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# ON THE *-PRODUCT

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## INTRODUCTION

Thanks to the papers of Bayen et al.[1,2] Gerstenhaber's deformation theory for algebras[3] has found its place in mathematical physics. It has been shown that we may regard quantization as a deformation of the usual associative algebra $\mathfrak{A}$ of $\mathcal{C}^\infty$ functions on phase space.

We investigate associative one-parameter deformations of algebra $\mathfrak{A}$ in the case when the new algebra law ( $*$-product) is given by a formal series expansion $\sum_{n=0}^{\infty} t^n F_n$. The deformation parameter $t = i\hbar$ ($\hbar$ is Planck's constant). The 2-cochains $F_n$ are given by $n$-bidifferential operators of order $\leq n$ in each argument. The requirement of associativity is so strong for the deformation class under consideration that it determines the only function form for the $*$-product. It is an exponential function.

We would like to remark that the unicity of the exponent *as a function of Poisson's bracket* for the flat Poisson manifold has been found by Bayen et al.[1, theorem 3]. The case of general symplectic manifold (for linear connection $\Lambda$ such that $\nabla\Lambda = 0$) has been also considered. Furthermore, notice earlier result by Mehta[4] in connection with deformation of the Poisson Lie algebra $\mathcal{C}^\infty$ functions on $R^2$.

We study the deformations on two-dimensional phase space only. It is easy to generalize these results to multidimensional cases. The main result of the paper is given by theorem 8.

## 1. DEFORMATIONS OF THE ALGEBRA $\mathfrak{A}$

Infinitely differentiable functions $\mathfrak{N}$ on phase space $\mathfrak{W}$ form the associative algebra $\mathfrak{A}$ over a field $\mathfrak{Q}$ with an ordinary composition law. We denote the Kronecker product of m-linear spaces isomorphic with the linear space underlying $\mathfrak{A}$ by $\mathfrak{A}^{(m)}$. By an m-dimensional $\mathfrak{A}$-cochain of $\mathfrak{A}$ an m-linear mapping $f^m$ : $\mathfrak{A}^{(m)} \to \mathfrak{A}$ is meant. We assume that the cochains are given by differential operators. To formulate the results we use the following (maybe somewhat "nonstandard") definition.

*Definition 1*

An s-*differentiable* m-*cochain* $F_s$ with values in $\mathfrak{A}$, is an m-linear mapping from $\mathfrak{A}^{(m)}$ to $\mathfrak{A}$ that can be written

$$F_s(u, v, \dots w) = \sum_{\substack{k_1+k_2+\dots+k_m=2s \\ i_1+i_2+\dots+i_m=s}} \lambda^{k_1, k_2, \dots k_m}_{i_1, i_2, \dots i_m}(s)\, u^{(k_1)}_{i_1}\, v^{(k_2)}_{i_2} \dots w^{(k_m)}_{i_m}, \qquad (1.1)$$

where $u, v, w \in \mathfrak{N}$, $\lambda^{k_1, k_2, \dots k_m}_{i_1, i_2, \dots, i_m} \in \mathfrak{Q}$ and

$$u_i^{(k)} = \partial^k u / \partial q^i \partial p^{k-i}. \qquad (1.2)$$

The vector space $\mathrm{Hom}(\mathfrak{A}^{(m)}, \mathfrak{A})$ of all this differentiable m-cochains will be denoted by $\mathbf{C}^m(\mathfrak{A}, \mathfrak{A})$. $\mathbf{C}^0(\mathfrak{A}, \mathfrak{A})$ is identical to $\mathfrak{A}$.

Let us consider an associative algebra $\mathfrak{A}_t$ which is a result of one-parameter deformation of $\mathfrak{A}$. The composition law in $\mathfrak{A}_t$ is defined through the bilinear function

$$f_t(u, v) = u * v = \sum_{n=0}^{\infty} t^n F_n(u, v), \qquad (1.3)$$

where $u, v \in \mathfrak{N}$, $F_0(u, v) = uv$, $F_n(u, v) \in \mathbf{C}^2(\mathfrak{A}, \mathfrak{A})$. The associativity requirement for algebra $\mathfrak{A}_t$

$$f_t(f_t(u, v), w) = f_t(u, f_t(v, w)) \qquad (1.4)$$

is equivalent to having

$$\sum_{\substack{n+m=\ell \\ n, m \geq 0}} \{F_m[F_n(u, v), w] - F_m[u, F_n(v, w)]\} = 0 \qquad (1.5)$$

for all $u, w, w \in \mathfrak{N}$ and all $\ell = 0, 1, 2, \dots$

We consider the coboundary operator $\delta_m$ according to the classical definition of Hochschild /5/. Operator $\delta_m$ acts on the set of all differentiable $\mathfrak{A}$-cochains according to the rule $\delta_m$: $\mathbf{C}^m(\mathfrak{A}, \mathfrak{A}) \to \mathbf{C}^{m+1}(\mathfrak{A}, \mathfrak{A})$

$$\delta_m f(u_1, \dots, u_{m+1}) = u_1 f(u_2, \dots, u_{m+1}) + \sum_{i=1}^{m} (-1)^i f(u_1, \dots, u_i u_{i+1}, \dots, u_{m+1})$$

$$- (-1)^{m+1} f(u_1, \dots, u_m) u_{m+1}. \qquad (1.6)$$

It is the case that $\delta_{m+1}\delta_m = 0$. $\mathbf{Z}^m(\mathfrak{A}, \mathfrak{A})$ is defined to be the kernel of $\delta_m$, $\mathbf{B}^m(\mathfrak{A}, \mathfrak{A})$ to be the image of $\delta_{m-1}$ for $m \geq 1$, and to be zero for $m = 0$.

We shall use also a special form of the Lie product introduced in /6/. Let $f^m \in \mathbf{C}^m(\mathfrak{A}, \mathfrak{A})$, $g^n \in \mathbf{C}^n(\mathfrak{A}, \mathfrak{A})$, then homomorphism $[f^m, g^n]$: $\mathbf{C}^m(\mathfrak{A}, \mathfrak{A}) \otimes \mathbf{C}^n(\mathfrak{A}, \mathfrak{A}) \to \mathbf{C}^{m+n-1}(\mathfrak{A}, \mathfrak{A})$

$$[f^m, g^n] = \sum_{i=0}^{m} (-1)^{i(n-1)} f^m \circ_i g^n - (-1)^{(m-1)(n-1)} \sum_{i=0}^{n-1} (-1)^{i(m-1)} g^n \circ_i f^m \qquad (1.7)$$

is defined.



Here

$$f^m \circ_i g^n (u_0, \dots, u_{i-1}, v_0, \dots v_{n-1}, u_{i+1}, \dots u_{m-1})$$

$$= f^m(u_0, \dots, u_{i-1}, g^n(v_0, \dots, v_{n-1}), u_{i+1}, \dots, u_{m-1}). \qquad (1.8)$$

The bracket product makes the direct sum $\oplus_n \mathbf{C}^n(\mathfrak{A}, \mathfrak{A})$ into a graded Lie ring

$$[f^m, g^n] = -(-1)^{(m-1)(n-1)} [g^n, f^m], \qquad (1.9)$$

$$(-1)^{(m-1)(\ell-1)} [[f^m, g^n], h^\ell] + (-1)^{(m-1)(n-1)} [[g^n, h^\ell], f^m]$$

$$+ (-1)^{(\ell-1)(n-1)} [[h^\ell, f^m], g^n] = 0. \qquad (1.10)$$

We remark that $\delta_m$ is a right derivation with respect to the bracket product

$$\delta [f^m, g^n] = (-1)^{(n-1)} [\delta f^m, g^n] + [f^m, \delta g^n]. \qquad (1.11)$$

## 2. ALGEBRA OF DIFFERENTIAL OPERATORS $\mathfrak{D}^{(m)}$

We denote by $\{x_n^{(m)}\}$ the operators acting on $\mathfrak{N}$ such that

$$x_n^{(m)}(u) = u_n^{(m)} \qquad (2.1)$$

for all $u \in \mathfrak{N}$.

These operators constitute a basis of the vector space which we make into an algebra $\mathfrak{D}$ by giving on this space the composition law

$$x_n^{(m)} \cdot x_{n'}^{(m')} = x_{n+n'}^{(m+m')} \qquad (2.2)$$

Let us denote the direct sum of the m-algebras isomorphic with $\mathfrak{D}$ by $\mathfrak{D}^{(m)}$. We suppose that the s-differentiable m-cochains $F_s(u, v, \dots, w)$ (see (1.1)) are given by the differential operators $F_s \in \mathfrak{D}^{(m)}$, i.e.,

$$F_s = \sum_{\substack{k_1+k_2+\dots k_m = 2s \\ i_1+i_2+\dots i_m = s}} \lambda^{k_1, k_2, \dots, k_m}_{i_1, i_2, \dots, i_m}(s)\, x^{(k_1)}_{i_1} \cup x^{(k_2)}_{i_2} \cup \dots \cup x^{(k_m)}_{i_m} \qquad (2.3)$$

Therefore, we may speak about the product between any two m-cochains.

*Definition 2*

Let $f^m, g^m \in C^m(\mathfrak{A},\mathfrak{A})$. Then there exists the product $h^m = f^m g^m$ such that $h^m \in C^m(\mathfrak{A},\mathfrak{A})$, and the operator corresponding to $h^m$ is equal to the product of operators $f^m$ and $g^m$ in $\mathfrak{D}^{(m)}$.

By induction we get

Proposition 3

$$(x \cup x)_n^{(m)} = \sum_{i=0}^{m} \sum_{j=0}^{j} C_n^j C_{m-n}^{i-j} \theta(n-j)\theta(m-n-i+j) x_j^{(i)} \cup x_{n-j}^{(m-i)}, \qquad (2.4)$$

where $\theta(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$ , $C_n^i = \dfrac{n!}{i!(n-i)!}$ .

We may now state

Theorem 4

*There is only one formal function of the operator* $F = \lambda_1 x_1^{(1)} \cup x_0^{(1)} + \lambda_2 x_0^{(1)} \cup x_1^{(1)}$ *that generates a formal deformation of the associative algebra* $\mathfrak{A}$. *It is the exponential function.*

Proof: Let us expand the functions f(F) in the formal series in t

$$f(F) = \sum_{n=0}^{\infty} a_n t^n F^n \qquad (2.5)$$

where

$$F^n = \sum_{k=0}^{n} C_n^k \lambda_1^{n-k} \lambda_2^k x_{n-k}^{(n)} \cup x_k^{(n)} . \qquad (2.6)$$

and

$$a_0 = a_1 = 1. \qquad (2.7)$$

We search for such functions f for which (1.4) is satisfied formally. Let us write the sum

$$U_\ell = \sum_{\substack{n,m=0 \\ n+m=\ell}} F^m \circ_{i=0} F^n$$

$$= \sum_{m=0}^{\ell} a_m a_{\ell-m} \sum_{s=0}^{m} \sum_{k=0}^{\ell-m} C_m^s C_{\ell-m}^k \lambda_1^{\ell-k-s} \lambda_2^{k+s} (x_{\ell-m-k}^{(\ell-m)} \cup x_k^{(\ell-m)})_{m-s}^{(m)} \cup x_s^{(m)} .$$



Taking into account the equality

$$\sum_{m=0}^{\ell} \sum_{s=0}^{m} \sum_{k=0}^{\ell-m} f(m,s,k) = \sum_{\sigma=0}^{\ell} \sum_{k=0}^{\sigma} \sum_{m=0}^{\ell} \theta(m-k)\theta(\ell-m-\sigma+k) \cdot$$

$$\cdot f(m, m-k, \ell-m-\sigma+k)$$

we obtain

$$U_\ell = \sum_{m,\sigma=0}^{\ell} \sum_{k=0}^{\sigma} a_m a_{\ell-m} C_m^k C_{\ell-m}^{\sigma-k} \theta(m-k)\theta(\ell-m-\sigma+k) \cdot$$

$$\cdot \lambda_1^{\sigma} \lambda_2^{\ell-\sigma} (x_{\sigma-k}^{(\ell-m)} \cup x_{\ell-m-\sigma+k}^{(\ell-m)})_k^{(m)} \cup x_{m-k}^{(m)} .$$

Similarly,

$$T_\ell = \sum_{\substack{n,m=0 \\ n+m=\ell}} F^m \circ_{i=1} F^n = \sum_{m,\sigma=0}^{\ell} \sum_{k=0}^{\sigma} a_m a_{\ell-m} C_m^k C_{\ell-m}^{\sigma-k} \theta(m-k) \cdot$$

$$\cdot \theta(\ell-m-\sigma+k) \lambda_1^{\sigma} \lambda_2^{\ell-\sigma} x_k^{(m)} \cup (x_{\sigma-k}^{(\ell-m)} \cup x_{\ell-m-\sigma+k}^{(\ell-m)})_{m-k}^{(m)} .$$

Since the coefficients $\lambda_1$ and $\lambda_2$ are arbitrary, the condition (1.4) for the series f(F) can be written

$$\sum_{k=0}^{\sigma} \sum_{m=0}^{\ell} a_m a_{\ell-m} C_m^k C_{\ell-m}^{\sigma-k} \theta(m-k)\theta(\ell-m-\sigma+k)[$$

$$(x_{\sigma-k}^{(\ell-m)} \cup x_{\ell-m-\sigma+k}^{(\ell-m)})_k^{(m)} \cup x_{m-k}^{(m)} - x_k^{(m)} \cup (x_{\sigma-k}^{(\ell-m)} \cup x_{\ell-m-\sigma+k}^{(\ell-m)})_{m-k}^{(m)}] = 0.$$

Using (2.4), we have

$$\sum_{k=0}^{\sigma} \sum_{m=0}^{\ell} a_m a_{\ell-m} C_m^k C_{\ell-m}^{\sigma-k} \theta(m-k)\theta(\ell-m-\sigma+k) \sum_{i=0}^{m} \sum_{j=0}^{i}$$

$$[C_k^{j} C_{m-k}^{i-j}\theta(m-k-i+j)\theta(k-j)x_{\sigma-k+j}^{(\ell-m+i)} \cup x_{\ell-m-\sigma+2k-j}^{(\ell-i)} \cup x_{m-k}^{(m)}$$

$$-C_{m-k}^{j} C_k^{i-j}\theta(k-i+j)\theta(m-k-j)x_k^{(m)} \cup x_{\sigma-k+j}^{(\ell-m+i)} \cup x_{\ell-\sigma-j}^{(\ell-i)}] = 0.$$

In the second term we make consequent replacements $j \to i-j$, $i \to m-i$, $j \to j-i$ and get

$$\sum_{k=0}^{\sigma}\sum_{m=0}^{\ell} a_m a_{\ell-m} C_m^{k} C_{\ell-m}^{\sigma-k}\theta(\ell-m-\sigma+k)\theta(m-k) \sum_{i,j=0}^{m}$$

$$[C_k^{j} C_{m-k}^{i-j}\theta(i-j)\theta(m-k-i+j)\theta(k-j)\, x_{\sigma-k+j}^{(\ell-m+i)} \cup x_{\ell-m-\sigma+2k-j}^{(\ell-i)} \cup x_{m-k}^{(m)}$$

$$-C_k^{j-i} C_{m-k}^{m-j}\theta(j-i)\theta(j-k)\theta(k-j+i)x_k^{(m)} \cup x_{\sigma-k+m-j}^{(\ell-i)} \cup x_{\ell-\sigma+j-m}^{(\ell-m+i)}] = 0.$$

Since each of the terms $x_j^{(i)} \cup x_\ell^{(k)} \cup x_n^{(m)}$ has different nature, the last equation gives rise to

$$a_m a_{\ell-m} C_m^{k} C_{\ell-m}^{\sigma-k} C_k^{j} C_{m-k}^{i-j}$$

$$= a_{\ell-m+i} a_{m-i} C_{\ell-m+i}^{\sigma-k+j} C_{m-i}^{k-j} C_{\sigma-k+j}^{\sigma-k} C_{\ell-m+i-\sigma+k-j}^{k-\sigma+\ell-m},$$

or

$$a_m a_{\ell-m}\, m!(\ell-m)! = a_{\ell-m+i} a_{m-i} (\ell-m+i)!\,(m-i)!$$

Taking into account conditions (2.7), we obtain the only solution

$$a_m = 1/m!$$

which was to be proved.

Finally, let us consider

Lemma 5

Let $G \in \mathbf{C}^2(\mathcal{A},\mathcal{A})$ and $\chi \in \mathbf{C}^1(\mathcal{A},\mathcal{A})$. Then we have $[G,\chi] = G\delta\chi$. The result is a straightforward consequence of definition 2 and formula (1.7).



## 3. THE MAIN RESULT

We shall now derive an interesting generalization of a result of theorem 4. According to this theorem $\exp(tF)$ is the only associative formal function of $F$ and we have identically

$$\delta F^n = \frac{1}{2}\sum_{i=1}^{n-1} C_n^{i}\, [F^i, F^{n-i}]. \qquad (3.1)$$

These equations have such a structure that for any $n = 1,2,\ldots$ we may add to $F^n$ an arbitrary 2-coboundary operator $\delta\phi^n \in \mathbf{B}^2(\mathcal{A},\mathcal{A})$ multiplied by an arbitrary coefficient $a_n \in \mathcal{Q}$

$$F^n \to F^n + a_n \delta\phi^n. \qquad (3.2)$$

In our case $\delta\phi^n$ has the form

$$-\delta\phi^n = \sum_{i=1}^{2n-1}\sum_{j=0}^{i} C_n^{j} C_n^{i-j}\ \theta(n-i+j)\theta(n-j)\, x_j^{(i)} \cup x_{n-j}^{(2n-i)}. \qquad (3.3)$$

Naturally, one may ask what charges take place in higher powers of $F$. We start with the following lemma.

Lemma 6

*Let* $F \in \mathbf{Z}^2(\mathcal{A},\mathcal{A})$ *satisfies the equations (3.1), then for any* $\chi \in \mathbf{C}^1(\mathcal{A},\mathcal{A})$ *we have*

$$\delta[\underbrace{\ldots}_{\ell}[F^k,\chi],\chi]\ldots],\chi] = \sum_{i=1}^{k-1} C_k^{i}[\ldots[F^i,[F^{k-i},\chi]],\ldots],\chi]$$

$$+[\ldots[F^k,\delta\chi],\chi],\ldots],\chi]+[\ldots[F^k,\chi],\delta\chi],\ldots],\chi]+ \qquad (3.4)$$

$$+[\ldots[F^k,\chi],\chi],\ldots],\delta\chi].$$

$$(\ell = 1,2,3,\ldots).$$

This result is obtained by induction. The properties of the bracket product are also used.

Next statement answers our question.

Proposition 7

*Let* $F \in \mathbf{Z}^2(\mathcal{A},\mathcal{A})$ *satisfies the equations (3.1). If we make a replacement* $F^n \to F^n + a_n\delta\phi^n$, *then for higher powers of* $F$ *we would have*

$$F^{in+k} \to F^{in+k} + a_n C_{in+k}^{n}\, \delta\phi^n [F^{(i-1)n+k} + a_n C_{(i-1)n+k}^{n} \frac{\delta\phi^n}{2}.$$

$$
[F^{(i-2)n+k} + a_n C^n_{(i-2)n+k}\frac{\delta\phi^n}{3}[\ldots[F^{(i-j)n+k} + a_n C^n_{(i-j)n+k}\frac{\delta\phi}{j+1}\;[\ldots
$$

$$
[F^{n+k} + a_n C^n_{n+k}\frac{\delta\phi^n}{i}]\ldots] \tag{3.5}
$$

*here* $k = 0,1,\ldots, n-1;\ i = 1,2\ldots;\ j = 1,2,\ldots, i-1.$

The proof of this suggestion is also accomplished by induction. We omit it for brevity. Note that lemmas 5,6 are needed for the proof. The result (3.5) practically asserts that formal function $\exp(tF + a_n\frac{t^n}{n!}\delta\phi^n)$ generates an associative deformation of algebra $\mathfrak{A}$. It is clear that this function is the only possible one.

We may now formulate the main theorem of this paper.

Theorem 8

*Exponential is the only formal function of* $\mathfrak{F} = \frac{t}{2}P - \sum_{n=1}^{\infty} a_n \delta\phi^n \frac{t^n}{n!}$ *that generates associative deformations for algebra* $\mathfrak{A}$ *on* $\mathfrak{C}$.

Here $P = x_1^{(1)} \cup x_0^{(1)} - x_0^{(1)} \cup x_1^{(1)}$ is Poisson's bracket operator. $\{a_n\}$ is an infinite sequence of constants.

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О *-произведении


Изучаются ассоциативные деформации обычной алгебры $\mathcal{C}^\infty$ функций на фазовой плоскости, когда коцепи задаются посредством дифференциальных операторов. Параметр деформации $t = i\hbar$. Показано, что любая деформация этого вида есть экспонента.



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On the * Product


We study associative deformations of the usual algebra of $\mathcal{C}^\infty$ functions on phase plane, when the cochains are given by differential operators. The deformation parameter $t = i\hbar$. It is shown that any deformation of this kind is a formal exponential function.



The investigation has been performed at the Laboratory of Nuclear Problems, JINR.

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