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# ON THE STORAGE OF ULTRACOLD NEUTRONS IN MAGNETIC TRAPS

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# ON THE STORAGE OF ULTRACOLD NEUTRONS IN MAGNETIC TRAPS

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О хранении ультрахолодных нейтронов в магнитных ловушках

Рассмотрено поведение ультрахолодных нейтронов (УХН) в поле магнитного шестиполюсника. Получены оценки времени удержания УХН в этом поле и учтено влияние поля тяжести на время удержания.

**Препринт Объединенного института ядерных исследований.**
**Дубна, 1974**

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On the Storage of Ultracold Neutrons
in Magnetic Traps

The storage of ultracold neutrons (UCN) in the hexapole magnetic field is considered. The storage time estimates are obtained and the influence of the gravitational field is taken into account.

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The present paper is considering the motion of an ultracold neutron (UCN) in the hexapole magnetic field. The problem is directly connected with the containment experiments. First it was considered in terms of semiclassical methods [1] and it has been shown that the neutron can be well stored in the hexapole magnetic field. Here we shall give a more exhaustive consideration of the problem. The obtained results are consistent with conclusions made in ref. [1] and at the same time indicate some peculiarities in the behaviour of UCN stored in the magnetic trap. In the first part of the paper the problem is formulated more precisely and the storage time of UCN on the low-lying levels is found. In the second part the storage of UCN on higher levels is estimated, and in the third part gravitational forces are included into consideration.

1. Schrödinger equation for the neutron in magnetic field:

$$i h \frac{\partial}{\partial t}\Psi = \left(-\frac{h^2}{2m_n}\Delta - \vec{\mu}_n \vec{H}\right)\Psi . \qquad (1)$$

Here $\Psi$ is the spinor wave function, $\vec{\mu}_n = -1.91\cdot\mu_N\vec{\sigma}$, $\mu_N$ is Bohr nuclear magneton. The axis of the hexapole is directed along $z$ and $\vec{\sigma}=(\sigma_x,\sigma_y)$ are Pauli matrices. The magnetic field may be presented as follows: $\vec{H}=A r^2\vec{e}$, $r^2=x^2+y^2$, $\vec{e}=(e_x, e_y)$ is the unit vector, $e_x=\sin 2\phi$, $e_y=\cos 2\phi$. In this paper we shall deal only with cylindrical system of references, so $\Delta$ looks as follows $\Delta = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} = \Delta_r + \frac{1}{r^2}\partial_\phi^2$. It should be noted that according to eq. (1) the magnetic field confines the neutron to a finite area if its spin is parallel to H or, in other words, if $\Psi$ is the eigenfunction of the operator $\vec{e}\vec{\sigma}$: $\vec{e}\vec{\sigma}\Psi = \Psi$. But the operator $\vec{e}\vec{\sigma}$ does not commute with $\partial^2_\phi$, and it means that the quantum number is not conserved. It means, generally speaking, that spin reverses spontaneously, and neutron leaves the trap.

For the convenience let us introduce the following time and coordinate units: $t_0 = \sqrt{2m_n/\mu_n A}$, $r_0 = \sqrt[4]{h^2/2m_n\mu_n A}$ and then turn to the dimensionless variables. The function

$$\Psi = \frac{1}{\sqrt{2\pi}} e^{i\sigma_z\phi} e^{im\phi} \psi_m(r,t) \qquad (2)$$

is substituted into eq. (1). Here $\psi_m$ is the spinor function again, but now independent of the angle $\phi$. Really, the first factor in eq. (2) causes spin rotation after the field. On its separation there appears no term in eq. (1) dependent on $\phi$, and so the separation of the variables is permissible. The second factor together with the constant $1/\sqrt{2\pi}$ forms the eigenfunction with respect to $\phi$, normalized over unity. The equation for the function $\psi_m(r,t)$ turns into:

$$i\frac{\partial}{\partial t}\psi_m(r,t) = (-\Delta_r + \frac{1}{r^2}(\sigma_z + m)^2 + r^2\sigma_y)\psi_m(r,t). \qquad (3)$$

To make clearer the physical meaning of the problem it is convenient to expand $\psi_m(r,t)$ in spinors, which are the eigenfunctions of the matrix $\sigma_y$

$$\psi_m(r,t) = f_{1m}(r,t)\cdot u_1 + f_{2m}(r,t)u_2, \qquad (4)$$

where

$$u_1^+u_1 = u_2^+u_2 = 1,\ u_1^+u_2 = 0,\quad \sigma_y u_{1,2} = \pm u_{1,2}.$$

Each component of eq. (4) follows the magnetic field H due to the first factor in the expression (2). Therefore the $f_{1m}(r,t)$ function describes the part of the wave function with the spin parallel to the field, and the $f_{2m}(r,t)$ function describes the part of the wave function having the spin antiparallel to the field. If there was no $\sigma_z$ matrix



in eq. (3), then it could be separated in two independent equations for $f_{1m}$ and $f_{2m}$. Therefore, if at some moment the $f_{2m}(r,t)$ function is equal to zero, it will be equal to zero at any other moment too, and the neutron will be polarized along the field. However, the equations for $f_{1,2m}(r,t)$ are related because of $\sigma_z$:

$$i\frac{\partial}{\partial t} f_{1,2m} = \mathcal{L}_{1,2}\cdot f_{1,2} + V\cdot f_{2,1};\quad \mathcal{L}_{1,2} = -\Delta_r + \frac{1+m^2}{r^2} \pm r^2; \qquad (5)$$

$$V = \frac{2m}{r^2}.$$

It is important to note that $\mathcal{L}_1$ has the discrete spectrum only, while $\mathcal{L}_2$ has only a continuous one. In case $m=0$ the equations will be independent and the neutron with spin parallel to the field will be confined by the field for ever. We shall expand the $f_{1,2m}(r,t)$ functions in eigenfunctions of the $\mathcal{L}_{1,2}$ operators

$$f_{1m} = \sum_n a_{mn}(t)e^{-iE_{mn}t}\chi_{mn}(r);\quad f_{2m} = \int a_{mp}(t)e^{-ip^2t}\chi_{mp}(r)p\,dp$$

$$\int \chi^*_{mn}(r)\chi_{mn'}(r)r\,dr\,\omega = \delta_{nn'};\ \int \chi^*_{mp}(r)\chi_{mp'}(r)r\,dr = 2\delta(p^2 - p'^2)$$

$$\int |f_{2m}(r,t)|^2 r\,dr = \int |a_{mp}(t)|^2 p\,dp. \qquad (6)$$

The equations for the $\chi$ functions turn into Whittaker's ones /2/:

$$\frac{\partial^2}{\partial X^2} g_{1,2} + (\mp\frac{1}{4} + \frac{\lambda}{X} + \frac{1/4 - \mu^2}{X^2})g_{1,2} = 0;\ \mu^2 = \frac{m^2+1}{4};\ \lambda = \frac{E}{4} \qquad (7)$$

when the variables $\chi_{mn,p} = g_{1,2}/\sqrt{X}$ and $X = r^2$ are introduced, and their solution may be written in the following form:

$$g_1(x)=c_1X^{\nu}e^{-X/2}F(\nu-\lambda,2\nu,X);\; g_2(x)=c_2x^{\nu}e^{-iX/2}F(\nu+i\lambda,2\nu,X);\quad (8)$$

$$\nu=\mu+\frac{1}{2},$$

where $F(a,b,\zeta)$ is the confluent hypergeometrical function. The finiteness requirement for $g_1(X)$ for all $X$ gives the spectrum $\nu-\lambda=-n$:

$$E_{mn}=E_N=4(\nu+n)=4N. \quad (9)$$

Normalization constants $c_{1,2}$ found according to ref. /3/ (see pages 86 and 696) are equal to:

$$c_1=\frac{1}{\Gamma(2\nu)}\sqrt{\frac{2\Gamma(2\nu+m)}{n!}};\; c_2=\frac{1}{\sqrt{2\pi}}e^{\pi\lambda/2}\frac{|\Gamma(\nu-i\lambda)|}{\Gamma(2\nu)}. \quad (10)$$

To solve the equations (5) coefficients $a_{mn,p}(t)$ should be found. In principle it is possible to seek for the stationary solution, but it seems to the author that the nonstationary one is more appropriate since in stationary state the particle must be bound and unbound simultaneously. Let us suppose that at the moment $t=0$ the neutron is in the bound state $E_N$ having the total quantum number $N=n+\nu$, then $a_N=1$, and all other terms of expansion (6) are equal to zero. The variation of the coefficients is found with the help of perturbation theory:

$$i\frac{d}{dt}a_{mn}(t)=0;\quad i\frac{d}{dt}a_{mp}(t)=V_{pn}e^{i(p^2-E_N)t}, \quad (11)$$

where

$$-V_{pn}=\int_0^{\infty}\chi^*_{mp}(r)\frac{2m}{r^2}\chi_{mn}(r)\,r\,dr=mc_1c_2\int_0^{\infty}X^{2\nu-2}e^{-X(1-i)/2}\times$$

$$\times F(-n,2\nu,X)F(\nu-i\lambda,2\nu,-iX)\,dX. \quad (12)$$



The probability of spin reversal and, consequently, of the transition to the unbound state is:

$$W(t)=\int|f_{2m}(r,t)|^2r\,dr=\frac{\pi}{2}m^2c_1^2c_2^2|J|^2\,t=a\,t, \quad (13)$$

where $J$ stands for the integral in eq.(12). First we shall consider the decay of the lowest states: $n=0$, $m=1$ (the state $n=0$, $m=0$ is absolutely stable). The dimensionless coefficient $a$ ahead the dimensionless time $t$ in eq. (13) is about unity since its parameters are of the same order. Therefore the decay constant for the low-lying states is:

$$\tau=t_0/a\approx t_0=\sqrt{2m_n/\mu_nA}. \quad (14)$$

Ultracold neutrons have energy $E_{UCN}\approx 10^{-19}$ erg and they may be kept in the field of about $\sim 10$ kgs. If the radius of the magnetic trap is about 10 cm, then $A$ should be about 100 gs/cm$^2$. With such $A$, $r_0\sim 10^{-2}$ cm and $t_0\sim 10^{-1}$ sec, i.e., the storage time of low-lying states is $\tau\sim 10^{-1}$ sec.

2. Now let us consider high energy states with quantum numbers $2N\approx 2n+m\approx\frac{E_{UCN}}{E_0}\approx 10^7$, $E_0=2h/t_0\approx 10^{-26}$ erg being the space between discrete levels. It is not convenient to use for these states the exact wave functions in matrix elements, since the computational difficulties are enormous. It is better to take the wave functions in quasi-classical approximation. According to ref. /4/ the wave functions for discrete and continuous spectra in quasi-classical approximation look as follows:

$$\chi_{mn,p}(r)=\sqrt{2/\pi}(X\cdot q_{mn,p}(x))^{-1/2}\cos(\theta_{mn,p}(x)-\pi/4). \quad (15)$$

Normalization is chosen as in (6) and the letter symbols in eq. (15) stand for:

$$\theta_{mn,p}(X)=\int_{X^-_{mn,p}}^{X}q_{mn,p}(X')dX';\; q_{mn,p}(X)=\sqrt{\mp\frac{1}{4}+\frac{\lambda}{x}-\frac{m^2+1}{4x^2}};$$

$$\lambda = \frac{E_{mn,p}}{4}; \quad X = r^2, \tag{16}$$

$X_{mn,p}$ being the left turn point for the discrete and continuous motion. When the indices m are omitted the matrix element takes the next form:

$$V_{pn} = (2m/\pi) \int_{X_n^-}^{X_n^+} \frac{dX}{X^2\sqrt{q_n(X)\, q_p(X)}} \cos(\theta_n(X) - \frac{\pi}{4}) \cos(\theta_p(X) - \frac{\pi}{4}), \tag{17}$$

where $X_n^+$ is the right turn point of the discrete state. Let us introduce the following symbols:

$$X = Ez, \ \beta = m/E, \ s_{n,p} = \sqrt{\mp z^2 + z - \beta}, \ \tilde{\theta}_{n,p}(z) = \int_{z_{n,p}^-}^{z} s_{n,p}(z') \frac{dz'}{z'},$$

then, for $m \gg 1$ and $E \gg 1$ one has:

$$V_{ph} \simeq (4\beta/\pi) \int_{z_n^-}^{z_n^+} \frac{dz}{z\sqrt{s_n s_p}} \cos(E\tilde{\theta}_n - \frac{\pi}{4}) \cos(E\tilde{\theta}_p - \frac{\pi}{4}). \tag{18}$$

We shall consider first the case when $m \sim E$, i.e., $\beta \simeq 1/2$ ($\beta$ cannot be large than 1/2). Then $z_n^- = (1 - \sqrt{1-4\beta^2})/2$ and $z_p^- = \frac{(\sqrt{1+4\beta^2}-1)}{2} < z_n^-$ are far from each other, and the matrix element is determined by the last factor of the integrand which oscillates at high frequency near the turning points. So $V_{pn}$ may be estimated:

$$V_{ph} \sim \int_0^\infty \frac{d\xi}{\sqrt[4]{\xi}} \cos(E\xi) \approx E^{-3/4} \tag{19}$$

from which it follows that the storage time of high energy

$$\tau = |V_{ph}|^{-2} t_0 \simeq \left(\frac{E}{E_0}\right)^{3/2} t_0 \simeq 10^9 \ \text{sec}. \tag{20}$$



Now we assume that $\beta \ll 1$ and $\beta^3 E < 1$. Then, near the turning point $X_n^-$

$$\cos(E\tilde{\theta}_n - \frac{\pi}{4})\cos(E\tilde{\theta}_p - \frac{\pi}{4}) = \frac{1}{2}[\cos E(\tilde{\theta}_n - \tilde{\theta}_p) - \sin E(\tilde{\theta}_n + \tilde{\theta}_p)].$$

In the integrand we retain only the first term, since the second one oscillates at high frequency. Moreover the first term may be assumed to be 1/2, since

$$E(\tilde{\theta}_n - \tilde{\theta}_p) \simeq E\beta^4 \int_{\beta^2}^{z} \frac{dy}{y\sqrt{y-\beta^2}} \sim E\beta^3 < 1. \tag{21}$$

As a result we have for $V_{pn}$ the following expression:

$$V_{pn} \simeq \frac{2}{\pi}\beta \int_{\beta^2}^{\infty} \frac{dx}{x\sqrt{x-\beta^2}} = 2. \tag{22}$$

Consequently, the storage time of the states m, n with $m^3 < n^2$ is

$$\tau \approx t_0 \approx 10^{-1} \text{sec}. \tag{23}$$

**3.** Now we shall estimate the influence of the gravitational field on the ultracold neutron storage time in magnetic trap. In the presence of the earth-gravity field the potential term of Schrödinger's equation $U = m_n g x$ should be taken into account (or in dimensionless units $U = Gr\cos\phi$, where $G = \frac{g t_0^2}{r_0} \simeq 10^3$). The potential U does not contain spin operators, so it cannot reverse the neutron spin, but it mixes up the states with different m and n, in particular, the stable states having high m with those having low m. But states with low m, as follows from the above considerations, decay quickly. Therefore the storage time of longlived states becomes less. Thus, we shall estimate the influence of U on the storage time in the following way: under assumption that at the initial moment the neutron is on the longlived level with

$m'=m$, we shall find out with the help of perturbation theory how long it takes the neutron to jump to the level with $m'=m-1$, and thus we shall find the velocity of the neutron motion along the $m$ axis. Then the period of time in which $m$ decreases to $m \sim E^{2/3}$ may be determined. And since the neutron in the state with small $m$ reverses the spin in $\tau \approx t_0$, the determined period will be resultant storage time of the initial state.

So, we shall use the perturbation theory to determine the probability of transition from the state $m'=m$ to $m'=m-1$. Similarly to (11) one can write:

$$i\frac{d}{dt} a_{m-1,n}(t) = U_{m-1,m} \exp[-i(E_{m-1}-E_m)t]. \qquad (24)$$

Making use of the quasiclassical functions we obtain:

$$U_{m-1,m} \approx \frac{2}{\pi} G\sqrt{E_m}\,\frac{1}{2}\int \frac{dz\cdot\sqrt{z}}{\sqrt{s_m\cdot s_{m-1}}}\cos(E_m\tilde{\theta}_m - \frac{\pi}{4})\cos(E_{m-1}\tilde{\theta}_{m-1} - \frac{\pi}{4}). \qquad (25)$$

As far as $E_m, s_m, \theta_m$ and $E_{m-1}, s_{m-1}, \theta_{m-1}$ respectively differ from each other by a relative value $\sim 1/E_m$, they may be assumed to be equal and the product of cosines in eq. (25) can be put equal to 1/2. As a result we have:

$$a_{m-1,n} \approx -iG\sqrt{E_{mn}}\; t\;. \qquad (26)$$

The equation holds when $|a_{m-1,n}|<1$. From it the transition time is determined:

$$\tau_1 \approx \frac{1}{G\sqrt{E}}. \qquad (27)$$

The value of $\tau_1$ is small. It justifies a posteriori the neglection of the exponent in eq. (24) when looking for the solution of eq. (26).

The motion along the $\beta = \frac{m}{E}$ axis is characterized by velocity:

$$v = \frac{\delta\beta}{\tau_1} = \frac{G}{\sqrt{E}} \qquad (28)$$



Since the potential $U$ gives transitions both to the $\Delta m=+1$ state and to $\Delta m=-1$ one, the motion along the $\beta$ axis cannot be considered a directed one. It is better to describe it with the diffusion formula:

$$(\Delta\beta)^2 \approx Dt\,, \qquad (29)$$

where $D$ is the diffusion coefficient: $D \approx v\cdot\delta\beta = GE^{-3/2}$. Assuming $\Delta\beta$ in eq. (29) being equal to 1/2 we shall find the time in which the system reaches the state with $\beta \approx 0$:

$$t \approx \frac{t_0}{4D} \approx \frac{E^{3/2}}{4G} t_0 \approx 10^6 \text{ sec}. \qquad (30)$$

Of course the estimations obtained are pure qualitative.

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