сообщения
объединенного
института
ядерных
исследований
дубна

E4-83-421

A.I.Georgieva*, P.P.Raychev*, R.P.Roussev

INTERACTING VECTOR BOSON MODEL OF COLLECTIVE NUCLEAR STATES.

TENSORIAL STRUCTURE OF THE HAMILTONIAN

* Bulgarian Academy of Sciences, Institute of Nuclear Research and Nuclear Energy, Sofia

1983

1. Introduction

It has been shown recently [1] that the Hamiltonian of the Interacting Vector Boson Model of collective nuclear states can be expressed in terms of two interacting vector bosons, which form a "pseudospin" doublet. It is shown also there that the noncompact Sp(12,R)-group is the group of dynamical symmetry for the most general one- and two-boson Hamiltonian. As a first approximation of the model it is assumed in (1,3) that the Hamiltonian H should conserve the number of bosons (this approximation seems to be reasonable for the description of the low-lying collective states in nuclei). In this case H can be expressed only in terms of the generators of the maximal compact subgroup of Sp(12,R), namely the group U(6). The latter necessarily includes the group O(3), generated by the angular momentum operators, and it has been shown in [1,2] that the following chains of subgroups are possible in the Interacting Vector Boson Model:

$$\begin{tikzcd}
 & U(6) \arrow[dl] \arrow[d] \arrow[dr] & \\
O(6) \arrow[d] \arrow[dr] & SU(3) \oplus SU(3) \arrow[d] & U(3) \oplus U(2) \arrow[d] \\
SU(3) \oplus O(2) \arrow[d] & O(3) \oplus O(3) \arrow[dd] & U(3) \oplus (U(1) \oplus U(1)) \arrow[ddl] \\
O(3) \oplus O(2) \arrow[dr] & & \\
 & O(3) &
\end{tikzcd} \tag{1.1}$$

In some particular cases [3] (when H is subject to some additional restrictions) the Hamiltonian can be expressed in terms of the independent Casimir operators of one of the chains of subgroups of the reduction scheme (1.1). Then the Hamiltonian can be diagonalized automatically in a U(6)-basis labelled by the quantum numbers of the representations of the corresponding subgroups of the chain. These limiting cases are of a great interest, because they give a set of comparatively simple analytical solutions, which can be easily compared with the corresponding experimental nuclear data. However, it turns out that only a very restricted number of nuclei can be described within these limiting cases. Most nuclei have intermediate spectra, which can be described only by a diagonalization of the full Hamiltonian of the model.

In this paper we are going to discuss an approach for an analytical diagonalization of the full Hamiltonian of the model. This approach is based on the idea, that the Hamiltonian can be constructed as tensor operators, which transform according to the irreducible representations of one of the chains of decomposition (1.1), namely the chain:

$$\begin{array}{c} U(6) \supset U(3) \oplus U(2) \supset U(3) \oplus (U(1) \oplus U(1)) \\ \cup \\ O(3) \supset O(2) \ . \end{array} \tag{1.2}$$

This chain of subgroups is very convenient, because the Interacting Vector Boson Model is a generalization of the model, used for the description of deformed even-even nuclei in the framework of the broken SU(3)-symmetry /4,5/. The latter makes use of the well-known basis of Bargmann and Moshinsky /6/, which is very appropriate for the calculation of the matrix elements of the physical observables, such as angular momentum, quadrupole momentum and so on /7/.

The representation of the Hamiltonian and the physical observables as a combination of irreducible tensor operators transforming

according to (1.2) makes possible the calculation of their matrix elements through a direct application of the generalized Wigner-Eckart theorem /8/. In this way the problem is reduced to the calculation of the corresponding isoscalar factors and reduced matrix elements in the basis of decomposition (1.2).

2. Tensorial structure of the U(6)-generators and their bilinear forms

We recall that the Interacting Vector Boson Model Hamiltonian, which conserves the number of bosons, can be expressed in the following way /1/:

$$\begin{aligned} H = {} & \sum_{\alpha\beta} h_0(\alpha,\beta) A^0(\alpha,\beta) + \\ & + \sum (-1)^M (2J+1) \begin{Bmatrix} 1 & 1 & J \\ 1 & 1 & L \end{Bmatrix} V^J(\alpha\beta;\gamma\delta) A^L_M(\alpha,\gamma) A^L_{-M}(\beta,\delta), \end{aligned} \tag{2.1}$$

where $h_0(\alpha,\beta)$ and $V^J(\alpha,\beta;\gamma\delta)$ are phenomenological constants and

$$A^J_M(\alpha,\beta) = \sum C^{1\,1\,J}_{m\,n\,M} u^+_m(\alpha) u_n(\beta) \qquad J=0,1,2 \tag{2.2}$$

are the generators of the group U(6).

The operators $u^+_m(\alpha)$ and $u_m(\alpha)$ in (2.2) ($m=0,\pm1$) are creation and annihilation operators of vector bosons with a projection of the "pseudospin" $\alpha = \pm\frac{1}{2}$. These operators satisfy the standard boson commutation relations.

The operators $u^+_m(\alpha)$ (or $u_m(\alpha)$) can be considered as components of a vector in the 6-dimensional space, which transform according to the irreducible representation $[1,0,0,0,0,0]_6 \equiv [1]_6$ of the group U(6) / $u_m(\alpha)$ transform according to the corresponding conjugate representation $[1,1,1,1,1,0]_6 \equiv [0,0,0,0,0,-1]_6 \equiv [1]^*_6$. These irreducible representations become reducible if one passes to one of the chains of subgroups in (1.1). This means that along with the quantum numbers characterizing the representations of U(6), the operators $\mathcal{U}^{+}_{m}(\alpha)$ and $\mathcal{U}^{m}(\alpha)$ are characterized by the quantum numbers of the subgroups of the corresponding chain of subgroups. In our case the most natural chain of subgroups is the chain (1.2) because of the **sense of the quantum numbers m and α. The chain (1.2) is very suitable** because it is a particular case (n=3) of the unitary scheme /6,8,9/

$$U\,(3(n-1)) \supset SU(3) \times U(n-1) \qquad\qquad (2.3)$$
$$\bigcup$$
$$O(3) \times O(n-1) \supset S(n)\,,$$

where S(n) is the permutation group. This scheme has been entirely investigated in /9/ and all further caloulations of the present paper are based on the algebraio technique developed in /9/.

The only possible representation of the direct product of U(3) $\otimes$ U(2) belonging to the representation $[1]_6$ of U(6) is $[1]_3 \cdot [1]_2$, i.e., $[1]_6 = [1]_3 \cdot [1]_2$ *). According to the reduction rules for decomposition $U(3) \supset O(3)$ the representation $[1]_3$ of U(3) contains the representation $(1)_3$ of the group O(3) giving the angular momentum

---

*) In this paper we use a simplified notation for the representations of the U(r)-groups

$$[\lambda_1, \lambda_2, \ldots\ldots, \lambda_r] \equiv [\lambda_1, \lambda_2, \ldots\ldots \lambda_{r-n}]_r$$

if $\lambda_{r-n+1}, \ldots\ldots, \lambda_r = 0$,

where

$$\lambda_1 \geq \lambda_2 \geq \ldots\ldots \geq \lambda_{r-n}.$$

of the bosons l=1 (with a projection m=0,±1). The representation $[1]_2$ of U(2) defines the "pseudospin" of the bosons $T = \frac{1}{2}$, whose projection is given by the corresponding representation of $U(1) \oplus U(1)$, i.e., $\alpha = \pm\frac{1}{2}$.

Further, when convenient, we shall use also the **notation:**

$$p^{+}_{m} = \mathcal{U}^{[1]_6}_{[1]_3 [1]_2\, m\ \alpha=\frac{1}{2}}\,, \qquad\qquad (2.4a)$$

$$n^{+}_{m} = \mathcal{U}^{[1]_6}_{[1]_3 [1]_2\, m\ \alpha=-\frac{1}{2}}$$

and the corresponding conjugate operators

$$p_{m} = \mathcal{U}^{[1]^{*}_6\qquad \alpha=\frac{1}{2}}_{[1]^{*}_3 [1]^{*}_2\, m} = \mathcal{U}^{[1]^{*}_6}_{[1]^{*}_3 [1]^{*}_2\, m\ \alpha=\frac{1}{2}}\,, \qquad\qquad (2.4b)$$

$$n_{m} = \mathcal{U}^{[1]^{*}_6\qquad \alpha=-\frac{1}{2}}_{[1]^{*}_3 [1]^{*}_2\, m} = -\,\mathcal{U}^{[1]^{*}_6}_{[1]^{*}_3 [1]^{*}_2\, m\ \alpha=\frac{1}{2}}\,.$$

Formulae (2.4a,b) establish the following phase convention

$$\mathcal{U}^{[1]^{*}_6\qquad m\ \alpha}_{[1]^{*}_3 [1]^{*}_2} = (-1)^{m+\frac{1}{2}-\alpha}\,\mathcal{U}^{[1]^{*}_6}_{[1]^{*}_3 [1]^{*}_2\, -m\ -\alpha}\,. \qquad\qquad (2.5)$$

In this way the generators of the group U(6) (2.2) can be rewritten in the following way:

$$A^{L}_{M}(\alpha,\beta) = \sum_{\substack{1\ L}} C^{1\ 1\ L}_{m\ n\ M}\, \mathcal{U}^{[1]_6}_{[1]_3 [1]_2\, m\, \alpha}\, \mathcal{U}^{[1]^{*}_6\quad \beta}_{[1]^{*}_3 [1]^{*}_2\, n}\,, \qquad\qquad (2.6)$$

where $C^{1}_{m\ n\ M}$ are the Clebsch-Gordan coefficients for the decomposition $O(3) \supset O(2)$. It is evident from (2.6) that the operators $A^{L}_{M}(\alpha\beta)$ have clear tensorial properties only according to the de**composition $O(3) \supset O(2)$. The irreducible tensor operators according to the chain (1.2) can be expressed in terms of the operators (2.6) by**

$$T([1]_6[1]^*_6)[\chi]_6[\lambda]_3[2T]_2 L M T T_0) =$$

$$= \sum C^{[1]_6 \quad [1]^*_6 \quad [\chi]_6}_{[1]_3[1]_2\, [1]^*_3[1]^*_2\, [\lambda]_3[2T]_2} C^{[1]_3 \quad [1]_3 \quad [\lambda]_3}_{(1)_3 \quad (1)_3 \quad (L)_3} C^{\frac{1}{2} \quad \frac{1}{2} \quad T}_{\alpha \quad \beta \quad T_0} \; * \tag{2.7}$$

$$* \, \mathcal{U}^{[1]_6}_{[1]_3[1]_2\, m\alpha} \, \mathcal{U}^{[1]^*_6}_{[1]^*_3[1]^*_2\, n\beta} =$$

$$= \sum (-1)^{\frac{1}{2}+\beta} C^{[1]_6 \quad [1]^*_6 \quad [\chi]_6}_{[1]_3[1]_2\, [1]^*_3[1]^*_2\, [\lambda]_3[2T]_2} C^{[1]_3 \quad [1]^*_3 \quad [\lambda]_3}_{(1)_3 \quad (1)_3 \quad (L)} C^{\frac{1}{2} \quad \frac{1}{2} \quad T}_{\alpha \; -\beta \; T_0} A^L_M(\alpha\beta),$$

where the generalized Clebsch-Gordan coefficients are given as a product of isoscalar factors for the following chains of subgroups:

1) for the chain $U(6) \supset U(3) \oplus U(2)$

$$C^{[\chi']_6 \qquad [\chi'']_6 \qquad \beta \; [\chi]_6}_{[\lambda']_3[2T']_2 \quad [\lambda'']_3[2T'']_2 \quad [\lambda]_3[2T]_2} , \tag{2.8a}$$

2) for the chain $U(3) \supset O(3)$

$$C^{[\lambda']_3 \quad [\lambda'']_3 \quad [\lambda]_3}_{(L')_3 \quad (L'')_3 \quad (L)_3} , \tag{2.8b}$$

3) for the chain $U(2) \supset (U(1) \oplus U(1))$

$$C^{[T'] \quad [T''] \quad [T]}_{\alpha \quad -\beta \quad T_0} . \tag{2.8c}$$

The Hamiltonian (2.1) is given in terms of the U(6)-generators, which can be expressed as a linear combination of the irreducible tensor operators (2.7) by

$$A^L_M(\alpha,\beta) = (-1)^{\frac{1}{2}+\beta} \sum C^{[1]_6 \quad [1]^*_6 \quad [\chi]_6}_{[1]_3[1]_2\, [1]^*_3[1]^*_2\, [\lambda]_3[2T]_2} \; * \tag{2.9}$$

$$* \, C^{[1]_3 \quad [1]^*_3 \quad [\lambda]_3}_{(1)_3 \quad (1)_3 \quad (L)_3} C^{\frac{1}{2} \quad \frac{1}{2} \quad T}_{\alpha \; -\beta \; T_0} T(([1]_6[1]^*_6)[\chi]_6[\lambda]_3[2T]_2 L M T T_0).$$

Hence these generators transform according to the direct product of the U(6)-representations $[1]_6$ and $[1]^*_6$, namely

$$[1]_6 \times [1]^*_6 = [1,-1]_6 + [0]_6 , \tag{2.10}$$

where $[1,-1]_6 = [2,1,1,1,1,0]_6$. Along the chain $U(6) \supset U(3) \oplus U(2)$ the representations (2.10) contain the following representations of $U(3) \oplus U(2)$:

$$\begin{aligned} [0]_6 &= [0]_3 . [0]_2 \\ [1,-1]_6 &= [2,1]_3 . [2]_2 + [2,1]_3 . [0]_2 + [0]_3 . [2]_2 \end{aligned} \tag{2.11}$$

and along the chain $U(3) \supset O(3)$ the U(3)-representations in (2.11) contain the corresponding representations of O(3) /3/:

$$\begin{aligned} [2,1]_3 &= (2)_3 + (1)_3 \\ [0]_3 &= (0)_3 . \end{aligned} \tag{2.12}$$

The chain $U(2) \supset (U(1) \oplus U(1))$ of (1.2) gives the values of the "pseudospin" and its third projection

$$T = \frac{1}{2}(\lambda_1 - \lambda_2) \qquad T_0 = -\lambda_3 + \frac{1}{2}(\lambda_1 + \lambda_2), \tag{2.13a}$$

where the integers $\lambda_1 \geq \lambda_2$ determine the representations of U(2) $[\lambda_1, \lambda_2]_2$ and λ_3 runs the values

$$\lambda_1 \geq \lambda_3 \geq \lambda_2 . \tag{2.13b}$$

Hence, the U(2)-representation $[2]_2$ contains tensors with $T=1$, $T_0 = 0, \pm 1$, and the representation $[0]_2$ contains tensors with $T = 0$ and $T_0 = 0$.

The two-boson interaction in the Hamiltonian (2.1) is expressed as a linear combination of all possible scalar products (according to the group O(3)) of the generators of U(6). **Along the chain** (1.2) these scalar products transform as the tensors

$$T(([\chi']_6[\chi'']_6)\omega[\chi]_6[\lambda]_3[2T]_2\omega' L=0\,T\,T_0)=$$

$$=\sum T(([1]_6[1]^*_6)[\chi']_6[\lambda']_3[2T']_2 L M T' T_0')\,T(([1]_6[1]^*_6)[\chi'']_6[\lambda'']_3[2T'']_2 L{-M}\, T'' T_0'')\,* \tag{2.14}$$

$$C^{[\chi']_6\quad [\chi'']_6\quad \omega[\chi]_6}_{[\lambda']_3[2T']_2\ [\lambda'']_3[2T'']_2\ [\lambda]_3[2T]_2}\,C^{[\lambda']_3\ [\lambda'']_3\ [\lambda]_3}_{(L)_3\ (L)_3\ \omega' 0}\,C^{L\ L\ 0}_{M\ -M\ 0}\,C^{T'\ T''\ T}_{T_0'\ T_0''\ T_0},$$

where the isoscalar factors are of the type (2.8a-c), and the symbols ω and ω' indicate the extra sets of quantum numbers **needed to characterize the tensors, as the chain (1.2) is not a canonical one.** Formula (2.14) is in fact a definition of the U(6)-irreducible tensor operators transforming according to **the chain** (1.2), **which are** O(3)-scalars. Through an inverse transformation one can express the two-boson interaction in the Hamiltonian in terms of the irreducible tensor operators (2.14)

$$C^{L\ L\ 0}_{M\ -M\ 0}\,A^L_M(\alpha,\beta)\,A^L_{-M}(\gamma,\delta)=(-1)^{1+\beta+\gamma}\ \times$$

$$\sum C^{[1]_6\quad [1]^*_6\quad [\chi']_6}_{[1]_3[1]_2\ [1]^*_3[1]^*_2\ [\lambda']_3[2T']_2}\,C^{[1]_6\quad [1]^*_6\quad [\chi'']_6}_{[1]_3[1]_2\ [1]^*_3[1]^*_2\ [\lambda'']_3[2T'']_2}\,C^{[\chi']_6\quad [\chi'']_6\quad \omega\,[\chi]_6}_{[\lambda']_3[2T']_2\ [\lambda'']_3[2T'']_2\ [\lambda]_3[2T]_2}\ \times$$

$$\times\, C^{[1]_3\ [1]^*_3\ [\lambda']_3}_{(1)_3\ (1)_3\ (L)_3}\,C^{[1]_3\ [1]^*_3\ [\lambda'']_3}_{(1)_3\ (1)_3\ (L)}\,C^{[\lambda']_3\ [\lambda'']_3\ [\lambda]_3}_{(L)\ (L)\,\omega' 0}\,C^{\frac12\ \frac12\ T'}_{\alpha\ -\beta\ T_0'}\,C^{\frac12\ \frac12\ T''}_{\gamma\ -\delta\ T_0''}\ \times \tag{2.15}$$

$$C^{T'\ T''\ T}_{T_0'\ T_0''\ T_0}\,T(([\chi']_6[\chi'']_6)\omega[\chi]_6[\lambda]_3[2T]_2\omega' L=0\,T\,T_0).$$

It is obvious, that the tensors (2.14) (or (2.15)) transform according to the direct product of the U(6)-representations

$$([0]_6+[1,-1]_6)\times([0]_6+[1,-1]_6)=2[0]_6+4[1,-1]_6+[2,-2]_6$$
$$+[1,1,0,0,-1,-1]_6+[1,1,0,0,0,-2]_6+[2,0,0,0,-1,-1]_6, \tag{2.16}$$

where

$$\begin{aligned}
&[2,-2]_6\equiv[2,0,0,0,0,-2]_6=[4,2,2,2,2,0]_6\\
&[1,1,0,0,-1,-1]_6=[2,2,1,1,0,0]_6\\
&[1,1,0,0,0,-2]_6=[3,3,2,2,2,0]_6\\
&[2,0,0,0,-1,-1]_6=[3,1,1,1,0,0]_6.
\end{aligned} \tag{2.17}$$

The boson realization of the basis along chain (1.2) is determined by the most symmetric representations of U(6), namely $[N]_6$ (N-integer). It has been shown in /9/, that in this case, only tensors, which transform according to U(6)-representations of the type $[a,0,0,0,0,-b]_6$, generate symmetric U(6)-representations $[N']$. On the other hand the Hamiltonian (2.1) conserves the number of bosons which gives $[N]_6=[N']_6$. For this reason in this paper we shall discuss the irreducible structure only of the operators, which transform according to the first three representations in the **right-hand side** of (2.16). Having in mind the reduction rules (analogous to (2.11) and (2.12)) along chain (1.2) one can enumerate all the representations of (1.2), which contain the scalar representations of O(3) and appear in the bilinear forms of the generators of U(6). These representations and the corresponding irreducible tensor operators are listed in Table 1 where the scheme of coupling of the U(6)-representations is given by

$$(([1]_6[1]^*_6)[\chi']_6([1]_6[1]^*_6)[\chi'']_6)[\chi]_6[\lambda]_3[2T]_2 L=0\,T_0). \tag{2.18}$$

Table 1

No	U (6)			U(3)	U(2)	U(1)+U(1)	Tensors
	$[\chi']_6$	$[\chi'']_6$	$[\chi]_6$	$[\lambda]_3$	$[2T_2]$	T_0	
1	0	0	0	0	0	0	U_1^0
2	1,-1	1,-1	0	0	0	0	U_2^0
3	1,-1	1,-1	2,-2	0	0	0	U_3^0
4	1,-1	1,-1	2,-2	0	4	-2,0,2	$U_4^{-2}, U_4^0, U_4^{+2}$
5	1,-1	1,-1	2,-2	4,2	0	0	U_5^0
6	1,-1	1,-1	2,-2	4,2	2	-1,1	U_6^{-1}, U_6^{+1}
7	1,-1	1,-1	2,-2	4,2	4	-2,0,2	$U_7^{-2}, U_7^0, U_7^{+2}$
8	0	1,-1	1,-1	0	2	-1,1	U_8^{-1}, U_8^{+1}
9	1,-1	0	1,-1	0	2	-1,1	U_9^{-1}, U_9^{+1}
10	1,-1	1,-1	s 1,-1	0	2	-1,1	U_s^{-1}, U_s^{+1}
11	1,-1	1,-1	a 1,-1	0	2	-1,1	U_a^{-1}, U_a^{+1}

Table 2

$C\begin{matrix}[1,-1]_6 & [1,-1]_6 & [\chi]_6 \\ [\lambda']_3[2T']_2 & [\lambda'']_3[2T'']_2 & [42]_3[0]_2\end{matrix}$	$[\chi]_6$	
	$[2,-2]_6$	$[2,2,1,1]_6$
$[2,1]_3 . [2]_2 \times [2,1]_3 . [2]_2$	-1/2	-3/2
$[2,1]_3 [0]_2 \times [2,1]_3 [0]_2$	+3/2	-1/2

Table 3

$C\begin{matrix}[1,-1]_6 & [1,-1]_6 & [\chi]_6 \\ [\lambda']_3[2T']_2 & [\lambda'']_3[2T]_2 & [0]_3[4]_2\end{matrix}$	$[\chi]_6$	
	$[2,-2]_6$	$[2,2,1,1]_6$
$[2,1]_3 [2]_2 \times [2,1]_3 [2]_2$	+1/3	-2/2
$[0]_3 [2]_2 \times [0]_3 [2]_2$	+2/3	+1/3

3. Transformation properties of the independent Casimir operators

It has been shown in /2/, that the two-boson interaction in (2.1) can be expressed in terms of the second-order Casimir operators of the algebras belonging to the reduction scheme (1.1)

$$
\begin{aligned}
H_{int} = {} & \alpha_6 K_6 + \alpha_3 K_3 + \alpha_2 K_2 + \alpha_1 K_1 + \bar{\alpha}_3' \bar{G}_3' + \\
& + \beta_6 \Pi_6 + \bar{\beta}_3 \bar{\Pi}_3 + \bar{\beta}_3' \bar{\Pi}_3' + \beta_3 \Pi_3 + \beta_2 \Pi_2 + \\
& + \Gamma (A^0(p,n) + A^0(n,p)),
\end{aligned}
\qquad (3.1)
$$

where K_6, K_3, K_2, K_1 are the Casimir operators of the U-algebras in (1.1), G_3' is one of the Casimir operators of the SU(3) ⊕ SU(3) - algebra, Π_6, $\bar{\Pi}_3$, $\bar{\Pi}'$, Π_3, and Π_2 are the Casimir operators of the corresponding O-algebras (we recall that the O(3) ⊕ O(3) - algebra has two second-order Casimir operators noted in (3.1) by $\bar{\Pi}_3$ and $\bar{\Pi}_3$). The relation between the coefficients of (3.1) and the independent constants of (2.1) is also given in /2/.

All Casimir operators in (3.1) are bilinear forms of the U(6)-generators (2.2) (or (2.6)), i.e., they can be expressed as linear combinations of the irreducible tensor operators listed in Table 1. For this reason, however, one has to know the explicit values of the isoscalar factors, which are of the type (2.8a-c). First of all it should be noted, that the isoscalar factors can be expressed in the form of orthogonal matrices. Thus for example, the isoscalar factors $C\begin{matrix}[\chi']_6 & [\chi'']_6 & \omega & [\chi]_6 \\ [\lambda']_3[2T']_2 & [\lambda'']_3[2T''] & & [\chi]_6 [2T]_6\end{matrix}$ which correspond to the decomposition of the direct product $[\chi']_6 \times [\chi'']_6 = [\chi]_6$ according to the chain U(6) ⊃ U(3) ⊕ U(2) can be treated as matrix elements of an orthogonal matrix M_{ji}, where $i \equiv [\chi]_6$ and $j \equiv [\lambda']_3 [2T']_2 \times [\lambda'']_3 [2T]_2$.

We shall first enumerate the trivial cases of the isoscalar factors corresponding to one-dimensional matrices:

a) of the type (2.8a)

$$C\begin{matrix} [0]_6 & [0]_6 & [0]_6 \\ [0]_3[0]_2 & [0]_3[0]_2 & [0]_3[0]_2 \end{matrix}$$

$$C\begin{matrix} [1]_6 & [1]^*_6 & [\chi]_6 \\ [1]_3[1]_2 & [1]^*_3[1]^*_2 & [\lambda]_3[2T]_2 \end{matrix}$$

$$C\begin{matrix} [1,-1]_6 & [1,-1]_6 & [2,-2]_6 \\ [2,1]_3[2]_2 & [2,1]_3[2]_2 & [4,2]_3[4]_2 \end{matrix}$$

$$C\begin{matrix} [0]_6 & [1,-1]_6 & [1,-1]_6 \\ [0]_3[0]_2 & [0]_3[2]_2 & [0]_3[2]_2 \end{matrix}$$

$$C\begin{matrix} [1,-1]_6 & [0]_6 & [1,-1]_6 \\ [0]_3[2]_2 & [0]_3[0] & [0]_3[2]_2 \end{matrix}.$$

b) of the type (2.8b)

$$C\begin{matrix} [0]_3 & [0]_3 & [0]_3 \\ 0 & 0 & 0 \end{matrix}$$

$$C\begin{matrix} [1]_3 & [1]^*_3 & [\lambda]_3 \\ (1)_3 & (1)_3 & L \end{matrix}$$

whose absolute values are equal to 1.

The $U(3) \oplus U(2)$-representations $[0]_3[0]_2$ and $[4,2]_3.[2]_2$ belong to three different U(6)-representations determined by the direct product $[1,-1]_6 \times [1,-1]_6$. The corresponding isoscalar factors form three-dimensional matrices, whose values are given in Table 4 and Table 5. The last two columns of Table 5 are not filled up because the U(6)- representations $[3,3,2,2,2]_6$ and $[3,1,1,1]_6$ do not contain

Table 4

$C\begin{matrix} [1,-1]_6 & [1,-1]_6 & [\chi]_6 \\ [\lambda']_3 [2T']_2 & [\lambda'']_3 [2T'']_2 & [0]_3[0]_2 \end{matrix}$	$[\chi]_6$		
	$[0]_6$	$[2,-2]_6$	$[2,2,1,1]_6$
$[2,1]_3.[2]_2 \times [2,1]_3[2]_2$	$-2\sqrt{\frac{2.3}{7.5}}$	$-\frac{5}{2}\frac{1}{\sqrt{3.7}}$	$-\frac{1}{2.\sqrt{3.5}}$
$[2,1]_3.[0]_2 \times [2,1]_3.[0]_2$	$+2\sqrt{\frac{2}{7.5}}$	$-\frac{3}{2.\sqrt{7}}$	$-\frac{3}{2.\sqrt{5}}$
$[0]_3[2]_2 \times [0]_3[2]_2$	$-\sqrt{\frac{3}{7.5}}$	$+2\sqrt{\frac{2}{7.3}}$	$-2\sqrt{\frac{2}{3.5}}$

Table 5

$C\begin{matrix} [1,-1]_6 & [1,-1]_6 & [\chi]_6 \\ [\lambda']_3[2T']_2 & [\lambda'']_3[2T'']_2 & [4,2]_3[2]_2 \end{matrix}$		$[\chi]_6$	
	$[2,-2]_6$	$[3,3,2,2,2]_6$	$[3,1,1,1]_6$
$[2,1]_3.[0]_2 \times [2,1]_3.[2]_2$	0		
$[2,1]_3.[2]_2 \times [2,1]_3.[2]_2$	$\frac{1}{\sqrt{2}}$		
$[2,1]_3[0]_2 \times [2,1]_3[0]_2$	$\frac{1}{\sqrt{2}}$		

scalar O(3)-representations. The method of calculation of the isoscalar factors belonging to the first column of Table 5 will be published in a following paper. This method has also been used for the calculation of the values of the isoscalar factors given in Tables 2, 3 and 4; they coincide (up to a phase factor) with the corresponding values calculated in /9/.

The U(3)⊕U(2)-representation $[0]_3\,[0]_2$ belongs to the U(6)-representations $[1,-1]_6$, $[3,3,2,2,2]_6$ and $[3,1,1,1]_6$ but the corresponding orthogonal matrix is four-dimensional, because the representation $[1,-1]_6$ appears twice in the direct product $[1,-1]_6 \times [1,-1]_6$. The values of the isoscalar factors are given in Table 6, where, as in the case of Table 5, the last two columns are not filled up,

Table 6

$C\begin{pmatrix} [1,-1]_6 & [1,-1]_6 & [\chi]_6 \\ [\lambda']_3[2T']_2 & [\lambda'']_3[2T'']_2 & [0]_3\,[2]_2 \end{pmatrix}$	$[\chi]_6$	
	a $[1,-1]_6$	s $[1,-1]_6$
$[2,1]_3.[2]_2 \times [2,1]_3.[2]_2$	$\frac{\sqrt{8}}{3}$	0
$[2,1]_3.[0]_2 \times [2,1]_3.[2]_2$	0	$\frac{1}{\sqrt{2}}$
$[2,1]_3\,[2]_2 \times [2,1]_3.[0]_2$	0	$\frac{1}{\sqrt{2}}$
$[0]_3\,[2]_2 \times [0]_3\,[2]_2$	$\frac{1}{3}$	0

Table 7

$C\begin{pmatrix} [2,1]_3 & [2,1]_3 & [\lambda]_3 \\ (\ell)_3 & (\ell)_3 & (0)_3 \end{pmatrix}$	$[\lambda]_3 = [2,2,2]_3 = [0]_3$	$[\lambda]_3 = [4,2,0]_3$
$(1)_3 \times (1)_3$	$-\sqrt{\frac{3}{8}}$	$+\sqrt{\frac{5}{8}}$
$(2)_3 \times (2)_3$	$+\sqrt{\frac{5}{8}}$	$+\sqrt{\frac{3}{8}}$

because the representations $[3,3,2,2,2]_6$ and $[3,1,1,1]_6$ do not contain scalar O(3)-representations. The indices a and s take into account that the representations $[1,-1]_6$ appear twice in the direct product $[1,-1]_6 \times [1,-1]_6$.

The isoscalar factors $C\begin{pmatrix} [2,1]_3 & [2,1]_3 & [\lambda]_3 \\ (\ell)_3 & (\ell)_3 & 0 \end{pmatrix}$, which correspond to the coupling $[\lambda]_3 = [2,1]_3 \times [2,1]_3$ containing scalar O(3)-representations, form two-dimensional matrices whose matrix elements are listed in Table 7.

The isoscalar factors of the type (2.8c) according to the chain U(2)⊃U(1) ⊕ U(1) coincide with the usual Clebsch-Gordon coefficients given in [10].

Using the values of the isoscalar factors given in Tables 2-7 one can easily express the independent quadratic Casimir operators of formulae (3.1) in terms of the irreducible tensor operators of Table 1. First of all it should be pointed out that the Casimir operators K_6, K_3, K_2, K_1 and $\mathfrak{N}_3$ are diagonal in the basis of chain (1.2), and using the results obtained in [2] they can be written as

$$K_6 = \mathcal{N}^2 + 5\mathcal{N} = 6\,U_1^0 + 5\mathcal{N}$$

$$K_3 = \frac{1}{2}\mathcal{N}^2 + \mathcal{N} + 2T^2 = 2\,U_1^0 + \frac{16}{\sqrt{35}}\,U_2^0 - 6\sqrt{\frac{2}{7}}\,U_3^0$$

$$K_2 = \frac{1}{3}\mathcal{N}^2 + \frac{4}{3}T^2 = \frac{8}{3}U_1^0 + \frac{4}{3\sqrt{35}}\,U_2^0 - 4\sqrt{\frac{2}{7}}\,U_3^0 + \frac{2}{3}\mathcal{N} \qquad (3.2)$$

$$K_1 = \frac{1}{3}\mathcal{N}^2 + \frac{4}{3}T_0^2 = 2\,U_1^0 + \frac{2}{\sqrt{35}}\,U_2^0 - \frac{4}{3}\sqrt{\frac{2}{7}}\,U_3^0 + \frac{4}{3}\,U_4^0$$

$$\mathfrak{N}_3 = \frac{1}{2}L^2 = \frac{6}{\sqrt{35}}\,U_2^0 - \frac{9}{4}\sqrt{\frac{2}{7}}\,U_3^0 - \frac{3}{2}\sqrt{\frac{5}{2}}\,U_5^0.$$

It is evident that [2,3]

$$K_3 = \frac{3}{2}K_2 + \mathcal{N} \qquad (3.3)$$

which is due to the fact that the groups U(3) and U(2) are mutually complementary /2,3/. This leads to the relation

$$U_2^0 = \sqrt{\frac{5}{7}}\left(U_1^0 + N\right). \tag{3.4}$$

Using (3.2) and (3.3) one can express the irreducible tensor operators U_1^0, U_3^0, U_4^0 and U_5^0 either in terms of the Casimir operators $K_6, K_3, K_1, \mathcal{K}_3$ or in terms of the "pseudospin" operator T, its third projection T , the number of boson operator N and the angular momentum operator L.

The second-order Casimir operators $\mathcal{K}_6$, $\bar{\mathcal{K}}_3$, $\bar{\mathcal{K}}_3'$, $\mathcal{K}_2$, and G_3' of expression (3.1) are off diagonal in the basis of chain (1.2). However, they can be expressed in terms of the remaining irreducible tensor operators listed in Table 1. Having in mind that H_{int} /2/ is invariant if the p- and n-bosons are mutually substitutable (a "pseudospin" symmetry), for convenience we introduce the following operators:

$$\begin{aligned}
&J_4 = U_4^2 + U_4^{-2}\ ;\ J_6 = U_6^2 + U_6^{-2}\ ;\ J_7 = U_7^2 + U_7^{-2} \\
&J_8 = U_{10}^1 - U_{10}^{-1}\ ;\ J_9 = U_{11}^1 - U_{11}^{-1} \\
&J_s = U_s^1 - U_s^{-1}\ ;\ J_a = U_a^1 - U_a^{-1}.
\end{aligned} \tag{3.5}$$

With the help of (3.5) the off diagonal Casimir operators can be written as

$$\mathcal{K}_6 = \frac{7}{12}N^2 + \frac{7}{2}N + \frac{1}{3}L^2 - \frac{5}{6}T^2 - \frac{3}{2}T_0 - \sqrt{\frac{3}{2}}J_4 - \frac{\sqrt{5}}{2}U_7^0 - \sqrt{\frac{15}{2}}J_7$$

$$\begin{aligned}
&\bar{\mathcal{K}}_3 = \frac{1}{6}N^2 + N + \frac{2}{3}L^2 - \frac{1}{6}T^2 - \frac{1}{2}T_0^2 + \sqrt{\frac{3}{8}}J_4 + \sqrt{\frac{5}{2}}U_7^0 - \sqrt{\frac{15}{8}}J_7 \\
&\bar{\mathcal{K}}_3' = -2J_a - \frac{3}{\sqrt{2}}J_s + \sqrt{\frac{15}{2}}J_6 - 2\sqrt{3}\,\widetilde{U} \\
&\mathcal{K}_2 = \frac{2}{3}T^2 - \frac{2}{3}T_0^2 - \sqrt{\frac{2}{3}}J_4 \\
&G_3' = -\frac{16}{3}J_a - 4\sqrt{2}J_s - \frac{16}{\sqrt{3}}\widetilde{U},
\end{aligned} \tag{3.6}$$

where $\widetilde{U} = A^0(p,n) + A^0(n,p)$. Having in mind certain relations /2/ between the second-order Casimir operators and the bilinear forms of the U(6)-generators one can obtain the relation

$$J_8 + J_9 = \sqrt{2}J_a + 2J_s. \tag{3.7}$$

Then the Hamiltonian (3.1) can be expressed as

$$\begin{aligned}
H_{int} = &\left(\alpha_6 + \frac{1}{2}\alpha_3 + \frac{1}{3}\alpha_2 + \frac{1}{3}\alpha_1 + \frac{7}{12}\beta_6 + \frac{1}{6}\bar{\beta}_3\right)N^2 + \\
&+\left(5\alpha_6 + \alpha_3 + \frac{7}{2}\beta_6 + \bar{\beta}_3\right)N + \\
&+\left(2\alpha_3 + \frac{4}{3}\alpha_2 + \frac{5}{6}\beta_6 - \frac{1}{6}\bar{\beta}_3 + \frac{2}{3}\beta_2\right)T^2 + \\
&+\left(\frac{4}{3}\alpha_1 - \frac{3}{2}\beta_6 - \frac{1}{2}\bar{\beta}_3 - \frac{2}{3}\beta_2\right)T_0^2 + \left(\frac{1}{3}\beta_6 + \frac{2}{3}\bar{\beta}_3 + \frac{1}{2}\beta_3\right)L^2 + \\
&+\left(-\sqrt{\frac{3}{2}}\beta_6 + \frac{1}{2}\sqrt{\frac{3}{2}}\bar{\beta}_3 - \sqrt{\frac{2}{3}}\beta_2\right)J_4 + \sqrt{\frac{15}{2}}\bar{\beta}_3'J_6 + \\
&+\left(-\frac{\sqrt{5}}{2}\beta_6 + \frac{\sqrt{5}}{2}\right)U_7^0 + \left(-\sqrt{\frac{15}{2}}\beta_6 - \frac{1}{2}\sqrt{\frac{15}{2}}\bar{\beta}_3\right)J_7 + \\
&+\left(-8\bar{\alpha}_3' - 3\bar{\beta}_3'\right)\left(\frac{2}{3}J_a + \frac{1}{\sqrt{2}}J_s\right).
\end{aligned} \tag{3.8}$$

This representation of the Hamiltonian of the Interacting Vector-Boson Model makes it possible to use the generalized Wigner-Eckart theorem for its diagonalization, i.e.,

$$\left\langle \begin{matrix}[N]_6\,[n]_3\,[2t]_2 \\ \alpha L \quad t_0\end{matrix} \middle| T([\chi]_6[\lambda]_3[2T]_2 L=0\, T_0) \middle| \begin{matrix}[N]_6\,[n']_3\,[2t']_2 \\ \alpha' L \quad t_0'\end{matrix} \right\rangle =$$

$$= \langle N \| T^{\chi} \| N \rangle \sum_{\beta_3} C^{[\chi]_6 \qquad [N]_6 \qquad [N]_6}_{[\lambda]_3[2T]_2\; [n']_3[2t']_2\,\beta_3\; [n]_3[2t]_2} \times \qquad (3.9)$$

$$\times\; C^{[\lambda]_3\; [n']_3\; \beta_3 [n]_3}_{0 \quad \alpha' L' \quad \alpha L}\; C^{T\; t'\; t}_{T_0\; t_0'\; t_0}.$$

Here $[n]_3 = [n_1, n_2, 0]_3$ and $[2t] = [n_1 - n_2, 0]_2$ are the representations of the groups U(3) and U(2), respectively. The numbers $[n]_3$ and $[2t]$ labelling the representations of the direct product U(3)⊕U(2) belonging to a given U(6)-representation $[N]_6$ can be obtained by standard group-theoretical methods /3,9/. The index α (α') distinguishes states with equal L, which appear more than once in the decomposition U(3)⊃O(3), while the summation index β_3 in (3.9) indicates that some U(3)-representations appear more than once in the direct product $[\chi]_3 \times [n]_3$.

Unfortunately, there are no explicit analytical expressions for the isoscalar factors in (3.9), which correspond to the decompositions U(6) ⊃ U(3)⊕U(2) and U(3)⊃O(3). However, one can calculate them in some particular cases, which are of importance for the problems discussed in this paper. The reduced matrix elements in (3.9) have also to be calculated.

4. Conclusions

The results of the present paper show an explicit way for a direct diagonalization of the Hamiltonian using a basis along one of the main chains of the reduction scheme (1.1) (it is the chain (1.2) in our case). Furthermore, it is evident from (3.6) and (3.7) that the irreducible tensor operators U_1^0, U_2^0, U_3^0, U_4^0, and U_5^0 of Table 1 are diagonal in the basis (1.2). The operators U_7^0, J_7, and J_4 give the transition from chain (1.2) to the chain U(6)⊃O(6)⊃SU(3)+O(2)⊃ ⊃O(3)+O(2)⊃ O(3), while the inclusion of different types of linear combinations of the operators J_5, J_a, and J_6 gives the transitions to the remaining two chains of (1.1).

At last, it should be noted, that all results of this paper can be applied directly to the three-body problem, where (after the introduction of the Jacoby coordinates) the group U(6) appears in a quite natural way.

References

1. Georgieva A.I., Raichev P.P., Rusev R.P. - Communication of the JINR, P4-81-134, Dubna 1980.
2. Georgieva A.I., Raichev P.P., Roussev R. - J.Phys. G: Nucl.Phys., 1982, 8, p. 1377.
3. Georgieva A., Raychev P., Roussev R. - J.Phys.G: Nucl.Phys., 1983, 5, p. 521
4. Roussev R. - Description of the deformed even-even nuclei in the framework of the broken SU(3) - symmetry - D.Phil.Thesis, INRNE, 1980, Sofia.
5. Raychev P. and Roussev R. - Sov. J.Nucl.Phys., 1978, 27, p. 1501.

6. Alisauskas S., Raychev P., Roussev R. - J. Phys.G: Nucl.Phys., 1981, 7, p. 1213.
7. Raychev P., Roussev R. - J.Phys.G: Nucl.Phys., 1981, 7, p. 1226.
8. Kretzschmar M. - Z. f. Phys., 1960, 157, p. 433; 1960, 158, p. 284.
9. Vanagas V. - Algebraic Methods in Nuclear Theory, 1971, Mintis, Vilnius.
10. Varshalovich D.A., Moskalev A.N and Hersonsky V.K. - Quantum Theory of Angular Momentum, 1975, Nauka, Leningrad.

Received by Publishing Department
on June 17, 1983.

Георгиева А.И., Райчев П.П., Русев П.Р. E4-83-421

Модель взаимодействующих векторных бозонов
для коллективных ядерных состояний.
Тензорная структура гамильтониана

При помощи двух взаимодействующих векторных бозонов сконструировано самое общее одно- и двухчастичное взаимодействие. Группа динамической симметрии сохраняющего число бозонов гамильтониана есть группа U(6). Все возможные типы взаимодействия выражаются посредством неприводимых тензорных операторов, соответствующих приведению на цепочке

$$U(6) \supset U(3) \otimes U(2) \supset U(3) \otimes (U(1) \otimes U(1)) \supset O(3).$$

Приведены все необходимые изоскалярные факторы. Такое представление гамильтониана делает возможной его диагонализацию при помощи теоремы Вигнера-Эккарта.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1983

Georgieva A.I., Raychev P.P., Roussev R.P. E4-83-421

Interacting Vector Boson Model
of Collective Nuclear States.
Tensorial Structure of the Hamiltonian

The most general one- and two-body interaction is constructed by means of two interacting vector bosons. The group U(6) is the group of dynamical symmetry if the Hamiltonian conserves the number of bosons. All possible types of one- and two-boson interactions are expressed by irreducible tensor operators corresponding to the decomposition

$$U(6) \supset U(3) \otimes U(2) \supset U(3) \otimes (U(1) \otimes U(1)) \supset O(3).$$

The necessary isoscalar factors are calculated. This representation of the Hamiltonian makes possible its diagonalization through a direct application of the Wigner-Eckart theorem.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1983