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THE SCATTERING OF CHARGED PARTICLES ON THE SCREENED TOROIDAL SOLENOID

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1. The vector magnetic potentials (VMP) of the toroidal solenoid have been obtained in ref./1/. Here we aim to use these VMP to investigate the scattering of the charged zero-spin particles on the space domains with zero magnetic field strength H. In order to prevent particles to be trapped into the space regions where $H \neq 0$, the last ones are screened with the infinite repulsive potential of the suitable geometrical form.

We are planning the present consideration as follows. In section 2 we try to seek the multivalued generating function whose gradient is just VMP of ref./1/. It is shown that the behaviour of this function at large distances determines the asymptotics of the wave function. At the same time we got analytic expressions for some integrals and series containing Legendre functions, which we failed to find in standard handbooks, treatises and original publications. In section 3 the contribution of the screened magnetic field to the scattering amplitudes is estimated for two topologically nonequivalent screens; it comes out that it equals zero in both cases. In section 4 we compare our results with those of the pioneering investigation of ref./2/, find out some disagreement between them and point out definite reasons for it. Finally, in section 5 we have collected the above-mentioned sums and integrals.

2. As the toroidal coordinates μ, ϑ, φ are of great use in the following, we give necessary mathematical details. The cartesian coordinates are expressed through toroidal ones as follows:

$$x = \frac{a\,\mathrm{sh}\mu\cos\varphi}{\mathrm{ch}\mu - \cos\vartheta}, \quad y = \frac{a\,\mathrm{sh}\mu\sin\varphi}{\mathrm{ch}\mu - \cos\vartheta}, \quad z = \frac{a\sin\vartheta}{\mathrm{ch}\mu - \cos\vartheta}$$

$$(0 \le \mu \le \infty, \quad -\pi < \vartheta \le \pi, \quad 0 \le \varphi < 2\pi).$$

$$\mathrm{cth}\,\mu = \frac{r^2 + a^2}{2ra\sin\vartheta_s}, \quad \mathrm{ctg}\,\vartheta = \frac{r^2 - a^2}{2ra\cos\vartheta_s}. \qquad (2.1)$$

μ, ϑ, being expressed through radius r and polar angle ϑ_s, are:

It follows that μ, ϑ tend to zero as r goes to infinity:

$$\mu \sim \frac{2a}{r}\sin\vartheta_s, \quad \vartheta \sim \frac{2a}{r}\cos\vartheta_s. \qquad (2.2)$$

The toroidal coordinates have a property that neighbouring space points, which possess the same coordinates $\rho = \sqrt{x^2+y^2}$ and z differing by sign, have (for $\rho < a$) essentially different ϑ coordinates. For example, points $(\rho, \pm|\delta z|)$ have coordinates $\mu = \mathrm{arcth}\,\frac{2\rho a}{\rho^2+a^2}$, $\vartheta = \pm\left(\pi - \frac{2a|\delta z|}{|\rho^2 - a^2|}\right)$. This resembles the situation with a logarithmic function: being analytically continued from the positive x semiaxis, it takes values differing by $2\pi i$ depending on whether the continuation contour lies in the upper or lower semiplane.

For μ fixed and variables ϑ, φ the points x, y, z fill the entire surface of torus: $(\rho - a\cdot\mathrm{cth}\mu)^2 + z^2 = a^2/\mathrm{sh}^2\mu$. Let $\mu = \mu_0$ correspond to the torus T_0 on which the solenoidal winding is performed. Then, for $\mu < \mu_0$ $(> \mu_0)$ the point x, y, z lies outside (inside) T_0. The radii of the cross section R and of the axis line d are: $R = a/\mathrm{sh}\mu_0$, $d = a\cdot\mathrm{cth}\mu_0$. For $\mu \to \infty$ the torus degenerates into the filament of the radius a lying in the $z = 0$ plane. Outside the solenoid the VMP is free of circulation and it may be viewed as gradient of some function α /3-5/. As the $\int A_\varphi d\ell$ is different from zero for the closed contour C passing through the hole of T_0, α is the multivalued function of coordinates x, y, z.

Consider first thin solenoid ($\frac{R}{d} \ll 1$). Using the Coulomb gauge, and VMP given in /1/, one has the following defining relations for the α function:

$$\frac{1}{a}\frac{\partial\alpha_0}{\partial\vartheta} = \frac{A_\theta}{\mathrm{ch}\mu - \cos\vartheta} = \frac{g\pi}{\sqrt{2}}\cdot\exp(-2\mu_0)\cdot\frac{P_{-1/2}\cdot\cos\vartheta - P_{1/2}}{(\mathrm{ch}\mu - \cos\vartheta)^{1/2}} \qquad (2.3)$$

$$\frac{1}{a}\frac{\partial\alpha_0}{\partial\mu} = \frac{A_\mu}{\mathrm{ch}\mu - \cos\vartheta} = \sqrt{2}\,\pi g\cdot\exp(-2\mu_0)\cdot\frac{\sin\vartheta}{(\mathrm{ch}\mu - \cos\vartheta)^{1/2}}\cdot P^1_{-1/2}.$$

From now we mean an argument of the Legendre functions is $\mathrm{ch}\mu$, if it is not indicated explicitly. Constant g is expressed through the total number of coils and current strength:

$g = \frac{2nJ}{c}$; c is the light velocity.

Integrate (2.3):

$$\alpha_0(\vartheta,\mu) = \frac{\pi g a}{\sqrt{2}}\exp(-2\mu_0)\cdot\left(P_{-1/2}\int_0^\vartheta \frac{\cos\vartheta\,d\vartheta}{\sqrt{\mathrm{ch}\mu - \cos\vartheta}} - P_{1/2}\int_0^\vartheta \frac{d\vartheta}{\sqrt{\mathrm{ch}\mu - \cos\vartheta}}\right) = \qquad (2.4)$$

$$= \sqrt{2}\,\pi g a\cdot\exp(-2\mu_0)\cdot\sin\vartheta\cdot\int_0^\mu \frac{d\mu}{\sqrt{\mathrm{ch}\mu - \cos\vartheta}} - 2\pi g a\cdot\exp(-2\mu_0)\cdot\left(1 - \cos\frac{\vartheta}{2}\right)\cdot\mathrm{sgn}\,\vartheta.$$

This integrals may be reduced to the elliptical ones:

$$\alpha_0(\vartheta,\mu)=\sqrt{2}\,\pi g a\cdot\exp(-2\mu_0)\cdot\Big\{\frac{\operatorname{ch}\mu\,P_{-1/2}-P_{1/2}}{\sqrt{1+\operatorname{ch}\mu}}\cdot F\Big(\delta,\sqrt{\frac{2}{1+\operatorname{ch}\mu}}\Big)-$$
$$-P_{-1/2}\cdot\Big[\sqrt{1+\operatorname{ch}\mu}\cdot E\Big(\delta,\sqrt{\frac{2}{1+\operatorname{ch}\mu}}\Big)-\frac{\sin\vartheta}{\sqrt{\operatorname{ch}\mu-\cos\vartheta}}\Big]\Big\}. \qquad (2.5)$$

Here $\delta=\arcsin\cdot\Big[\frac{(1+\operatorname{ch}\mu)(1-\cos\vartheta)}{2\sqrt{\operatorname{ch}\mu-\cos\vartheta}}\Big]^{1/2}.$

It follows from the last expression and (2.2) that:

$$\alpha_0\sim-\pi g\exp(-2\mu_0)\cdot\frac{a^3}{r^2}\cdot\cos\vartheta_s \qquad \text{for } r\to\infty. \qquad (2.6)$$

So, at large distances α_0 falls as $1/r^2$. Now develop integrands in (2.5) under $\cos n\vartheta$, $\sin n\vartheta$:

$$\alpha_0=ga\cdot\exp(-2\mu_0)\cdot\Big\{-2\vartheta+\sum\frac{\sin n\vartheta}{n}\cdot\big[P_{-1/2}(Q_{n+1/2}+Q_{n-3/2})-2P_{1/2}\cdot Q_{n-\frac{1}{2}}\big]\Big\}= \qquad (2.7)$$
$$=-2ga\cdot\exp(-2\mu_0)\cdot\Big[\sum\sin n\vartheta\cdot\int_0^{\mu}P^1_{-1/2}\cdot(Q_{n+\frac{1}{2}}-Q_{n-3/2})d\mu+\pi\cdot\big(1-\cos\tfrac{\vartheta}{2}\big)\cdot\operatorname{sgn}\vartheta\Big].$$

Taking into account the relation $\sum\frac{\sin n\vartheta}{n(n^2-1/4)}=2\vartheta-2\pi(1-\cos\frac{\vartheta}{2})\operatorname{sgn}\vartheta$ and comparing the coefficients at $\sin n\vartheta$, one gets the following analytical expression for the integral from the Legendre functions:

$$\int_0^{\mu}P^1_{-1/2}\cdot Q^1_{n-1/2}\cdot\operatorname{sh}\mu\cdot d\mu=\frac{1}{4}\Big(\frac{1}{4n^2}-1\Big)\cdot\big[P_{-\frac{1}{2}}\cdot(Q_{n+\frac{1}{2}}+Q_{n-\frac{3}{2}})-2P_{\frac{1}{2}}\cdot Q_{n-\frac{1}{2}}\big]+\frac{1}{4n^2}. \qquad (2.8)$$

From (2.7) it follows that the integral $\oint_C A_\ell\, d\ell$ equals $a\int_{-\pi}^{\pi}A_\theta\frac{d\vartheta}{\operatorname{ch}\mu-\cos\vartheta}=\int_{-\pi}^{\pi}\frac{\partial\alpha}{\partial\vartheta}d\vartheta=-4\pi g a\cdot\exp(-2\mu_0)$, i.e., the magnetic field flow, if contour C passes through the hole of solenoid, and zero otherwise.

For the finite solenoid the direct integration of the VMP components is very cumbersome. The task is simplified very much if one extracts first from α the linear dependence in ϑ:

$$\alpha=a\cdot\gamma\cdot\vartheta+a\cdot\sqrt{\operatorname{ch}\mu-\cos\vartheta}\cdot\tilde{\alpha}. \qquad (2.9)$$

Here $\tilde{\alpha}$ is a function to be found, and the constant γ is to be determined from the correct value of the magnetic field flow: $\gamma=g\cdot(1-\operatorname{cth}\mu_0)$. Outside the solenoid $(\mu<\mu_0)$ function α satisfies the equation:

$$\Delta\alpha=0 \qquad (2.10)$$

which coincides with the gauge condition $\operatorname{div}\vec{A}=0$. Inserting there (2.9), one gets equation for $\tilde{\alpha}$:

$$\frac{\partial^2\tilde{\alpha}}{\partial\mu^2}+\operatorname{cth}\mu\,\frac{\partial\tilde{\alpha}}{\partial\mu}+\frac{\partial^2\tilde{\alpha}}{\partial\vartheta^2}=\frac{\gamma\cdot\sin\vartheta}{(\operatorname{ch}\mu-\cos\vartheta)^{3/2}}. \qquad (2.11)$$

We seek the solution developing both sides in $\sin n\vartheta$:

$$\tilde{\alpha}=\sum\alpha_n(\mu)\cdot\sin n\vartheta. \qquad (2.12)$$

The expansion coefficients $\alpha_n(\mu)$ have the following form:

$$\alpha_n(\mu)=\Lambda_n\cdot P_{n-1/2}-2\tilde{\gamma}\cdot n\cdot(Q_{n-1/2}\cdot C_n+P_{n-1/2}\cdot\mathcal{D}_n), \qquad (2.13)$$

where we put for brevity: $\tilde{\gamma}=\frac{2\sqrt{2}\,\gamma}{\pi}$, $C_n=\int_0^{\mu}P_{n-1/2}\cdot Q_{n-1/2}\cdot\operatorname{sh}\mu\cdot d\mu$, $\mathcal{D}_n=\int_{\mu}^{\mu_0}(Q_{n-1/2})^2\operatorname{sh}\mu\, d\mu$. These coefficients contain arbitrary constants Λ_n. This is due to the fact that the gauge condition is only necessary, but not sufficient one. Λ_n may be fixed by using the vector equation $\vec{A}=\operatorname{grad}\alpha$, valid outside the solenoid. As a result, one has two following conditions on $\tilde{\alpha}$:

$$\big[(1-\operatorname{ch}\mu\cdot\cos\vartheta)\partial_\mu-\sin\vartheta\cdot\operatorname{sh}\mu\,\partial_\theta-\tfrac{1}{2}\operatorname{sh}\mu\cdot\cos\vartheta\big]\tilde{\alpha}=\frac{A_\rho+\gamma\cdot\sin\vartheta\cdot\operatorname{sh}\mu}{\sqrt{\operatorname{ch}\mu-\cos\vartheta}}, \qquad (2.14)$$
$$\big[(1-\operatorname{ch}\mu\cdot\cos\vartheta)\partial_\theta+\sin\vartheta\cdot\operatorname{sh}\mu\,\partial_\mu+\tfrac{1}{2}\sin\vartheta\cdot\operatorname{ch}\mu\big]\tilde{\alpha}=\frac{A_z-\gamma(1-\operatorname{ch}\mu\cdot\cos\vartheta)}{\sqrt{\operatorname{ch}\mu-\cos\vartheta}},$$

A_ρ and A_z being the cylindrical components of VMP were written explicitly in refs. /1/. Inserting them into (2.14), one obtains the following finite-difference equations for Λ_n:

$$(\Lambda_n-\tfrac{1}{2}\Lambda_{n-1}-\tfrac{1}{2}\Lambda_{n+1})\cdot P^1_{n-1/2}-2\cdot\tilde{\gamma}\cdot\big\{Q^1_{n-1/2}\cdot\big[nC_n-\tfrac{1}{2}(n-1)C_{n-1}-\tfrac{1}{2}(n+1)C_{n+1}\big]+$$
$$+P^1_{n-1/2}\cdot\big[n\mathcal{D}_n-\tfrac{1}{2}(n-1)\mathcal{D}_{n-1}-\tfrac{1}{2}(n+1)\mathcal{D}_{n+1}\big]\big\}=\tfrac{1}{2}\tilde{\gamma}\cdot\operatorname{sh}\mu\,(Q_{n-3/2}-Q_{n+1/2})+$$
$$+R^1_n(\mu); \qquad (2.15)$$

$$\big[n\Lambda_n-\tfrac{1}{2}(n-1)\Lambda_{n-1}-\tfrac{1}{2}(n+1)\Lambda_{n+1}\big]\cdot P_{n-1/2}-$$
$$-2\tilde{\gamma}\big\{\cdot Q_{n-1/2}\cdot\big[n^2C_n-\tfrac{1}{2}(n+1)(n+\tfrac{1}{2})C_{n+1}-\tfrac{1}{2}(n-1)(n-\tfrac{1}{2})C_{n-1}\big]+$$
$$+P_{n-1/2}\cdot\big[n^2\mathcal{D}_n-\tfrac{1}{2}(n+1)(n+\tfrac{1}{2})\mathcal{D}_{n+1}-\tfrac{1}{2}(n-1)(n-\tfrac{1}{2})\mathcal{D}_{n-1}\big]\big\}=-\frac{\tilde{\gamma}}{2}A_n-R^0_n(\mu).$$

Here: $A_n = 2Q_{n-1/2} - ch\mu(Q_{n+1/2} + Q_{n-3/2})$ for $n \neq 0$, $A_0 = Q_{-1/2} - ch\mu Q_{1/2}$.
Functions R_n^0, R_n^1 were obtained earlier in ref./1/:

$$R_n^0 = \tilde{g}\cdot[(n+1/2)Q_{n+1/2}(ch\mu_0) - (n-1/2)Q_{n-3/2}(ch\mu_0)]\cdot Q_{n-1/2}(ch\mu_0)\cdot P_{n-\frac{1}{2}},$$

$$R_0^0 = \frac{1}{2}\cdot\tilde{g}\cdot Q_{1/2}(ch\mu_0)\cdot Q_{-1/2}(ch\mu_0)\cdot P_{1/2}.$$

$$R_n^1 = -\tilde{g}\cdot[Q_{n+1/2}(ch\mu_0) - Q_{n-3/2}(ch\mu_0)]\cdot Q_{n-1/2}(ch\mu_0)\cdot P_{-1/2}$$

$$\tilde{g} = \frac{2\sqrt{2}g}{\pi}.$$

As Λ_n do not depend upon μ, they could be calculated by fixing μ in(2.15).The simplest way to do it is to consider the case of small μ. By collecting coefficients at the first nonvanishing degree of μ, one gets the following system of equations:

$$\beta_{n+1} + \beta_{n-1} - 2\beta_n = -\frac{2n\tilde{\gamma}}{(n^2-1/4)^2} + 2\tilde{g}\cdot[Q_{n+1/2}(ch\mu_0) - Q_{n-3/2}(ch\mu_0)]\cdot Q_{n-1/2}(ch\mu_0), \tag{2.16}$$

$$(n+\tfrac{1}{2})\beta_{n+1} + (n-1/2)\beta_{n-1} - 2n\beta_n = -\frac{\tilde{\gamma}}{n^2-1/4} + 2\tilde{g}\cdot Q_{n-1/2}(ch\mu_0)\cdot$$
$$\cdot[(n+\tfrac{1}{2})Q_{n+1/2}(ch\mu_0) - (n-1/2)Q_{n-3/2}(ch\mu_0)],$$

where $\beta_n = \Lambda_n - 2n\tilde{\gamma}\mathcal{D}_n^0$, $\mathcal{D}_n^0 = \int_0^{\mu_0}(Q_{n-\frac{1}{2}})^2 sh\mu d\mu$. These recurrence relations are valid for $n \geq 1$. For $n=0$, one has the single relation:

$$\beta_1 = 4\tilde{\gamma} + 2\tilde{g}\,Q_{1/2}(ch\mu_0)Q_{-1/2}(ch\mu_0).$$

These equations are satisfied if:

$$\beta_{n+1} - \beta_n = \frac{\tilde{\gamma}}{(n+1/2)^2} + 2\tilde{g}\cdot Q_{n-1/2}(ch\mu_0)\cdot Q_{n+1/2}(ch\mu_0). \tag{2.17}$$

From this one finds Λ_n:

$$\Lambda_n = 2\tilde{\gamma}n\cdot\mathcal{D}_n^0 + \tilde{\gamma}\cdot\sum_{k=0}^{n-1}(k+\tfrac{1}{2})^{-2} + 2\tilde{g}\cdot\sum_{k=0}^{n-1}Q_{k-\frac{1}{2}}(ch\mu_0)\cdot Q_{k+\frac{1}{2}}(ch\mu_0). \tag{2.18}$$

Above we have used the independence Λ_n of μ and extracted them from the zero μ limit of (2.15). As these Eqs. should be valid for arbitrary μ, they may be viewed as recurrence relations for C_n, $\mathcal{D}_n$. In fact, excluding the just found Λ_n from (2.15), one obtains the following equations for the integrals:

$$[nF_n - \tfrac{1}{2}(n-1)F_{n-1} - \tfrac{1}{2}(n+1)\cdot F_{n+1}]\cdot P^1_{n-\frac{1}{2}} - [nC_n - \tfrac{1}{2}(n-1)C_{n-1} - \tfrac{1}{2}(n+1)\cdot$$
$$\cdot C_{n+1}]\cdot Q^1_{n-\frac{1}{2}} + \frac{n}{2(n^2-\frac{1}{4})^2}\cdot P^1_{n-1/2} = \frac{1}{4}sh\mu\,(Q_{n-3/2} - Q_{n+1/2}),$$

$$[n^2\cdot F_n - \tfrac{1}{2}(n-1)(n-\tfrac{1}{2})F_{n-1} - \tfrac{1}{2}(n+1)(n+\tfrac{1}{2})\cdot F_{n+1}]\cdot P^1_{n-1/2} - \tag{2.19}$$
$$-[n^2C_n - \tfrac{1}{2}(n-1)(n-1/2)C_{n-1} - \tfrac{1}{2}(n+1)(n+\tfrac{1}{2})\cdot C_{n+1}]\cdot Q^1_{n-1/2} +$$
$$+\frac{1}{4n^2-1}\cdot P^1_{n-1/2} + \frac{1}{4}\left[ch\mu(Q^1_{n+1/2} + Q^1_{n-3/2}) - 2Q^1_{n-1/2}\right] = 0,$$
$$(n \neq 0).$$

Here $F_n = \int_0^\mu (Q_{n-\frac{1}{2}})^2 sh\mu d\mu$. For $n=0$ we have an additional relation:

$$(F_1 + 2)P_{-1/2} - C_1\cdot Q_{-1/2} = Q_{-1/2} - ch\mu\cdot Q_{1/2}. \tag{2.19a}$$

Inserting coefficients Λ_n given by (2.18) into (2.13), one gets:

$$d_n(\mu) = 2\tilde{\gamma}n\cdot(P_{n-1/2}\cdot F_n - Q_{n-1/2}\cdot C_n) + P_{n-1/2}\cdot\Big[\tilde{\gamma}\cdot\sum_{k=0}^{n-1}\frac{1}{(k+\frac{1}{2})^2} +$$
$$+ 2\tilde{g}\cdot\sum_{k=0}^{n-1}Q_{k-1/2}(ch\mu_0)\cdot Q_{k+1/2}(ch\mu_0)\Big]. \tag{2.20}$$

Now we make use of the fact that the function $\tilde{\alpha}$ is known for the thin solenoid (Eqs. (2.4), (2.5), (2.7)). The coefficients d_n^0 are found from the relations similar to (2.12):

$$\tilde{\alpha} = \sum d_n^0\cdot\sin n\vartheta = \frac{g\cdot\exp(-2\mu_0)}{\sqrt{ch\mu - \cos\vartheta}}\cdot\sum^\infty\frac{\sin n\vartheta}{n}\cdot\left[P_{-\frac{1}{2}}(Q_{n+\frac{1}{2}} + Q_{n-\frac{3}{2}}) - 2P_{\frac{1}{2}}\,Q_{n-\frac{1}{2}}\right]. \tag{2.21}$$

From this, one obtains $d_n^0(\mu)$:

$$d_n^0(\mu) = -\frac{1}{4}\cdot\tilde{\gamma}_0\cdot S_n, \quad \tilde{\gamma}_0 = -2\tilde{g}\cdot\exp(-2\mu_0),$$
$$S_n = \sum_m\frac{1}{m}\left[(Q_{m+\frac{1}{2}} + Q_{m-\frac{3}{2}})\cdot P_{-\frac{1}{2}} - 2\cdot Q_{m-\frac{1}{2}}P_{\frac{1}{2}}\right]\cdot(Q_{m-n-\frac{1}{2}} - Q_{m+n-\frac{1}{2}}). \tag{2.22}$$

It is clear that $d_n(\mu)$ should coincide with $d_n^0(\mu)$ as $\mu_0 \to \infty$. From this fact one gets the relation:

$$2\cdot n\cdot(P_{n-\frac{1}{2}}F_n - Q_{n-\frac{1}{2}}C_n) + \left[\sum_{k=0}^{n-1}\frac{1}{(k+1/2)^2} - \frac{\pi^2}{2}\right]\cdot P_{n-1/2} = -\frac{1}{4}\cdot S_n. \tag{2.23}$$

In the limit of small μ, it goes over to the following one:

$$\sum_{k=0}^{n-1}\frac{1}{(k+1/2)^2} - \frac{\pi^2}{2} = -\frac{1}{4}\sum_{m=1}^{\infty}\frac{1}{m}\,\frac{1}{m^2-1/4}\cdot\left(\frac{1}{m+n-1/2} + \frac{1}{m+n-3/2} + \ldots + \frac{1}{m-n-1/2}\right)$$

that permits one to get explicit values for some numerical series.

For example for $n=1$, one has:

$$\sum_{m=1}^{\infty}\frac{1}{(m^2-\frac{1}{4})^2}=\pi^2-8$$

that is well known. For $n=2$: $\sum_{k=1}^{\infty}\frac{1}{4k^2-9}=\frac{1}{18}$, etc. In general, the following relation $(n\geq 1)$ takes place: $\sum_{k=1}^{\infty}\frac{1}{k^2-(n+\frac{1}{2})^2}=\frac{1}{2(n+\frac{1}{2})^2}$.

In the opposite limit ($\mu\to\infty$), one gets from (2.23) two identities:

$$2n\int_0^{\infty}(Q_{n-\frac{1}{2}})^2\,sh\mu\cdot d\mu=\frac{\pi^2}{2}-\sum_{k=0}^{n-1}\frac{1}{(k+\frac{1}{2})^2},$$

$$\sum_{k=1}^{n}\frac{\Gamma(k-\frac{1}{2})\cdot\Gamma(n-k+\frac{1}{2})}{\Gamma(k+1)\cdot\Gamma(n-k+1)}=2\sqrt{\pi}\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}.$$

First of these coincides with the well-known relation /6/

$$\int_0^{\infty}(Q_\nu)^2\,sh\mu\,d\mu=\frac{1}{2\nu+1}\,\Psi'(\nu+1).$$

The second one looks like the Dougoll formula /6/, but does not reduce to it.

Eqs. (2.19) and (2.23) permit one to calculate explicitly all the sums S_n and integrals F_n and C_n. We demonstrate this evaluating them, taking $n=1,2$ as an example. Combine (2.19a) and (2.23) $n=1$:

$$(F_1+2)P_{-1/2}-C_1\cdot Q_{-1/2}=Q_{-1/2}-ch\mu\,Q_{1/2},$$

$$(F_1+2)P_{1/2}-C_1\cdot Q_{1/2}=\frac{1}{4}\pi^2\cdot P_{1/2}-\frac{1}{8}S_1$$

From this, express F_1 and C_1 through S_1 :

$$C_1=\frac{1}{8}P_{-1/2}\cdot(\pi^2\cdot P_{1/2}-\frac{1}{2}S_1)-\frac{1}{2}\cdot P_{1/2}\cdot(Q_{-1/2}-ch\mu\cdot Q_{1/2}),$$

$$F_1=\frac{1}{8}Q_{-1/2}\cdot(\pi^2P_{1/2}-\frac{1}{2}S_1)-\frac{1}{2}Q_{1/2}\cdot(Q_{-1/2}-ch\mu\,Q_{1/2})-2. \tag{2.24}$$

Further, set $n=1$ in (2.19):

$$C_2\cdot Q^1_{1/2}-F_2\cdot P^1_{1/2}=\frac{1}{4}sh\mu(Q_{-1/2}-Q_{3/2})-\frac{8}{9}P^1_{1/2}+C_1\cdot Q^1_{1/2}-F_1\cdot P^1_{1/2}, \tag{2.25}$$

$$\frac{3}{2}(C_2Q^1_{1/2}-F_2\cdot P^1_{1/2})=-\frac{1}{3}P^1_{1/2}-\frac{1}{4}\cdot[ch\mu(Q^1_{3/2}+Q^1_{-1/2})-2Q^1_{1/2}]+C_1Q^1_{1/2}-F_1\cdot P^1_{1/2}.$$

Taking the ratio of two last equations, one obtains an explicit value for S_1 :

$$S_1=2\pi^2\cdot P_{1/2}-16\cdot ch\mu\frac{Q^1_{-1/2}}{Q^1_{1/2}\cdot P_{-1/2}-P^1_{1/2}Q_{-1/2}}. \tag{2.26}$$

Turning now back to (2.24), we find out analytical expressions for integrals F_1, C_1 :

$$F_1=ch\mu\frac{Q_{-1/2}\cdot Q^1_{-1/2}}{Q^1_{1/2}\cdot P_{-1/2}-P^1_{1/2}\cdot Q_{-1/2}}-\frac{1}{2}Q_{1/2}(Q_{-1/2}-ch\mu\,Q_{1/2})-2, \tag{2.27}$$

$$C_1=ch\mu\,\frac{P_{-1/2}\cdot Q^1_{-1/2}}{Q^1_{1/2}\cdot P_{-1/2}-P^1_{1/2}\cdot Q_{-1/2}}-\frac{1}{2}P_{1/2}\cdot(Q_{-1/2}-ch\mu\cdot Q_{1/2}).$$

Now make one more step and get F_2, C_2. At first, express F_2 and C_2 through S_2 using one of the Eqs. (2.25) and (2.23) for $n=2$:

$$C_2\cdot Q^1_{1/2}-(F_2+\frac{10}{9})P^1_{1/2}=-\frac{1}{4}\cdot sh\mu\cdot(Q_{-1/2}+Q_{3/2}), \tag{2.28}$$

$$C_2\,Q_{3/2}-(F_2+\frac{10}{9})P_{3/2}=\frac{1}{16}S_2-\frac{\pi^2}{8}P_{3/2}.$$

Further set $n=2$ in (2.19) and exclude F_3, C_3. This is always possible because they are contained in the same linear combination $F_3P^1_{1/2}-C_3Q^1_{1/2}$.
The resulting relation is:

$$C_2\cdot Q^1_{3/2}-F_2\cdot P^1_{3/2}=\frac{1}{2}(C_1\cdot Q^1_{3/2}-F_1\cdot P^1_{3/2})+\frac{1}{9}P^1_{3/2}+\frac{1}{6}(ch\mu\,Q^1_{1/2}-Q^1_{3/2}). \tag{2.29}$$

Inserting into it F_2, C_2 from (2.28), one gets S_2. Turning back to (2.28), one obtains F_2, C_2.

After these preliminaries, exclude F_n, C_n from (2.20) by using (2.23), and get $d_n(\mu)$:

$$d_n(\mu)=\frac{d^0_n(\mu)}{1-\exp(-2\mu_0)}+2\tilde{g}\cdot\beta^0_n(\mu_0)\cdot P_{n-1/2}, \tag{2.30}$$

where $\beta^0_n=\sum_{k=0}^{n-1}Q_{k-\frac{1}{2}}(ch\mu_0)\cdot Q_{k+\frac{1}{2}}(ch\mu_0)-\frac{\pi^2}{4}(cth\mu_0-1)=$

$$=-\sum_{k=n}^{\infty}Q_{k-1/2}(ch\mu_0)\cdot Q_{k+1/2}(ch\mu_0).$$

As a result of these manipulations, one escapes complicated integration procedure and expresses the generating function d for the finite solenoid through its counterpart d_0 for the thin solenoid, which is known explicitly (Eqs. (2.4), (2.5)):

$$d(\vartheta,\mu)=\frac{d_0(\vartheta,\mu)}{1-\exp(-2\mu_0)}+2\cdot\tilde{g}\,\sqrt{ch\mu-\cos\vartheta}\cdot\sum_{n=1}^{\infty}\beta^0_n\cdot P_{n-1/2}\cdot\sin n\vartheta. \tag{2.31}$$

Using this relation (2.2) and asymptotical behaviour of the Legendre functions, one obtains:

$$d(\vartheta,\mu)\underset{r\to\infty}{\sim}-\frac{\pi g}{4}\,\frac{ch\mu_0}{sh^3\mu_0}\,\frac{a^3}{r^2}\cdot\cos\vartheta_s.$$

Comparing this with (2.6), one sees that in both cases the function α falls as $1/r^2$, as r tends to infinity.

3. Here we consider scattering of the charged zero-spin particles on the circulation free VMP of the toroidal solenoid. In order to keep incoming particles out of the region where $H \neq 0$, one screens this field using the infinite repulsive potential of the suitable geometrical form. We shall mainly make use of the two topologically nonequivalent screens: the spherical one, containing solenoid T_0 as a whole and the toroidal one containing T_0 and differing from it negligibly. The repulsive potentials being infinitely high, the wave functions vanish on the boundary of the impenetrable screen (as well as inside it).

So, one has the Schroedinger equation:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x_k}-\frac{ie}{\hbar c}A_k\right)^2\Psi + V\Psi = E\cdot\Psi, \tag{3.1}$$

where $V=\infty$, $\Psi=0$ inside the screen and at its boundary, and $V=0$ otherwise. This equation may be slightly modified:

$$\Delta\Psi - \frac{2ie}{\hbar c}A_k\frac{\partial\Psi}{\partial x_k} - \frac{e^2}{\hbar^2c^2}A^2\Psi + (k^2-V)\Psi = 0. \tag{3.2}$$

Here it was taken into account that $\operatorname{div}\vec{A}=0$. Eq. (3.2) is equivalent in turn to the Lippman-Schwinger equation:

$$\Psi = \Psi_0 + \int G_0(\vec{r},\vec{r}\,')\cdot V_1(\vec{r}\,')\cdot\Psi(\vec{r}\,')\,dV', \tag{3.3}$$

where Ψ_0 and G_0 are the wave and Green functions in the absence of the electromagnetic field (emf for brevity).

$$(\Delta - V + k^2)\Psi_0 = 0, \quad (\Delta - V + k^2)G_0 = \delta^3(\vec{r}-\vec{r}\,'). \tag{3.4}$$

Moreover, we set in (3.3):

$$V_1 = \frac{2ie}{\hbar c}A_k\frac{\partial}{\partial x_k} + \frac{e^2}{\hbar^2c^2}A^2. \tag{3.5}$$

The following expression satisfies formally Eqs. (3.1), (3.2):

$$\Psi = \Psi_0\cdot\exp\left(\frac{ie}{\hbar c}\alpha\right). \tag{3.6}$$

The function α determined by the relation $\vec{A}=\operatorname{grad}\alpha$ has been calculated in the previous section.

Now, we discuss some consequences of (3.6). It is well known /5,7/, that the Aharonov-Bohm (A.B.) effect arises only if the single-valued functions Ψ are used. We demonstrate this taking solution (3.6) as an example. It is evident that if Ψ_0 in (3.6) is everywhere a single-valued and continuous solution of the Schroedinger equation in the absence of the emf, then Ψ undergoes a finite jump as one passes the circle of radius a lying in the $z=0$ plane:

$$\Psi(\rho<a, z=\varepsilon) - \Psi(\rho<a, z=-\varepsilon) = \Psi_0(\rho,0)\cdot\exp\left[\frac{2ie}{\hbar c}\pi g a\,(\operatorname{cth}\mu_0-1)\right].$$

As the generating function α decreases as $1/r^2$ for $r\to\infty$, the asymptotic behaviour of Ψ and Ψ_0 differs in the same order; this in turn means that the magnetic field does not contribute to the scattering amplitude if one uses a multivalued solution (3.6) of the Schroedinger equation. If one requires the total wave function Ψ to be single valued and continued, one must impose on Ψ_0 (which is the Schroedinger equation solution in the absence of the emf) the following boundary condition:

$$\Psi_0(\rho<a, z=\varepsilon) = \Psi_0(\rho<a, z=-\varepsilon)\cdot\exp\left[\frac{2iea}{\hbar c}\pi g(\operatorname{cth}\mu_0-1)\right]. \tag{3.7}$$

Now, surround solenoid with the impenetrable sphere of the radius $R_1 > d+R$. Then, inside this sphere and on its boundary Ψ_0 equals zero. Relation (3.7) is satisfied trivially ($0=0$), and Ψ, given by (3.6), is a single valued and continuous solution of the Schroedinger equation. Ψ_0 is given by:

$$\Psi_0 = \exp(ikz) - \sqrt{\frac{\pi}{2kr}}\cdot\sum(2l+1)\cdot i^l\, H^1_{l+\frac12}(kr)\,\frac{J_{l+1/2}(kR_1)}{H^1_{l+1/2}(kR_1)}\,P_l(\cos\vartheta_s) \tag{3.8}$$

and has the following asymptotic behaviour as $r\to\infty$:

$$\Psi_0 \sim \exp(ikz) + \frac{1}{r}\exp(ikr)\cdot f(\vartheta_s), \quad \text{where}$$
$$f(\vartheta_s) = \frac{i}{k}\sum(2l+1)\frac{J_{l+1/2}(kR_1)}{H^1_{l+\frac12}(kR_1)}\cdot P_l(\cos\vartheta_s). \tag{3.9}$$

Repeating word in word the previous considerations, one concludes that the scattering amplitude in the presence of the emf is the same as (3.9).

The situation is more complicated if the solenoid T_0 is embedded into the screen of the toroidal form. For simplicity take it to coincide with the solenoid T_0. We restrict ourselves to the first order perturbation theory with respect to potentials A_k. This is achieved if one uses in the right-hand side of (3.3) the function Ψ_0 instead of the total wave function Ψ:

$$\Psi = \Psi_0 + \frac{2ie}{\hbar c}\int G_0(\vec{r},\vec{r}')\, A_k \frac{\partial \Psi_0}{\partial x'_k}\, dV' . \tag{3.10}$$

Here integration takes place outside the solenoid T_0, where $A_k = \frac{\partial \alpha}{\partial x_k}$. Further, one has:

$$A_k \frac{\partial \Psi_0}{\partial x_k} = \frac{\partial \alpha}{\partial x_k}\cdot\frac{\partial \Psi_0}{\partial x_k} = \frac{1}{2}\Delta(\alpha\cdot\Psi_0) + \frac{1}{2}(k^2 - V)\cdot\Psi_0\cdot\alpha \cdot$$

Then:

$$\Psi = \Psi_0 + \frac{ie}{\hbar c}\int G_0\cdot\left[\Delta(\alpha\Psi_0) + (k^2 - V)\Psi_0\right] dV' . \tag{3.11}$$

Now integrate (3.11) twice in parts and make use of Eq. (3.4) for the Green function G_0:

$$\Psi = \Psi_0 + \frac{ie}{\hbar c}\alpha(\vec{r})\cdot\Psi_0(\vec{r}) + \frac{ie}{\hbar c}\int \mathrm{div}\left[G_0\cdot \mathrm{grad}(\alpha\Psi_0) - \alpha\Psi_0\cdot \mathrm{grad}\, G_0\right] dV'. \tag{3.12}$$

Using the Gauss theorem one can replace the volume integration by the one over the surface containing this volume. This surface consists of two parts: the surface of the torus T_0 and one of the infinite radius sphere S_R. The integral over the torus surface vanishes since both Ψ_0 and G_0 equal zero there. So, there remains an integral over S_R:

$$R^2\int\left[G_0\frac{\partial(\alpha\Psi_0)}{\partial R} - \alpha\Psi_0\frac{\partial G_0}{\partial R}\right] d\Omega' .$$

Since at infinity: $\Psi_0 \sim \exp(ikz)$, $\alpha \sim -\frac{\pi g}{4}\frac{\mathrm{ch}\,\mu_0}{\mathrm{sh}^3\mu_0}\frac{a^3}{R^2}\cos\vartheta_s$,

$G_0 \sim -\frac{1}{4\pi}\frac{\exp(ikR)}{R}\cdot\exp(-ik\,\vec{r}\cdot\vec{n}')$, the last expression

has the asymptotic form:

$$-\frac{g}{16}\frac{kea^3}{\hbar c}\frac{\mathrm{ch}\,\mu_0}{\mathrm{sh}^3\mu_0}\frac{\exp(ikR)}{R}\int\cos\vartheta'_s(1+\cos\vartheta'_s)\cdot\exp[ik(R\cos\vartheta'_s - \vec{r}\cdot\vec{n}')]\,d\Omega'$$

and tends to zero as $R\to\infty$. So, in the first order perturbation theory

$$\Psi = \Psi_0 + \frac{ie}{\hbar c}\alpha(\vec{r})\cdot\Psi_0(\vec{r}) , \tag{3.13}$$

where Ψ_0 is the single-valued and continuous solution of the Schroedinger equation without emf. From this it follows that the asymptotic form of Ψ and Ψ_0 is the same and the contribution of the H into the scattering amplitude vanishes.

The same result is obtained if one calculates the scattering amplitude directly from (3.10). Assume that for $r\to\infty$ the Green function G_0 coincides with the plane-wave one $-\frac{\exp(ik|\vec{r}-\vec{r}'|)}{4\pi|\vec{r}-\vec{r}'|}$. Then, one finds out the following expression for the scattering amplitude:

$$f(\vec{n}) = -\frac{ie}{2\pi\hbar c}\cdot\int\exp(-ik\vec{n}\cdot\vec{r}')\cdot A_k\frac{\partial\Psi_0}{\partial x'_k}\, dV' . \tag{3.14}$$

Using the same trick as before, one reduces this volume integral to the surface one:

$$f(\vec{n}) = -\frac{ke}{16\hbar c}\cdot g a^3\frac{\mathrm{ch}\,\mu_0}{\mathrm{sh}^3\mu_0}\int d\Omega'\cdot\cos\vartheta'_s(\cos\vartheta'_s + \vec{n}\cdot\vec{n}')\cdot\exp[ikR(\cos\vartheta'_s - \vec{n}\cdot\vec{n}')].$$

Here $\vec{n}$ and $\vec{n}'$ are unit vectors of the observation and current integration points. After one angle integration the last expression reduces to

$$f(\vartheta_s) = -\frac{\pi e}{4\hbar c}g a^3\frac{\mathrm{ch}\,\mu_0}{\mathrm{sh}^3\mu_0}\;\frac{1+\cos\vartheta_s}{\sin\vartheta_s}\;\frac{1}{R}\int_0^1\frac{x^2dx}{\sqrt{1-x^2}}\cdot J_0(kRx\sin\vartheta_s)\cdot\cos\left[kR(1-\cos\vartheta_s)\sqrt{1-x^2}\right], \tag{3.15}$$

For any finite (but different from zero) value of ϑ_s the scattering amplitude vanishes as $R\to\infty$, so we get the previous result. Both derivations assumed that asymptotic behaviour of G_0 and Ψ_0 is the same as their plane-wave counterparts. This takes place, for example, for scattering on the impenetrable sphere.

Earlier we have used the Gauss theorem in order to escape the volume integration in (3.10), because the behaviour of the wave function in the neighbourhood of the solenoid is essential in this case. The last one is not known explicitly because equation $(\Delta+k^2)\Psi=0$ is not separated in the toroidal coordinates. We consider now a concrete example showing that the use of natural approximations for the Green and wave function in the neighbourhood of the solenoid could lead to a wrong result. Take the thin solenoid embedded in a thin impenetrable screen. This may be chosen as an oblate ellipsoid of revolution with a large ratio of semiaxes (this is convenient because equation $(\Delta+k^2)\Psi=0$ is separated in the spheroidal coordinates). From the above consideration it follows that the magnetic field gives a zero contribution to the scattering amplitude. Now calculate the same amplitude in the high-energy approximation [3,8]. For this we make in the Lippman-Schwinger equation

$$\Psi = \Psi_0 + \int G_0(\vec{r},\vec{r}\,')\, V_1(\vec{r}\,')\cdot\Psi(\vec{r}\,')\, dV'$$

the following typical for this method simplifications: change Green function G_0 with a plane wave function $-\frac{1}{4\pi}\frac{\exp(ik|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|}$; instead of Ψ under the integral make use of its high energy approximation: $\exp(ikz+\frac{ie}{\hbar c}\int_{-\infty}^{z} A_z dz)$. Then, the scattering amplitude equals:

$$f(\vec{n}) = \frac{eK}{2\pi\hbar c}\int \exp(i\vec{q}\,\vec{r}\,')\, A_z \exp\Big(\frac{ie}{\hbar c}\int_{-\infty}^{z'} A_z dz\Big)\, dV'. \tag{3.16}$$

Here $\vec{q}=\vec{K}-\vec{K}'$ is the momentum transfer; $\vec{K}'=K\cdot\vec{n}$, $\vec{K}=K\cdot\vec{e}_z$. As for high energy the small angle scattering dominates, the vector $\vec{q}$ may be considered to be perpendicular to the initial vector $\vec{K}$ [8]; so it lies in the xy plane: Then:

$$f(\vec{n}) = \frac{eK}{2\pi\hbar c}\int d^2\vec{\rho}\cdot\exp(i\vec{q}\,\vec{\rho})\cdot\int_{-\infty}^{\infty} dz'\cdot A_z \exp\Big(\frac{ie}{\hbar c}\int_{-\infty}^{z'} A_z dz\Big) =$$

$$= -\frac{iK}{2\pi}\int d^2\vec{\rho}\cdot\exp(i\vec{q}\,\vec{\rho})\cdot\Big[\exp\Big(\frac{ie}{\hbar c}\int_{-\infty}^{\infty} A_z dz\Big)-1\Big] =$$

$$= -iK\cdot\Big[\exp\Big(\frac{ie\Phi}{\hbar c}\Big)-1\Big]\cdot\int_0^d J_0(q\rho)\,\rho\, d\rho = -\frac{iKd}{q}\cdot\Big[\exp\Big(\frac{ie\Phi}{\hbar c}\Big)-1\Big]\cdot J_1(qd).$$

Here Φ is the magnetic field flow.

Here, we took into account that major contribution to the integral $\int_{-\infty}^{\infty} A_z dz$ comes from the integration path lying outside the screen. In fact, estimate, for example, $\int_{-R}^{R} A_z dz$ for $\rho = 0$. The potential A_z equals [1]:

$$A_z(\rho=0, z) = \frac{\pi g}{2}\frac{R^2 d}{(z^2+d^2)^{3/2}}$$

Then:

$$\frac{\int_{-R}^{R} A_z(0,z)}{\int_{-\infty}^{\infty} A_z(0,z)} = \frac{R}{\sqrt{d^2+R^2}}$$

that is negligible for the thin $(\frac{R}{d} \ll 1)$ solenoid.

So, a naive application of the high energy approximation leads to a wrong result. This is due to the nonvalidity of this approximation

in the treated concrete example (there is space region with an infinite value of the potential) and to using of the simplified wave and Green functions, which do not satisfy the proper boundary conditions.

4. The lack of space forces us to leave here the detailed discussion of the A.B. effect aside . Instead of that we make some comments concerning interesting in many respects ref.[2]. It is shown there that the contribution of the magnetic field to the scattering amplitude differs from zero if the screen has the form of the torus into which the solenoid T_0 is embedded. The proof contains qualitative arguments as well as high energy and first order perturbation calculations. Consider them step by step.

A qualitative consideration is due to the interference that takes place between particles passing the hole of the solenoid and outside it. The interference in turn is due to a phase difference between the corresponding wave functions. This reasoning presumes implicitly the multivaluedness of the used wave functions. However, as we mentioned earlier, the singlevaluedness of these functions is the necessary condition for the existence of the A.B. effect [5,7]. The representation of the wave function in the form $\Psi=\Psi_0\cdot\exp(\frac{ie}{\hbar c}\int A_l dl)$ is correct if under Ψ_0 one means the multivalued wave function in the absence of emf. As a result, the above cited qualitative reasoning does not work and in fact proves nothing. So, a concrete calculation is needed. The misleading character of these qualitative arguments has been discussed in [5] also.

At the end of the preceding section we demonstrated by the concrete example nonadequacy of the naive version of the high energy approximation. As was used in [2] in the same context, it has the same drawbacks.

We point out three reasons why the results of [2] concerning the first order perturbation theory seem unsatisfactory to us: first, plane wave functions are used instead of those satisfying proper boundary conditions; second, VMP used in [2] have delta-type singularities outside the solenoid due to the singular gauge transformation. This means that the neglected quadratic in potentials terms are more singular than the linear ones taken into account; third, infinite value of the magnetic field inside the solenoid in the absence of screening (due to using of the plane wave functions).

Finally, some notes concerning recent intensive discussion on the existence of the A.B. effect. If one means under this the essential role of the electromagnetic potentials in quantum mechanics, the existence of the A.B. effect is without any doubt. For example, it gives rise to the nonzero scattering cross section of the charged particles on the VMP of the cylindrical solenoid. Note, that the generating function α in this case equals $\operatorname{arc\,tg}\frac{y}{x}$ up to constant and does not decrease at large distances. This does not exclude that for certain configurations of the screened magnetic field the scattering amplitude on the VMP could vanish. This resembles the absence of scattering on the Bargmann potentials.

5. Here we collect the integrals and series of the Legendre functions obtained in this paper and in /1/ and which we failed to find out in mathematical handbooks /6,9/, treatises /10/ and original publications /11/:

I. $\sum (n+\tfrac{1}{2})^2 Q_{n-1/2}\, Q_{n+1/2} = \frac{\pi^2}{32}\,\frac{\operatorname{ch}\mu}{\operatorname{sh}^3\mu}$,

II. $\sum (-1)^n Q_{n-1/2}\, Q_{n+1/2} = \frac{\pi}{8}\,\frac{1}{\sqrt{\operatorname{ch}\mu}}\, Q_{1/2}\left(\frac{1+\operatorname{ch}^2\mu}{2\operatorname{ch}\mu}\right)$,

III. $\sum Q_{n-1/2}\cdot Q_{n+1/2} = \frac{\pi^2}{4}\cdot(\operatorname{cth}\mu-1)$,

IV. $C_n - C_{n+1} = (n+1/2)\, Q_{n-1/2}\, Q_{n+1/2} \qquad \left(C_n = \int_\mu^\infty (Q^1_{n-1/2})^2 \frac{d\mu}{\operatorname{sh}\mu}\right)$,

V. $\mathcal{D}_n - \mathcal{D}_{n+1} = (n+1/2)\, P_{n-1/2}\, Q_{n+1/2} \qquad \left(\mathcal{D}_n = \int_\mu^\infty P^1_{n-1/2}\, Q^1_{n-1/2} \frac{d\mu}{\operatorname{sh}\mu}\right)$,

VI. $\displaystyle\int_0^d \frac{dx}{\sqrt{x}} \left[\rho\, Q^1_{-1/2}\left(\frac{r^2+x^2}{2\rho x}\right) - x\, Q^1_{1/2}\left(\frac{r^2+x^2}{2\rho x}\right)\right] \frac{1}{\left[\left(\frac{r^2+x^2}{2\rho x}\right)^2-1\right]^{1/2}} =$

$= -\rho\sqrt{d}\cdot Q_{1/2}\left(\frac{r^2+d^2}{2d\rho}\right) \qquad (r^2 = \rho^2 + z^2)$,

VII. $\displaystyle\int_0^\mu P^1_{-\frac{1}{2}}\cdot Q^1_{n-\frac{1}{2}}\cdot \operatorname{sh}\mu\cdot d\mu = \frac{1}{4}\left(\frac{1}{4n^2}-1\right)\cdot\left[P_{-\frac{1}{2}}(Q_{n+\frac{1}{2}}+Q_{n-\frac{3}{2}}) - 2P_{\frac{1}{2}}Q_{n-\frac{1}{2}}\right] + \frac{1}{4n^2}$, $(n \geq 1)$

VIII. $\displaystyle\sum_{k=1}^{n} \frac{\Gamma(k-\frac{1}{2})\,\Gamma(n-k+\frac{1}{2})}{\Gamma(k+1)\cdot\Gamma(n-k+1)} = 2\sqrt{\pi}\,\frac{\Gamma(n+1/2)}{\Gamma(n+1)}$,

IX. $\displaystyle\int_0^\mu P_{\frac{1}{2}}\, Q_{\frac{1}{2}} \operatorname{sh}\mu\cdot d\mu = \operatorname{ch}\mu\,\frac{P_{-\frac{1}{2}}\, Q^1_{-1/2}}{Q^1_{1/2}\, P_{-1/2} - P^1_{1/2}\, Q_{-1/2}} - \frac{1}{2}P_{1/2}(Q_{-1/2} - \operatorname{ch}\mu\, Q_{1/2})$,

X. $\displaystyle\int_0^\mu (Q_{1/2})^2 \operatorname{sh}\mu\, d\mu = \operatorname{ch}\mu\,\frac{Q_{-1/2}\, Q^1_{-1/2}}{Q^1_{1/2}\, P_{-1/2} - P^1_{1/2}\, Q_{-1/2}} - \frac{1}{2}Q_{\frac{1}{2}}(Q_{-\frac{1}{2}} - \operatorname{ch}\mu\, Q_{\frac{1}{2}}) - 2$,

XI. $\displaystyle\sum_{n=1}^\infty \frac{1}{n}\cdot\left[(Q_{n+\frac{1}{2}}+Q_{n-\frac{3}{2}})\cdot P_{\frac{1}{2}} - 2\cdot Q_{n-\frac{1}{2}}\cdot P_{\frac{1}{2}}\right]\cdot(Q_{n-\frac{1}{2}} - Q_{n+\frac{1}{2}}) =$

$= 2\pi^2\cdot P_{1/2} - 16\cdot\operatorname{ch}\mu\,\dfrac{Q'_{-1/2}}{Q'_{1/2}\, P_{-1/2} - P'_{1/2}\cdot Q_{-1/2}}$.

Some words on the derivation of these formulas. Expr. I,II,IV, V were obtained in /1/ by comparing two different solution methods of the same equations with the same boundary conditions. Expr. III was also derived there by equating the magnetic field flow to the linear integral along the contour passing through the torus hole. It was proved independently,too. Expr. VII-XI have been obtained in the present paper from the identity of the generating functions for the thin and finite solenoid if the thichness of the last one tends to zero. Finally, Expr. VI was obtained as follows. Take from /1/ the cylindrical component A of the VMP outside the solenoid T_0:

$$\frac{R^2 z g}{4}\int_0^{2\pi} \frac{\cos\varphi\, d\varphi}{(r^2+d^2-2d\rho\cos\varphi)^{3/2}} = -\frac{R^2 g}{4}\frac{\partial}{\partial z}\int_0^{2\pi}\frac{\cos\varphi\, d\varphi}{(r^2+d^2-2d\rho\cos\varphi)^{1/2}}.$$

On the other hand A_ρ equals $\frac{\partial^2 \alpha}{\partial\rho\,\partial z}$, where the function α (do not mix it with the generating function introduced in sect. 2) permits different representations:

$$\alpha = \iint_{\substack{\rho'\leq d\\ z'=0}} \frac{dx'\,dy'}{|\vec r - \vec r'|} = \int_0^d\int_0^{2\pi} \frac{\rho'\,d\rho'\,d\varphi}{[r^2+\rho'^2-2\rho\rho'\cos\varphi)^{1/2}} = \frac{2}{\sqrt{d}}\int_0^d \sqrt{\rho'}\,d\rho'\cdot Q_{\frac{1}{2}}\left(\frac{r^2+\rho'^2}{2\rho\rho'}\right) =$$

$$= 2\pi\left(\sqrt{z^2+d^2}-|z|\right) - 2\sqrt{d}\int_0^\rho Q_{\frac{1}{2}}\left(\frac{r^2+d^2}{2d\rho}\right)\frac{d\rho}{\sqrt{\rho}} = \sqrt{\operatorname{ch}\mu-\cos\vartheta}\,\sum d_n(\mu)\cdot\cos n\vartheta,$$

$$d_n = (-1)^n\cdot\frac{4a}{1+\delta_{n,0}}\cdot\left[Q_{n-\frac{1}{2}}\int_0^\mu \frac{P_{n-\frac{1}{2}}\operatorname{sh}\mu\, d\mu}{(1+\operatorname{ch}\mu)^{3/2}} - P_{n-\frac{1}{2}}\int_\infty^\mu \frac{Q_{n-\frac{1}{2}}\operatorname{sh}\mu\, d\mu}{(1+\operatorname{ch}\mu)^{3/2}}\right].$$

It follows from this that the integrals

$$\int_0^{2\pi}\frac{\cos\varphi\, d\varphi}{(r^2+d^2-2d\rho\cos\varphi)^{1/2}} \qquad \frac{1}{d}\int_0^d\int_0^{2\pi}\frac{\rho'\,d\rho'\,(\rho-\rho'\cos\varphi)\,d\varphi}{(r^2+d^2-2d\rho\cos\varphi)^{3/2}}$$

may differ utmost by the arbitrary function of ρ. As they have the same limit as $z\to\infty$, they in fact coincide everywhere. By integrating with respect to φ, one obtains VI.

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Афанасьев Г.Н. E4-84-65

Рассеяние заряженных частиц на экранированном тороидальном соленоиде

Рассмотрено рассеяние заряженных бесспиновых частиц на экранированном магнитном поле тороидального соленоида. Показано, что магнитное поле не дает вклада в амплитуду рассеяния для двух топологических неэквивалентных форм экрана. Получены аналитические выражения для ряда сумм и интегралов от лежандровских функций.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Afanasiev G.N. E4-84-65

The Scattering of Charged Particles on the Screened Toroidal Solenoid

The scattering of charged particles on the screened magnetic field of the toroidal solenoid is considered. It is shown that the magnetic field does not contribute to the scattering amplitude for two topologically non-equivalent screenings. The analytic expressions for some integrals and series containing Legendre functions are obtained.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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