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REPRESENTATIONS WHICH SATISFY IDENTITIES ASSOCIATED WITH TENSOR OPERATORS.

Second-Degree Tensor Operators which Transform under the Representations $(2\Lambda_1)$ of so(2n,c) and (Λ_2) of sp(2n,c)

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1. Introduction

The existence of relations between the generators of linear representations of Lie algebras is a well-known fact and has been pointed out in connection with various physical problems (e.g.[1-10]).

In the last years, several methods for the construction of such identities have been given[11-18]. A procedure for the construction of the classical equivalent of these relations, i.e., of the relations between the generators of a Poisson bracket realization of a Lie algebra, has been pointed out in[19]. This procedure according to which the polynomial identities are associated with tensors with respect to subrepresentations of ($ad^{\otimes k})_s$ has been applied in[20] to the determination of second-degree polynomial identities for the nonexceptional Lie algebras A_n $(n \geq 3)$, B_n $(n \geq 2)$, C_n $(n \geq 2)$, D_n $(n \geq 5)$. From these "classical" identities, the corresponding "quantum" identities satisfied by the generators of linear representations can be easily obtained by symmetrization.

It is to be expected that these identities contain an amount of information about the representationswhich satisfy them. In previous papers[21,22], we proved that for a class of linear representations of the Lie algebras so ($2n$, R) and sp($2n$, R) which appear in the formulation of quantum field theory in terms of $O(n)$ -invariant pseudo-spin operators[23] and in the study of collective motion in nuclei[24], respectively, boson realizations of the Holstein-Primakoff-type can be obtained from the corresponding polynomial identities.

The aim of the present and of subsequent works is to prove that the polynomial identities satisfied by the generators of a linear representation of a Lie algebra provide information concerning the Dynkin indices of the representation.

The statement "a set of relations $P_k(X_1^{(\rho)}, \ldots, X_{\dim L}^{(\rho)}) = 0$ $(k \in \mathcal{K})$ is verified by the generators $X_i^{(\rho)}$ of a representation ρ " can be proved in different ways. If expressions for the generators $X_i^{(\rho)}$ $(i = 1, \ldots, \dim L)$ are known, to prove the statement amounts to a trivial verification. If such explicit expressions are not available, to prove the statement means to prove that relations $P_k(X_1^{(\rho)}, \ldots, X_{\dim L}^{(\rho)}) v_i = 0$ are true for any basis vector v_i , $i = 1, \ldots, \dim \rho$

of the representation space V_ϱ of ϱ. This condition can however be relaxed if the polynomials P_k $(k \in \mathcal{K})$ are generators of a tensor operator; in this case it is sufficient to prove the relation on the highest-weight vector of the representation ϱ.

To be more specific, let us first remind the definition of an irreducible tensor operator.

Let $\varrho : L \longrightarrow \mathrm{End}\,V$ be a linear representation of the Lie algebra L on the linear space V_ϱ. Let σ be an irreducible representation of L on the linear space W_σ. An irreducible tensor operator of type σ with respect to the representation ϱ is a linear mapping $t : W_\sigma \rightarrow \mathrm{End}\,V_\varrho$ such that (denoting the commutator by $[\ ,\ ]$)

$$[\,\varrho(x), t(w)] = t(\sigma(x)w) \tag{1.1}$$

for any $x \in L$ and $w \in W_\sigma$. Introducing a basis $\{w_j,\ j = 1,\dots,\dim\sigma\}$ in W_σ and denoting $t_j = t(w_j)$, Eq.(1.1) becomes

$$[\,\varrho(x), t_j] = \sum_{k=1}^{\dim\sigma} \sigma_{jk}(x)\, t_k \tag{1.2}$$

(for any $x \in L$ and any $j = 1,2,\dots,\dim\sigma$).

Let now the linear space W_σ be generated by a set of polynomials in $\dim L$ indeterminates $\{P_k(\xi_1,\dots,\xi_{\dim L});\ k=1,\dots,\dim\sigma\}$ which transform under the irreducible representation σ of L as

$$\sigma(x_i)\, P_j = \sum_{k=1}^{\dim\sigma} \sigma^i_{jk}\, P_k \tag{1.3}$$

($j = 1,\dots,\dim\sigma$; $i = 1,\dots,\dim L$) and let

$$t(P_j) \equiv P_j(X_1^{(\varrho)},\dots,X_{\dim L}^{(\varrho)}), \tag{1.4}$$

where $X_i^{(\varrho)} \equiv \varrho(x_i)$. Then

$$[X_i^{(\varrho)}, P_j(X_1^{(\varrho)},\dots,X_{\dim L}^{(\varrho)})] = \sum_{k=1} \sigma^i_{jk}\, P_k(X_1^{(\varrho)},\dots,X_{\dim L}^{(\varrho)}). \tag{1.5}$$

Let $v_\varrho \in V_\varrho$ be the vector corresponding to the highest weight of representation ϱ, assumed irreducible and finite-dimensional. If

$$P_k(X_1^{(\varrho)},\dots,X_{\dim L}^{(\varrho)})\, v_\varrho = 0 \quad (\text{any } k = 1,\dots,\dim\sigma) \tag{1.6}$$

then also

$$P_k(X_1^{(\varrho)},\dots,X_{\dim L}^{(\varrho)})\, X_i^{(\varrho)} v_\varrho = 0 \tag{1.7}$$

for any $k = 1,\dots,\dim\sigma$ and any $i = 1,\dots,\dim L$. This follows by writing

$$P_k X_i^{(\varrho)} v_\varrho = -[X_i^{(\varrho)}, P_k]\, v_\varrho + X_i^{(\varrho)} P_k v_\varrho \tag{1.8}$$

and taking into account the tensor operator property (1.5) and condition (1.6). But $X_{i_1}^{(\varrho)} X_{i_2}^{(\varrho)} \dots X_{i_k}^{(\varrho)} v_\varrho$ generate, for $\dim V_\varrho < \infty$, $i_1, i_2,\dots,i_k \in \{1,2,\dots,\dim L\}$ and k sufficiently large the whole space V_ϱ and this tells us that the validity of relations (1.6) is sufficient for having

$$P_k(X_1^{(\varrho)},\dots,X_{\dim L}^{(\varrho)})\, v = 0 \tag{1.9}$$

for any $v \in V_\varrho$.

The representations ϱ and σ are, in general, different; in particular, this will result from the subsequent theorem.

In the following, the tensors of type σ will be generated by second-degree homogeneous polynomials; As proved in [19,20], such tensorial sets, the elements of which vanish if the indeterminates in the polynomials are replaced by generators of a Poisson bracket realization or by the generators of a linear representation of the Lie algebra L, are provided by subrepresentations of the symmetric part of the direct square of the adjoint representation of L : $(\mathrm{ad} \otimes \mathrm{ad})_s$.

For the semisimple Lie algebras of types C_n and D_n which we shall analyse in the present paper, the Clebsch-Gordan series of $(\mathrm{ad}\otimes\mathrm{ad})_s$ are [25,26]:

$$\text{for } C_n\ (n \geq 2):\quad (\mathrm{ad}\otimes\mathrm{ad})_s = (0) \oplus (\Lambda_2) \oplus (4\Lambda_1) \oplus (2\Lambda_2) \tag{1.10}$$

$$\text{for } D_n\ (n \geq 5):\quad (\mathrm{ad}\otimes\mathrm{ad})_s = (0) \oplus (2\Lambda_1) \oplus (\Lambda_4) \oplus (2\Lambda_2) \tag{1.11}$$

where by Λ_i, $i = 1,\dots,n$ we denoted the fundamental weight system of a Lie algebra of rank n.

In [20] we obtained explicit expressions for the tensors which transform under the subrepresentations in (1.10), (1.11).

What we have to do is to apply such tensors (P_k , $k = 1, \ldots, \dim \sigma$) which depend on the generators of the algebra L to a vector v and ask that v be the highest weight vector v_ρ of a representation ρ and that $P_k v_\rho = 0$ for any $k = 1, \ldots, \dim \sigma$.

What will result is

(i) information concerning the weight of representation ρ : its Dynkin indices will be deduced.

(ii) information concerning the highest weight vector v_ρ .

Otherwise stated, the first point tells us that the tensorial set T_σ determines the representation ρ .

2. Determination of the representations ρ for which the tensor operators $\{T_\sigma ; \sigma = (2\Lambda_1)\}$ of $so(2n, C)$ and $\{T_\sigma ; \sigma = (\Lambda_2)\}$ of $sp(2n, C)$ vanish

We shall treat the two algebras $so(2n,C)$ and $sp(2n,C)$ simultaneously and use for their structure relations as well as for the expressions of the tensor operators expressions which unify both cases ; in the unified formulae, the two algebras are identified by the values taken by a parameter ε

$$\varepsilon = \begin{cases} +1 & \text{for } so(2n,C) \\ -1 & \text{for } sp(2n,C) \end{cases} \tag{2.1}$$

The structure relations of the two algebras expressed in Cartan-Weyl bases are then

$$[A_{ij}, A_{kl}] = \delta_{jk} A_{il} - \delta_{il} A_{kj} \tag{2.2}$$

$$[A_{ij}, B_{kl}] = \delta_{jk} B_{il} - \varepsilon \delta_{jl} B_{ik} \tag{2.3}$$

$$[A_{ij}, C_{kl}] = \varepsilon \delta_{il} C_{jk} - \delta_{ik} C_{jl} \tag{2.4}$$

$$[B_{ij}, C_{kl}] = -\delta_{jk} A_{il} - \delta_{il} A_{jk} + \varepsilon \delta_{ik} A_{jl} + \varepsilon \delta_{jl} A_{ik} \tag{2.5}$$

$$[B_{ij}, B_{kl}] = [C_{ij}, C_{kl}] = 0 . \tag{2.6}$$

The identities which result by equating to zero the expressions of the tensor operators which transform under the subrepresentation ($2\Lambda_1$) of ($ad \otimes ad$)$_s$ for $so(2n,C)$ and of the tensor operators which transform under the subrepresentation (Λ_2) of $sp(2n,C)$ for $sp(2n,C)$ are (cf [20] and Appendix):

$$\underset{\sim}{A}\underset{\sim}{B} - \underset{\sim}{B}\underset{\sim}{A}^t = -\varepsilon (\underset{\sim}{A}\underset{\sim}{B} - \underset{\sim}{B}\underset{\sim}{A}^t)^t \tag{2.7}$$

$$\underset{\sim}{C}\underset{\sim}{A} - \underset{\sim}{A}^t\underset{\sim}{C} = -\varepsilon (\underset{\sim}{C}\underset{\sim}{A} - \underset{\sim}{A}^t\underset{\sim}{C})^t \tag{2.8}$$

$$\underset{\sim}{A}^2 + ((\underset{\sim}{A}^t)^2)^t - \underset{\sim}{B}\underset{\sim}{C} - (\underset{\sim}{C}\underset{\sim}{B})^t = \frac{1}{n} Tr\left[\underset{\sim}{A}^2 + ((\underset{\sim}{A}^t)^2)^t - \underset{\sim}{B}\underset{\sim}{C} - (\underset{\sim}{C}\underset{\sim}{B})^t\right] \cdot \underset{\sim}{I} \tag{2.9}$$

In formulae (2.7-9) we denoted by $\underset{\sim}{X}$ the matrix (X_{ij}) and by $\underset{\sim}{X}^t$ the transposed of $\underset{\sim}{X}$.

In previous papers [21,22] we proved that the spinorial representations ($k\Lambda_{n-1}$) and ($k\Lambda_n$) of so $(2n, R)$ and the representation ($k\Lambda_n$) of $sp(2n, R)$, defined in [23,24] satisfy the relations (2.7-9).

The subsequent theorem proves that the spinorial representations ($k\Lambda_{n-1}$) and ($k\Lambda_n$) of $so(2n, C)$ and the representation ($k\Lambda_n$) of $sp(2n, C)$ are the <u>only</u> representations which satisfy these relations.

With the notation introduced above we can now state the following

<u>Theorem</u>. Let the Lie algebra L be so $(2n, C)$ with $n \geq 5$ and let us consider the second-degree tensor operator $T_{(2\Lambda_1)}$ which transforms under the representation $\sigma = (2\Lambda_1)$ of L . Then:

a1) if the action of $T_{(2\Lambda_1)}$ on (states of) a representation ρ of L vanishes, then either $\rho = (k\Lambda_{n-1})$ or $\rho = (k\Lambda_n)$, i.e., ρ is a spinorical representation of $so(2n, C)$.

a2) The highest weight vector v_ρ of the representation ρ satisfies the conditions

$$A_{ij} v_\rho = 0 \qquad \text{for} \quad i > j \qquad \text{if} \quad \rho = (k\Lambda_n)$$
$$C_{n,n-i} A_{n,n-i} v_\rho = C_{in} v_\rho = C_{ni} v_\rho = 0 \qquad \text{for} \quad i = 1,2,\ldots,n-1 \qquad \text{if} \quad \rho = (k\Lambda_{n-1}),$$

Let the Lie algebra L be $sp(2n, C)$ with $n \geq 2$ and let us consider the second-degree tensor operator $T_{(\Lambda_2)}$ which transforms under the representation $\sigma = (\Lambda_2)$ of L. Then:

b1) if the action of $T_{(\Lambda_2)}$ on (states of) a representation ρ of L vanishes, then $\rho = (k\Lambda_n)$.

b2) The highest weight vector v_ρ of the representation ρ satisfies the conditions $A_{ij}\, v_\rho = 0$ for $i > j$.

P r o o f. Let us remark that for both algebras under consideration:

i) the operators A_{ii}, $i = 1, \ldots, n$ are generators of the Cartan subalgebra; hence, denoting by v_ρ the highest-weight vector of representation ρ, we have

$$A_{ii}\, v_\rho = f_i\, v_\rho \, , \tag{2.10}$$

where f_i denotes the i-th component of the weight;

ii) the operators A_{ij} with $i<j$ and B_{ij} with arbitrary i and j are raising operators, hence

$$A_{ij}\, v_\rho = 0 \qquad \text{for} \quad i<j \tag{2.11}$$

and

$$B_{kl}\, v_\rho = 0 \qquad (\text{any } k, l)\, ; \tag{2.12}$$

iii) the operators A_{ij} with $i>j$ and C_{kl} with orbitrary k and l are lowering operators.

We shall examine successively the effect of applying the relations (2.7),(2.9) and (2.8) to a highest-weight vector.

Both sides of Eq.(2.7) applied to the highest weight vector v_ρ vanish as a consequence of (2.3) and (2.12). This relation gives no information.

Let us consider Eq. (2.9). We have

$$\begin{aligned} &[\, \underset{\sim}{A}^2 + ((\underset{\sim}{A}^t)^2)^t - \underset{\sim}{B}\underset{\sim}{C} - (\underset{\sim}{C}\underset{\sim}{B})^t\,]_{ij}\, v_\rho = \\ &= 2\, (\underset{\sim}{A}^2 + (Tr\, \underset{\sim}{A})\underset{\sim}{I} - \varepsilon \underset{\sim}{A})_{ij}\, v_\rho \end{aligned} \tag{2.13}$$

and identity (2.9) applied to v_ρ becomes

$$(\underset{\sim}{A}^2 - \varepsilon \underset{\sim}{A})_{ij}\, v_\rho = \frac{\delta_{ij}}{n}\, Tr\, (\underset{\sim}{A}^2 - \varepsilon \underset{\sim}{A})\, v_\rho \tag{2.14}$$

a relation valid for any i and j. We shall examine separately the different possible relations between i and j.

Let us consider first the case $i = j$ and denote

$$\frac{1}{n}\, Tr\, (\underset{\sim}{A}^2 - \varepsilon \underset{\sim}{A})\, v_\rho = c\, v_\rho \, . \tag{2.15}$$

Relation (2.14) becomes, taking into account (2.2), (2.10),(2.11),

$$\begin{aligned} &\Big(\sum_{l=1}^{n} A_{il} A_{li} - \varepsilon A_{ii} \Big) v_\rho = \Big(f_i^2 - \sum_{l>i} A_{il} A_{li} - \varepsilon f_i \Big) v_\rho \\ &= \Big(f_i^2 + (n - i - \varepsilon) f_i - \sum_{l=i+1}^{n} f_l \Big) v_\rho = c\, v_\rho \end{aligned} \tag{2.16}$$

whence

$$\sum_{l=i+1}^{n} f_l = f_i^2 + (n - i - \varepsilon)\, f_i - c \tag{2.17}$$

and

$$\sum_{l=i+2}^{n} f_l = f_{i+1}^2 + (n - i - 1 - \varepsilon)\, f_{i+1} - c\, . \tag{2.18}$$

Subtracting the second equality from the first, we get the set of equations

$$(f_i - f_{i+1})(f_i + f_{i+1} + n - i - \varepsilon) = 0 \qquad (i = 1, 2, \ldots, n-1)\, . \tag{2.19}$$

The system (2.19) admits several solutions; we have to retain only those solutions which are compatible with the conditions which have to be satisfied by the components f_i of the weight, namely

$$f_1 \geqslant f_2 \geqslant \cdots \geqslant f_n \geqslant 0 \qquad \text{for} \quad sp\,(2n, C) \tag{2.20}$$

and

$$f_1 \geqslant f_2 \geqslant \cdots \geqslant f_{n-1} \geqslant |f_n| \qquad \text{for} \quad so\,(2n, C)\, . \tag{2.21}$$

An admissible solution for both cases $\varepsilon = \pm 1$ is

$$f_1 = f_2 = \ldots = f_n \tag{2.22}$$

which results from the vanishing of the first factor in each product in (2.19).

Let us admit now that one of the other factors vanishes, i.e., that

$$f_i + f_{i+1} + n - i - \varepsilon = 0\, . \tag{2.23}$$

We have to distinguish two cases:

1. $\varepsilon = -1$. Then, for any $i = 1,2,\ldots, n-1$, condition (2.23) leads to $f_i + f_{i+1} < 0$ which is incompatible with condition (2.20). Hence, for $sp\,(2n, C)$ the solution (2.22) is the only admissible.

2. $\varepsilon = +1$. In this case, for any $i = 1,2,\ldots, n-2$ we have

$f_i + f_{i+1} < 0$ again in contradiction with (2.21). However, for $\varepsilon = +1$ and for $i = n-1$ equation (2.23) leads to

$$f_n + f_{n-1} = 0 \qquad (2.24)$$

which is admissible. This result leads in the case of the algebra $so(2n, C)$ to the existence of a second solution, namely

$$f_1 = f_2 = \dots = f_{n-1} = -f_n . \qquad (2.25)$$

Taking into account the relations between Dynkin indices ($m_1, \dots, m_n$) and weights ($f_1, \dots, f_n$) for a representation Λ of a rank n Lie algebra

$$\Lambda = \sum_{j=1}^{n} m_j \Lambda_j \quad m_j \text{ - nonnegative integers} \qquad (2.26)$$

namely

$$f_j = m_j + m_{j+1} + \dots + m_n \qquad (2.27)$$

for C_n algebras and

$$f_j = m_j + m_{j+1} + \dots + m_{n-2} + \tfrac{1}{2}(m_{n-1} + m_n) \qquad (2.28)$$

$$f_{n-1} = \tfrac{1}{2}(m_{n-1} + m_n) \quad , \quad f_n = \tfrac{1}{2}(-m_{n-1} + m_n)$$

for D_n algebras, we obtain that the representation ς the highest weight components of which satisfy Eqs.(2.22) is ($m_n \Lambda_n$) both for $sp(2n, C)$ and for $so(2n, C)$ and that the representation the weight components of which satisfy (2.25) is ($m_n \Lambda_{n-1}$). The highest weights corresponding to ($m_n \Lambda_n$) and ($m_n \Lambda_{n-1}$) are ($\frac{m_n}{2}, \frac{m_n}{2}, \dots, \frac{m_n}{2}$) and ($\frac{m_n}{2}, \frac{m_n}{2}, \dots, \frac{m_n}{2}, -\frac{m_n}{2}$), respectively for $so(2n, C)$.

Let us consider now the cases $i < j$ and $i > j$. In these cases Eq. (2.14) becomes

$$(\underset{\sim}{A}^2 - \varepsilon \underset{\sim}{A})_{ij} \, v_\varsigma = 0 . \qquad (2.29)$$

It is easy to prove that taking into account the relation (2.11) Eq. (2.29) is identically satisfied if $i < j$.

The case $i > j$ is more interesting because it provides information concerning the highest weight vector v_ς of representation ς . Eq. (2.29) becomes in this case

$$\sum_{i > \ell > j} A_{\ell j} A_{i\ell} v_\varsigma = [\varepsilon - (f_i + f_j) - (n-j)] A_{ij} v_\varsigma . \qquad (2.30)$$

We shall analyse this equation separately for the representations ($m_n \Lambda_n$),i.e., $f_1 = f_2 = \dots = f_n = \frac{m_n}{2}$ and ($m_n \Lambda_{n-1}$),i.e., $f_1 = f_2 = \dots = f_{n-1} = f_n = \frac{m_n}{2}$.

a) Let $i > j$ and $f_1 = f_2 = \dots = f_n = \frac{m_n}{2}$ we shall prove that, in this case,

$$A_{ij} v_\varsigma = 0 . \qquad (2.31)$$

For the case under consideration, relation (2.30) becomes

$$\sum_{i > \ell > j} A_{\ell j} A_{i\ell} v_\varsigma = (\varepsilon - m_n - (n-j)) A_{ij} v_\varsigma . \qquad (2.32)$$

We shall prove relation (2.31) by induction. Let $i = j+1$; the $l.h.s.$ in (2.32) vanishes and we have $\varepsilon - m_n - (n-j) \neq 0$. Otherwise $m_n = \varepsilon - (n-j)$ and this leads for $j < n-1$ to $m_n < 0$ and for $j = n-1$ and $\varepsilon = +1$ to $m_n = 0$, hence to $f_i = 0$ for all i.

Let us now admit that

$$A_{j+1,j} v_\varsigma = \dots = A_{j+p,j} v_\varsigma = 0 \qquad (2.33)$$

and prove that

$$A_{j+p+1,j} v_\varsigma = 0 . \qquad (2.34)$$

We have taken the Lie relation (2.2) and the induction condition (2.38) into account:

$$\sum_{j+p+1 > \ell > j} A_{\ell j} A_{j+p+1,\ell} v_\varsigma = -(p-1) A_{j+p+1,j} v_\varsigma . \qquad (2.35)$$

Eq.(2.32) becomes

$$[\varepsilon - m_n + (p-1) - (n-j)] A_{j+p+1,j} v_\varsigma = 0 . \qquad (2.35)$$

The numerical factor of $A_{j+p+1,j} v_\varsigma$ cannot vanish. Otherwise we should have

$$m_n = \varepsilon + p - 1 - (n-j) . \qquad (2.37)$$

But $1 \le p \le n-j-1$, whence $m_n \le \varepsilon - 2 < 0$, for $\varepsilon = \pm 1$ in contradiction with the condition $m_n \geq 0$. Relation (2.31) is proved.

b) Let $i > j$ and $f_1 = f_2 = \dots = f_{n-1} = -f_n = \frac{m_n}{2}$ Remind that this weight is obtained only for $\varepsilon = +1$.

If $i,j \leq n-1$, the previous proof keeps valid.

If $i = n$, $j = n-p$, equation (2.30) becomes, taking into account the Lie relation (2.2) and relation (2.31), for $i, j < n-1$

$$\sum_{n,\ l > n-p} A_{l,n-p} A_{n,l} v_\xi = -(p-1) A_{n,n-p} v_\xi = (\varepsilon - p) A_{n,n-p} v_\xi \tag{2.38}$$

whence

$$(\varepsilon - 1) A_{n,n-p} v_\xi = 0 . \tag{2.39}$$

This relation is verified for any $p < n$, because $\varepsilon = +1$, but it does no more allow one to conclude that $A_{n,n-p} v_\xi = 0$.

The only equation which remains to be analysed is (2.8).

If we apply to v_ξ the relation obtained by taking the matrix elements of relation (2.8) we obtain (using (2.4), (2.10))

$$(f_j - f_i) C_{ij} v_\xi + \sum_{l > j} C_{il} A_{lj} v_\xi - \sum_{l > i} C_{lj} A_{li} v_\xi = 0 . \tag{2.40}$$

This result does no more depend on ε because of a compensation of signs resulting from the fact that for $so(2n, C)$ we have $\varepsilon = +1$ and $C_{ji} = - C_{ij}$ while for $sp(2n, C)$ we have $\varepsilon = -1$ and $C_{ji} = C_{ij}$.

The following cases have to be considered separately

1) i, j , arbitrary, $\varepsilon = \pm 1$ and $f_1 = f_2 = \dots = f_n$.

In this case, using relation (2.31) it results that (2.40) is verified for any $i = j$ if v_ξ is the highest weight vector of representation (Λ_n). No condition for v_ξ emerges.

2) $i = j$, $\varepsilon = +1$ and $f_1 = f_2 = \dots = f_{n-1} = -f_n$ we obtain the conditions

$$C_{n,n-i} A_{n,n-i} v_\xi = 0 . \tag{2.41}$$

3) $i \neq j$, $\varepsilon = +1$, $f_1 = f_2 = \dots = f_{n-1} = -f_n$; if $i, j \leq n-1$ relation (2.40) is satisfied; no condition for v_ξ emerges in addition to (2.31).

If $i = n$, $j < n$, (2.40) becomes

$$C_{nj} v_\xi = 0 \quad (j = 1, 2, \dots, n-1) . \tag{2.42}$$

APPENDIX

In paper /20/, using for the generators of the $so(2n, C)$ algebra the basis M_{ij} ($M_{ji} = - M_{ij}$) $i,j = 1, 2, \dots, 2n$ defined by the Lie relations

$$[M_{ij}, M_{kl}] = \delta_{il} M_{jk} + \delta_{jk} M_{il} - \delta_{ik} M_{jl} - \delta_{jl} M_{ik} \tag{A.1}$$

we obtained the following expressions for the tensors which transform under the representation $(2\Lambda_1)$

$$\sum_{k=1}^{2n} M_{qk} M_{kr} - \frac{1}{2n} \delta_{qr} \sum_{l,m=1}^{2n} M_{lm} M_{ml} . \tag{A.2}$$

The expressions (A2) have been obtained also in /25/.

The transformation from the basis (2) to the Cartan Weyl basis is defined by $(i = \sqrt{-1})$:

$$\begin{aligned} M_{2k,2l-1} &= -\frac{i}{2} (B_{kl} + C_{kl} - A_{kl} - A_{lk}) \\ M_{2k-1,2l} &= -\frac{i}{2} (B_{kl} + C_{kl} + A_{kl} + A_{lk}) \\ M_{2k-1,2l-1} &= -\frac{1}{2} (B_{kl} - C_{kl} - A_{kl} + A_{lk}) \\ M_{2k,2l} &= \frac{1}{2} (B_{kl} - C_{kl} + A_{kl} - A_{lk}) . \end{aligned} \tag{A3}$$

These transformations applied to the tensor (A2) and followed by a symmetrization lead to tensor operators which transform under $(2\Lambda_1)$: The expressions of these tensor operators written in matrix form and ascribed to the condition that also their components vanish in the representation ξ lead to the conditions (2.7-9) (with $\varepsilon = +1$) which constituted the starting point of our considerations.

References

1. P.A.M.Dirac. Proc. Roy. Soc. A155, 447 (1936).
2. J.D.Louck. "Theory of Angular Momentum in N-Dimensional Space". Los Alamos Scientific Lab. Report La 2451 (1966); Amer. J. Phys. 31, 378 (1963).
3. R.H.Pratt, T.F. Jordan. Phys.Rev. 148, 1276 (1966).
4. C.E.Wulfman, Y.Takahata. J. Chem. Phys. 47, 488 (1967).

5. G. Györggi, S. Kövesi-Domokos. Nuovo Cim., 58B, 191 (1968).

6. G. Györgyi. Nuovo Cim. 62A, 449 (1969).

7. A.O. Barut. A. Böhm. J. Math. Phys. 11, 2938 (1970).

8. I. Gyuk. H. Umezawa. Phys.Rev. D3, 899 (1971).

9. L.P. Staunton. J. Math. Phys. 19, 1471 (1978); Phys.Rev. D8, 2446 (1973), D10, 1760 (1974), D13, 3269 (1976).

10. I.V. Komarov. Teor. Mat. Fiz. 53, 296 (1982).

11. Y. Lehrer Ilamed. Bull. Counc. Israel 5A, 197 (1956).

12. A.J. Bracken, H.S. Green.J. Math.Phys. 12, 2099 (1971).

13. H.S. Green. J.Math. Phys. 12, 2106 (1971).

14. K.C. Hannabuss. "Characteristic Equations for Semi-Simple Lie Groups", Preprint Mathematical Institute, Oxford (1972).

15. S. Okubo. J. Math.Phys. 16, 528 (1975).

16. S. Okubo. J. Math.Phys. 18, 2382 (1977).

17. D.M. O'Brien, A. Cant. A.L. Carey, Ann. Inst. H.Poincaré, 26, 405 (1977).

18. M.D. Gould. "Characteristic Identities for Semisimple Lie Algebras", Preprint, University of Adelaide (1983).

19. M. Iosifescu, H. Scutaru. J. Math. Phys. 25, 2856 (1984).

20. M. Iosifescu, H. Scutaru. "Polynomial Constraints Associated to Dynamical Symmetries". Preprint, Bucharest FT-253 (1984).

21. M. Iosifescu, H. Scutaru. "Boson Realizations from Quantum Constraints", Preprint, Bucharest FT-255 (1984).

22. M. Iosifescu, H. Scutaru. "Holstein-Primakoff Realization from Quantum Constraints" (Physics Letters A (1985) (to appear).

23. N. Papanicolaou. Annals of Phys. (N.Y.) 136, 210 (1981).

24. J. Deenen, C. Quesne. J. Math. Phys. 23, 2004 (1982).

25. P.D. Jarvis. J. Phys. A, Math. Gen., 12, 1 (1979).

26. S. Okubo. J. Math. Phys. 20, 586 (1979).

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Представления, удовлетворяющие тождествам, связанным с тензорными операторами. Тензорные операторы второй степени, преобразующиеся по представлениям $(2\Lambda_1)$ so (2n,c) и (Λ_2) sp (2n,c)

В предыдущих работах установлен метод для определения всех полиномиальных тождеств, которые удовлетворяются генераторами x_i представлений ρ алгебры Ли. Эти тождества имеют вид $T_\sigma^{(k)}(x_1,x_2,\dots,x_n)=0$, где $T_\sigma^{(k)}(x_1,x_2,\dots,x_n)$ - тензорный оператор преобразующийся по подпредставлению $\sigma \subset (\mathrm{ad}^{\otimes k})_s$. В настоящей работе определяются представления, для которых $T^{(2)}_{(2\Lambda_1)}(\rho)=0$ для so(2n,c) и $T^{(2)}_{(\Lambda_2)}(\rho)=0$ для sp(2n,c). Употребляется для этого условие $T_\sigma^{(2)}(\rho)\times \times v_\rho=0$, где v_ρ - вектор наибольшего веса ρ. Проверено, что в первом случае $\rho=(k\Lambda_{n-1})$ или $\rho(k\Lambda_n)$, а во втором $\rho=(k\Lambda_n)$. Получены также уравнения для v_ρ.

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Representations which Satisfy Identities Associated with Tensor Operators. Second-Degree Tensor Operators which Transform under the Representations $(2\Lambda_1)$ of so (2n,c) and (Λ_2) of sp (2n,c)

In previous works a method has been pointed out for the determination of all polynominal identities which can be satisfied by the generators x_i of representations ρ of a Lie algebra. These identities are $T_\sigma^{(k)}(x_1,\dots,x_n) = 0$, where $T_\sigma^{(k)}(x_1,\dots,x_n)$ is a tensor operator transforming under the subrepresentation σ of $(\mathrm{ad}^{\otimes k})_s$. The present work determines the representations ρ for which $T^{(2)}_{(2\Lambda_1)}(\rho) = 0$ for so(2n,c) and $T^{(2)}_{(\Lambda_2)}(\rho) = 0$ for sp (2n,c). They result from the condition $T_\sigma^{(2)}(\rho)v_\rho = 0$ (v_ρ = highest weight vector of ρ). It is proved that in the first case $\rho = (k\Lambda_{n-1})$ or $\rho = (k\Lambda_n)$ and that in the second $\rho = (k\Lambda_n)$. Equations for v_ρ are also determined

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