Lambda_1)}(p,q;r,s) = \{S_{pq},S_{rs}\}+\{S_{ps},S_{rq}\}+\{S_{pr},S_{qs}\}. \tag{3.2}$$

The generators of the Cartan-Weyl basis are related to the generators S_{ij} by the formulae

$$A_{ij}=S_{i+n,j}, \quad B_{ij}=S_{i+n,j+n}, \quad C_{ij}=S_{ij} \quad (i,j=1,\dots,n). \tag{3.3}$$

Their Lie relations are given in (I). Remind that like for $so(2n,C)$, A_{ii} are generators of the Cartan subalgebra of $sp(2n,C)$ while A_{ij} $(i<j)$ and B_{ij} are raising and A_{ij} $(i>j)$ and C_{ij} are lowering operators so that relations (2.5), (2.6) keep valid for $sp(2n,C)$.

Expressed in terms of the generators A_{ij}, B_{ij}, C_{ij} the components of the tensor operator $T_{(4\Lambda_1)}$ have no more a unique expression. They are

$$T_{(4\Lambda_1)}(p,q;r,s) = \{C_{pq},C_{rs}\}+\{C_{ps},C_{rq}\}+\{C_{pr},C_{qs}\} \quad \text{for } p,q,r,s<n \tag{3.4}$$

$$T_{(4\Lambda_1)}(p,q;r,s) = \{C_{pq},A_{s-n,r}\}+\{A_{s-n,p},C_{rq}\}+\{C_{pr},A_{s-n,q}\} \quad \text{for } p,q,r<n;\ s>n \tag{3.5}$$

$$T_{(4\Lambda_1)}(p,q;r,s) = \{C_{pq},B_{r-n,s-n}\}+\{A_{s-n,p},A_{r-n,q}\} + \{A_{r-n,p},A_{s-n,q}\} \quad \text{for } p,q<n \text{ and } r,s>n \tag{3.6}$$

$$T_{(4\Lambda_1)}(p,q;r,s) = \{A_{q-n,p},B_{r-n,s-n}\}+\{A_{s-n,p},B_{r-n,q-n}\} + \{A_{r-n,p},B_{q-n,s-n}\} \quad \text{for } p<n \text{ and } q,r,s>n \tag{3.7}$$

$$T_{(4\Lambda_1)}(p,q;r,s) = \{B_{p-n,q-n},B_{r-n,s-n}\}+\{B_{p-n,s-n},B_{r-n,q-n}\} + \{B_{p-n,r-n},B_{q-n,s-n}\} \quad \text{for } p,q,r,s>n. \tag{3.8}$$

Because of the properties (2.5) (2.6), the components (3.7) and (3.8) vanish identically when applied to the highest weight vector v_ξ.

The whole information concerning the representation ϱ is thus contained in the following three relations:

$$[\{C_{pq}, C_{rs}\} + \{C_{ps}, C_{rq}\} + \{C_{pr}, C_{qs}\}]\, v_\varrho = 0 \tag{3.9}$$

$$[\{C_{pq}, A_{s-n,r}\} + \{A_{s-n,p}, C_{rq}\} + \{C_{pr}, A_{s-n,q}\}]\, v_\varrho = 0 \tag{3.10}$$

$$[\{C_{pq}, B_{r-n,s-n}\} + \{A_{s-n,p}, A_{r-n,q}\} + \{A_{r-n,p}, A_{s-n,q}\}]\, v_\varrho = 0 \tag{3.11}$$

Theorem 2. The only representation ϱ of the algebra $sp(2n,C)$ on (the states of) which $T_{(4\Lambda_1)} = 0$ is the representation $\rho = (\Lambda_1)$ having the weight $f_1 = 1,\ f_2 = f_3 = \dots = f_n = 0$, i.e., the natural representation.

The highest weight vector v_ϱ satisfies the equations $A_{ij} v_\rho = C_{ij} v_\rho = 0$ (3.12) for all i, j for which $f_i = f_j = 0$.

Proof. Let us consider eq. (3.11) and take

$$p = q = r-n = s-n = i \tag{3.13}$$

we get

$$f_i\,(f_i - 1)\, v_\rho = 0, \tag{3.14}$$

i.e., f_i can take only the values 0 or 1. Let us take in Eq.(3.II)

$$p = r-n = i, \quad q = s-n = j \tag{3.15}$$

and let $i < j$. We obtain

$$f_j\,(f_i - 1)\, v_\varrho = 0 \qquad (i < j). \tag{3.16}$$

Let us now prove that only the representation Λ_1 satisfies all the equations (3.9– 11).

To do that let us take in eq. (3.11)

$$p = q = r-n = i \quad \text{and} \quad s-n = j. \tag{3.I7}$$

We get for $i \neq j$

$$(f_i - 1)\, A_{ji}\, v_\rho = 0 \tag{3.18}$$

which is identically verified for $j < i$ but leads for $j > i$ to the condition

$$A_{ji}\, v_\varrho = 0 \quad \text{for any } j > i \text{ if } f_i = 0. \tag{3.19}$$

Let us now admit in Eq. (3.11) the following values for the labels of the generator:

$$p = i, \qquad q = r-n = s-n = j \qquad (i \neq j).$$

In this case eq. (3.11) leads to

$$f_j\, A_{ji}\, v_\rho = 0 \tag{3.20}$$

whence

$$A_{ji}\, v_\varrho = 0 \text{ for any } i < j, \text{ if } f_j = 1. \tag{3.21}$$

Conditions (3.19), (3.21) lead to

$$A_{ji}\, v_\varrho = 0 \qquad \text{for any } i \neq j \tag{3.22}$$

unless

$$f_1 = 1, \quad f_2 = f_3 = \dots = f_n = 0. \tag{3.23}$$

Indeed, in this case relations (3.18) (3.20) are both verified without any supplementary assumption concerning $A_{ji}\, v_\varrho$.

A similar result holds for the generators C_{ij}. Let us consider Eq. (3.10) in which $p = i$ and $q = r = s-n = j$. In this case, we get for $i > j$

$$(f_j - 1)\, C_{ij}\, v_\varrho = 0, \tag{3.24}$$

i.e., for any i, j for which $f_i = f_j = 0$ we have $C_{ij}\, v_\rho = 0$.

Let us consider now Eq. (3.10) with $p = 1$ and $q = r = s-n = j$ ($j \neq i$). This case leads for $j > i$ to the relation

$$[4\,(f_j - 1)\, C_{ij} + 2\, C_{jj}\, A_{ji}]\, v_\varrho = 0. \tag{3.25}$$

Assume now again $f_j = 0$.

We have already proved that unless $f_1 = 1,\ f_2 = f_3 = \dots = f_n = 0$ we have $A_{ji}\, v_\rho = 0$, whence from Eq. (3.25) we get $C_{ij}\, v_\varrho = 0$ unless $i = 1$.

In conclusion, unless $f_1 = 1$ and $f_2 = f_3 = \dots = f_n = 0$ we have

$$C_{ij}\, v_\varrho = A_{ij}\, v_\varrho = 0$$

which is inadmissible if we remind that for any i, j we have $B_{ij}\, v_\rho = 0$. The only representation which satisfies Eqs. (3.9), (3.10), (3.11) is the natural representation Λ_1.

Theorem 3. a) There exist no representations ρ of the algebra $sp(2n,C)$, $n \geq 2$ on the states of which the second-degree $sp(2n,C)$– tensor operator $T_{(2\Lambda_2)}$ vanishes.

b) There exists a representation $\rho = (\Lambda_n)$ of the algebra $so(2n,C)$ $n \geq 5$ on the states of which the second-degree $so(2n,C)$ – tensor operator $T_{(2\Lambda_2)}$ vanishes.

The first result is obtained by proving that the vanishing of the corresponding polynomials in [5] leads either to the solution $f_1 = f_2 = \dots = f_n = 0$ or to incompatibility with the conditions the weight components f_i have to satisfy.

The second result is obtained by writing the tensor operator

$$T_{(2\Lambda_2)}(p,q;r,s) = \frac{1}{3}\left(2\{M_{pq}, M_{rs}\} - \{M_{ps}, M_{qr}\} - \{M_{pr}, M_{sq}\}\right)$$
$$+ \frac{1}{2n-2}\left(-\delta_{qr}\sum_{i=1}^{2n}\{M_{pi}, M_{is}\} + \delta_{qs}\sum_{i=1}^{2n}\{M_{pi}, M_{ir}\} + \delta_{pr}\sum_{i=1}^{2n}\{M_{qi}, M_{is}\} - \delta_{ps}\sum_{i=1}^{2n}\{M_{qi}, M_{ir}\}\right)$$
$$+ \frac{1}{(2n-1)(2n-2)}\left(\delta_{qr}\delta_{ps} - \delta_{qs}\delta_{pr}\right)\sum_{i,j=1}^{2n}\{M_{ij}, M_{ji}\} \quad ; \text{ for } \begin{cases} p = r = 2i \\ q = s = 2i-1 \end{cases},$$

in the Cartan-Weyl basis (cf. (I) Appendix). The equality $T_{(2\Lambda_2)} v_\rho = 0$ leads to

$$f_1 = f_2 = \dots = f_n = \frac{1}{2}.$$

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Преобразования, удовлетворяющие тождествам, связанным с тензорными операторами. Тензорные операторы второй степени, преобразующиеся по представлениям (Λ_4), $(2\Lambda_2)$ so (2n, c) и $(4\Lambda_1)$, $(2\Lambda_2)$ sp (2n, c)

Настоящая работа продолжает определение /начало в [6] / представлений ρ полупростой Ли алгебры L , для которых $T^{(2)}_\sigma(\rho) \equiv T^{(2)}_\sigma(x_1, \dots, x_n) = 0$, где $T^{(2)}_\sigma(x_1, \dots, x_n)$ – тензорный оператор, преобразующийся по представлению $\sigma \subset (ad \otimes ad)_s$, и x_i – генераторы ρ . Для L = so(2n,c) проверено, что, если $\sigma = (\Lambda_4)$, тогда $\rho = (k\Lambda_1)$ и, если $\sigma = (2\Lambda_2)$, тогда $\rho = (\Lambda_n)$.
Для L = sp(2n, c) проверено, что, если $\sigma = (4\Lambda_1)$, тогда $\rho = (\Lambda_1)$ и, если $\sigma = (2\Lambda_2)$, тогда уравнение $T^{(2)}_{(2\Lambda_2)}(\rho) = 0$ не имеет решений. (Λ_i – наибольший вес фундаментального представления (Λ_i)).

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Representations which Satisfy Identities Associated with Tensor Operators. Second-Degree Tensor Operators Transforming under the Representations: (Λ_4), $(2\Lambda_2)$ of so(2n,c) and $(4\Lambda_1)$, $(2\Lambda_2)$ of sp(2n,c)

The present work continues the determination (started in [6]) of the representations ρ of semisimple Lie algebras L for which $T^{(2)}_\sigma(\rho) = T^{(2)}_\sigma(x_1,\dots,x_n) = 0$, where $T^{(2)}_\sigma(x_1,\dots,x_n)$ is a tensor operator transforming under a subrepresentation σ of $(ad \otimes ad)_s$, and x_i are the generators of ρ. For L = so(2n, c) it is proved that if $\sigma = (\Lambda_4)$, then $\rho = (k\Lambda_1)$ and that if $\sigma = (2\Lambda_2)$, then $\rho = (\Lambda_n)$. For L = sp (2n, c) it is proved that if $\sigma = (4\Lambda_1)$, then $\rho = (\Lambda_1)$ and if $\sigma = (2\Lambda_2)$, then there exists no solution to the equation $T^{(2)}_{(2\Lambda_2)}(\rho) = 0$. ($\Lambda_i$ is the highest weight of the fundamental representation (Λ_i) (i = 1,...,n)).

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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