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NONUNIQUE SOLUTION OF THE SYSTEM OF INTEGRO-DIFFERENTIAL FADDEEV EQUATIONS IN THE CASE OF CONFINING OF s-WAVE POTENTIALS

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The Faddeev equations in the configurational space [1] are intensively utilized for studying different three-particle systems. First, this is due to the existence and uniqueness of solutions of these equations being established for a wide class of two-body potentials bound at large distances [2] and second, to a comparatively simple scheme of numerical solution characteristic of differential equations. Indeed, in the finite-difference approximation [1] as well as in the approximation of sought solutions by bicubic Hermitian splines [3] or those of the class C^2 [4], the initial problem is reduced to a system of linear equations with a band matrix. The efficiency of algorithms based on discretization of the differential Faddeev equations depends essentially on properly chosen mesh points. Such a choice, i.e., the construction of mesh points, is determined by qualitative properties of the solutions sought. These properties are necessary, in particular, when solving three-particle equations by variational methods and methods based on finite dimensional approximation of the functions [5] and operators [6].

The present paper is aimed at investigating qualitative properties of the wave function components of a bound state of three particles interacting via s-wave potentials. In the case of s-wave potentials $V_i(x_i)$, $i = 1,2,3$ and zero values of orbital moments $\ell = \lambda = L = 0$ the wave function of a bound three-particle state in polar coordinates $\rho^2 = x_i^2 + y_i^2$, $\phi_i = \mathrm{arctg}\, y_i/x_i$ has the form

$$\psi(\rho, \phi_i) = \rho^{-2} \mathrm{coses}\, 2\phi_i \, F(\rho, \phi_i). \qquad (1)$$

The function F is expressed through the Faddeev components Φ_i

$$F(\rho, \phi_i) = \Phi_i(\rho, \phi_i) + \sum_{k \neq i} \langle \rho, \phi_i | \hat{h} | \Phi_k \rangle \qquad (2)$$

which are the solution of the system of three equations

$$(\Delta_{\rho\phi_i} + E)\Phi_i(\rho, \phi_i) = V_i(\rho \cos\phi_i) F(\rho, \phi_i), \quad i = 1, 2, 3, \qquad (3)$$

where $\Delta_{\rho\phi_i}$ is the Laplacian in polar coordinates.

It is assumed that $\psi(\rho, \phi) \in C^2_\Omega \cap L_{2\Omega}$, $V_i(x) \in C^2_{[0, \pi/2]}$ where $\Omega = \{\rho, \phi, \rho \in [0, \infty], \phi \in [0, \frac{\pi}{2}]\}$ therefore system (3) is added by the boundary conditions

$$\Phi_i(\rho, \phi_i) = 0, \ \phi_i = 0, \frac{\pi}{2}, \ \rho \in [0, \infty], \tag{4}$$

$$\Phi_i(\rho, \phi_i) = 0, \ \rho = 0, \infty, \ \phi \in [0, \frac{\pi}{2}], \ i = 1, 2, 3. \tag{5}$$

From representations (1) and (2) there follows asymptotic behaviour

$$\Phi_i(\rho, \phi_i) \underset{\rho \to 0}{\to} \rho^{2+\epsilon}, \quad \epsilon \geq 0, \ i = 1, 2, 3. \tag{6}$$

The operator $\hat{h}$ acts on the angular variable only /5/. Its eigenfunctions $y_n(\phi) = \frac{2}{\sqrt{\pi}} \sin 2n\phi$, $n = 1, 2, \ldots$ satisfying boundary conditions (4) form an angular orthonormalized basis of system (3). The corresponding eigenvalues /5/ $\lambda_n(\gamma_{ik}) = \frac{\sin 2n\gamma_{ik}}{n \sin 2\gamma_{ik}}$ depend on the angles $\gamma_{ik} \in [0, \frac{\pi}{2}]$ the values of which are fixed only by the ratio of particle masses $\mathrm{tg}^2 \gamma_{ik} = \frac{m_j(m_1 + m_2 + m_3)}{m_i m_k}$ and by the definition satisfying the relations

$$\gamma_{ij} = \gamma_{ji}, \ \gamma_{ik} + \gamma_{jk} \geq \frac{\pi}{2}, \quad \gamma_{ij} + \gamma_{ik} + \gamma_{jk} = \pi, \tag{7}$$

$ijk = 123, 231, 312$.

Solution of system (3-5) is sought in the form

$$\Phi_i(\rho, \phi_i) = \sum_{n=1}^{\infty} f_{in}(\rho) y_n(\phi_i), \ i = 1, 2, 3, \tag{8}$$

then for unknown functions f_{in} we get an infinite system of the second-order differential equations. For unknown columns $f_n = (f_{1n}, f_{2n}, f_{3n})^T$ it is convenient to write down this system in the matrix form

$$(\Delta_\rho^n + E) I f_n(\rho) = \sum_{m=1}^{\infty} V_{nm}(\rho) A_m f_m(\rho), \ n = 1, \ldots \tag{9}$$

Here the symbol Δ_ρ^n denotes the differential operator $\frac{d}{\rho d\rho}(\rho \frac{d}{d\rho}) - \frac{4n^2}{\rho^2}$, I the unit matrix (3x3), and the matrix elements A_m and V_{nm} are defined respectively by

$$(A_m)_{ij} = \delta_{ij} + (1 - \delta_{ij}) \lambda_m(\gamma_{ij})$$

$$(V_{nm})_{ij} = \delta_{ij} \int_0^{\pi/2} dt\, y_n(t) y_m(t) V_i(\rho \cos t), \ i, j = 1, 2, 3, \ n, m = 1, 2, \ldots$$

Unknown functions f_{in} owing to (5,6) should satisfy

$$f_{in}(\rho) = 0, \ \rho = 0, \infty, \ f_{in}(\rho) \underset{\rho \to 0}{\to} \rho^{2+\epsilon}, \quad \epsilon \geq 0. \tag{10}$$

Using the properties of eigenvalues of the operator $\hat{h}$, which have been obtained in ref. /5/ and relations (7), one can easily prove that

$$\mathrm{rank} A_1 = 1, \ \mathrm{rank} A_2 = 2, \ \mathrm{rank} A_m = 3, \ m \geq 3, \tag{11}$$

if masses of all particles are finite and nonzero. Assume that some column f_k satisfies system (9,10) and simultaneously is a nontrivial solution of system $A_k f_k(\rho) = 0$. The latter, owing to (11), is possible only at $k = 1, 2$. In this case function (2)

$$F(\rho, \phi_i) = \sum_{m=1}^{\infty} (A_m \cdot f_m(\rho))_i \, y_m(\phi_i), \tag{12}$$

and consequently, the wave function (1) expressed in any set of polar coordinates ($i = 1, 2, 3$) will not contain terms $f_{jk}(\rho) y_k(\phi_i)$, $j = 1, 2, 3$ and system (9) will be separated into two parts. The first part determines the spectrum of eigenvalues E and has the form

$$(\Delta_\rho^n + E) I f_n(\rho) = \sum_{m \neq k}^{\infty} V_{nm} A_m f_m(\rho) \tag{13}$$

and the second one is the system of uncoupled inhomogeneous Bessel equations

$$(\Delta_\rho^k + E) I f_k = \sum_{m \neq k}^{\infty} V_{km} A_m f_m = \chi = (\chi_1, \chi_2, \chi_3)^T. \tag{14}$$

If the bound state energy is positive ($E > 0$), the set of solutions of system (10,14) is such

$$f_k(\rho) = a J_{2k}(t) + \frac{\pi}{2} [Y_{2k}(t) \int_0^\rho du\, u J_{2k}(u) \chi(u) - J_{2k}(t) \int_0^\rho du\, u\, Y_{2k}(u) \chi(u)]. \tag{15}$$

Here $t = \sqrt{E}\rho$, $a = (a_1, a_2, a_3)^T$ is the numerical column containing three elements. The functions (15) have asymptotic behaviours

$$f_{ik}(\rho) \underset{\rho \to 0}{\to} a_i \left(\frac{t}{2}\right)^{2k} / \Gamma(2k + 1) + O(\rho^{4+\sigma}), \ \sigma \geq 0, \tag{16}$$

$$f_{ik}(\rho) \underset{\rho \to \infty}{\to} O(t^{-1/2}), \ i = 1, 2, 3, \tag{17}$$

and obviously, satisfy the boundary conditions (10) at any values of a_i. The asymptotic behaviour of (16) of the summed functions (15), i.e., particular solutions of the inhomogeneous Bessel equations (14) and general solutions of the corresponding homogeneous equations ($\chi = 0$) are different. Therefore, the column a should satisfy the system $A_k a = 0$. At $k = 1 (k=2)$ according to equalities (11) only two (one) of the numbers a_1, a_2, a_3 are arbitrary.

If the bound state energy is negative, system (10,14) has obviously a single solution

$$f_k(\rho) = K_{2k}(t)\int_0^\rho du\, u\, I_{2k}(u)\chi(u) - I_{2k}(t)\int_0^\rho du\, u\, K_{2k}(u)\chi(u),$$

where $t = \sqrt{-E}\rho$, I_ν, K_ν are the modified Bessel functions. Indeed, in this case only a trivial solution of the homogeneous system of equations, corresponding to system (14), satisfies the boundary conditions (10).

We would like to emphasize that only a possibility is proved for the existence of a nonunique solution of system (3-5) in case of positive energy of the three-particle bound state. Namely, if the solutions of this system can be expressed as a sum $\Phi_i = U_i + S_i$, $i = 1,2,3$ so that the wave function (1) in any set of coordinates is expressed only through the term U_i, then the term $S_i = f_{ik}(\rho) y_k(\phi_i)$ has a quite definite angular dependence ($k = 1,2$) and the functions have the form of (15) and asymptotic behaviour (16), (17).

Let us prove that in case of identical particles and positive energy of their bound state there exists a nonunique solution of system (3-5). If particles are identical, all the angles γ_{ij} equal $\pi/3$, and the system (3-5) is reduced to one equation for the unknown function $\Phi(\rho, \phi)$. Representing this function as a series (8) we arrive at the system of equations (9) where I, f_n, V_{nm}, $A_m = 1 + 2\lambda_m(-\frac{\pi}{3})$ are the unit dimension matrices. The equality $A_k = 0$ is valid only at $k = 2$, i.e., of all the functions y_m, $m = 1,2\ldots$ only one $y_2(\phi)$ is mapped by the operator $(1 + 2\hat{h})$ into zero. The sum (12) and consequently the wave function (1) do not contain the term $S = f_2(\rho)y_2(\phi)$, and the part of the function f_2 (15) is determined up to the multiplication by an arbitrary number a. This property is independent of the form of confining potentials securing the existence of a bound state of the system of three particles with positive energy. Note that in solving the system (3-5) numerically one has to limit the interval of changing of variable ρ to a large but final value of R and to use instead of point boundary conditions (5) approximate ones $\Phi(R, \phi) = 0$. In this case the number a is automatically fixed by the condition $f_2(R) = 0$ if only $J_4(\sqrt{E}R) \neq 0$. Now we consider the system of three identical particles interacting via the potentials $V(x) = (\omega^2 x^2)/6$. In this case the system of equations (9) is

$$(\Delta_\rho^1 + E - \frac{\omega^2\rho^2}{4}) f_1(\rho) = 0, \tag{18}$$

$$(\Delta_\rho^n + E) f_n(\rho) = \frac{\omega^2\rho^2}{24}\sum_{m=n-1}^{n+1} A_m(1 + \delta_{nm}) f_m(\rho), \quad n \geq 3, \tag{19}$$

$$(\Delta_\rho^2 + E) f_2(\rho) = \frac{\omega^2\rho^2}{24}[3f_1(\rho) + f_3(\rho)] \tag{20}$$

and is supplemented by the boundary conditions (10). Equations (18,19) correspond to system (13); and eq. (20), to system (14). The three-particle bound-state energies $E_p = \omega(3 + 2p)$ are eigenvalues of eq. (18). They correspond to the eigenfunctions

$$f_1(\rho, p) = \rho^2 L_p^2(\frac{\omega\rho^2}{2})\exp(-\frac{\omega\rho^2}{4}), \quad p = 0, 1, \ldots \tag{21}$$

where L_p^2 are the Laguerre polynomials. The functions identically equal to zero

$$f_n(\rho) \equiv 0, \quad \rho \in [0, \infty], \quad n \geq 3 \tag{22}$$

satisfy, obviously, system (10,19). In this case the solutions of eq. (10,20) for the p-th level have the form of (15), where $k = 2$, $\chi(\rho) = \frac{\omega^2\rho^2}{8} f(\rho, p)$ i.e.,

$$f_2(\rho, p) = \frac{a}{\omega} J_4(t_p) + \frac{\pi\omega^2}{16}[Y_4(t_p)\int_0^\rho du\, u^3 J_4(u) f_1(u, p) - J_4(t_p)\int_0^\rho du\, u^3 Y_4(u) f_1(u, p)]. \tag{23}$$

Here $t_p = \sqrt{E_p}\rho$, a is an arbitrary number. The Faddeev components corresponding to the solution (21-23) are the sum of two terms

$$\Phi(\rho, \phi, p) = \frac{2}{\sqrt{\pi}}[f_1(\rho, p)\sin 2\phi + f_2(\rho, p)\sin 4\phi], \tag{24}$$

the first being of the type U and the second of the type S. The wave function (1) of the p-th state

$$\psi_p(\rho, \phi) = 3\rho^{-2} f_1(\rho, p) \tag{25}$$

does not contain the second terms of the components (24) and coincides up to the normalization factor with the solution obtained by the method of K-harmonics [7]. In the case of $p = 0$, i.e., for the ground state of the three-particle system, equalities (21), (23-25) reproduce the known result of ref. [8]. Thus, expressions (21,23-25) are the exact solution of the Faddeev system of equations not only for the ground state obtained in ref.[8], but for all other states too.

The reason for the existence of nonunique solutions of the three-particle equations in case of different particles and the existence of such solutions in case of identical ones is an arbitrary normalization of confining potentials[9]. If the potentials are limited at large values of relative distances ($V(x) \to B < \infty$, $x \to \infty$), they are normalized by the condition $B = 0$. The energy is reckoned from the level $E = 0$ and turns out to be negative for bound states and the components of the wave function are determined nonuniquely. For unlimited potentials ($V(x) \to \infty$, $x \to \infty$) there exists arbitrariness in the level of energy reckoning, and consequently, in the sign of the bound state energy. Indeed, assume that the bound state ψ energy E of the system of three particles interacting via s-wave confining potentials is positive. The Schrödinger equation

$$(H_0 + \sum_{i=1}^{3} V_i - E)\psi = 0 \qquad (26)$$

is in the form

$$(H_0 + \sum_{i=1}^{3} V_i' - E')\psi = 0, \quad V_i' = V_i - \frac{C}{3}, \quad E' = E - C. \qquad (27)$$

Choosing the number C large enough $C > E$ we get that the function ψ, i.e., the solution (26) corresponding to $E > 0$, is the solution of eq.(27) with negative energy $E' < 0$. So, if eq.(26) is reduced to the system (3-5), the components may have nonuniquely determined terms (15) giving no contribution to the wave function. If eq.(27) is reduced to this system, the components Φ_i do not contain such terms.

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Пупышев В.В. E4-85-313

О неоднозначном решении системы интегродифференциальных уравнений Фаддеева в случае запирающих s-волновых потенциалов

Показано, что в случае положительной энергии связанного трехчастичного состояния фаддеевские компоненты могут содержать слагаемые, которые определены с точностью до умножения на произвольное число и не дают вклада в волновую функцию. Если частицы тождественны, то такие слагаемые существуют и не являются квадратично-интегрируемыми функциями.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Pupyshev V.V. E4-85-313

Nonunique Solution of the System of Integro-Differential Faddeev Equations in the Case of Confining s-Wave Potentials

It is shown that in the case of positive energy of the bound three-particle state, the Faddeev components may contain terms determined up to the multiplication by an arbitrary number and do not contribute to the wave function. If particles are identical, such terms exist and are not the quadratically integrable functions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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