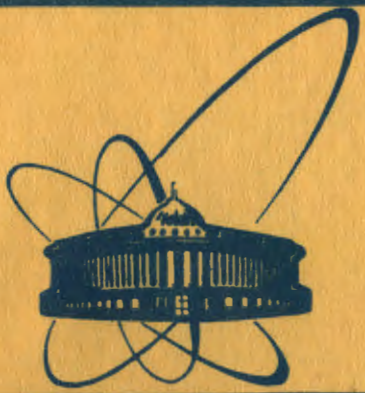

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THE MANY-BODY ANHARMONIC OSCILLATORS AND THE MATRIX CONTINUED FRACTIONS

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1982

1. INTRODUCTION

The complexity of the many-body problem quickly increases with the number of particles A. The microscopic description of the nuclear structure is a typical situation where we must use the efficient approximation techniques to be able to derive the measurable quantities from the first principles (two-body interaction $V(\bar{r})$) since the "exact" numerical solution of the underlying Schrödinger equation

$$H\psi = E\psi, \quad H = -\sum_{i=1}^{A} \frac{\hbar^2}{2m_i} \Delta_{\bar{r}_i} + \sum_{i>j=1}^{A} V(\bar{r}_i - \bar{r}_j) \qquad (1.1)$$

is extremely complicated. Also the perturbation approaches may start from the only solvable many-body with the harmonic -oscillator (HO) forces $V(\bar{r}) \sim \bar{r}^2$, in spite of the strong nature and various phenomenological shapes of the nucleon-nucleon forces, long range of the Coulombic interaction, etc. It meets therefore serious formal as well as practical difficulties. For example, the convergence can rarely be proved or achieved within the reasonably large truncated basis.

Usually, the simple-minded perturbation strategy is therefore being modified by the alternative (variational, etc.) techniques adapted to the specific nature of the particular question considered (for example, Faddeev-Yakubovsky equations for the few-body energies, exp S description of the structure of magic nuclei, etc.). In the present paper, we shall investigate the new possibility inspired by the recent exact solution of the one-body problem with the anharmonic oscillator (AHO) interactions using the formalism of the matrix continued fractions[1] (MCF).

The main idea of our approach consists in the possibility of replacing any "realistic" central force by the polynomial approximations

$$V(\bar{r}) = \sum_{m=1}^{p} g_m (\bar{r}^2)^m, \quad g_p > 0, \quad p \geq 1 \qquad (1.2)$$

with arbitrary precision, and to apply the same or slightly modified and complemented MCF method[2] for any $A \geq 2$, or, at least, for the few-body systems. We shall be interested in the bound states only, so that the necessary value of degree p in the AHO force Eq. (1.2) may be expected to be reasonably small owing to the spatial confinement of any sufficiently strongly bound system.

Our belief in the efficiency of the MCF strategy is supported by the encouraging numerical results of Ref.1 and by the consequent reinterpretation of the traditional effective-force concept[2]. In connection with the present wide-spread possibilities to employ in partical calculations the small and medium-size computers, our methematical formalism[2] using the interative algorithms working with the relatively small matrices seems to be a highly promising technique which might bridge the gap between the analytical and the purely numerical picture of the physical processes (cf. the popular and fruitful doorway- and hallway-state hypothesis used in a somewhat different contex[3]). From the analytic methods, we hope to inherit the advantages of the strict proofs and some useful properties of the classical continued fractions[4] surviving their matrix generalization (as an illustration, we may quote, e.g., Ref.[5]). In the numerical context, MCF formalism (we give its short review in Sec.II) may be expected to preserve merits of the Lanczos algorithm[6].

One of our main conclusions is an unusual observation that, once the anharmonicity p in Eq. (1.2) is fixed, the increase of A, especially the transition to the genuine many-body problem ($A=2 \to A=3 \to \ldots$) is surprisingly "smooth" from the technical point of view, first of all in the case of the identical particles. Because of the pedagogical reasons, we revert the presentation of the material and start from the distinguishable particles in Sec.III where the existence of the very MCF solvability of the AHO $A > 2$ Schrödinger equation is emphasized. In Sec.IV devoted to the identical particles, neglecting the spin variables for the sake of brevity, we show how the group theory helps to elucidate the structure of the optimal basis[7] and to minimize the dimensions of the continued-fraction matrices, i.e., the practical requirements concerning the computer capacity. In the simplest examples we consider as illustrations, this factor is irrelevant after all - we never need matrices larger than, say, 5x5 to reach quite a fair convergence of the three-bosonic ground-state energies for the elementary model-potentials with "core".

II. THE MATRIX CONTINUED-FRACTION METHOD

Linear equations of the Schrödinger type may be treated by various methods reaching from the analytic representations of ψ and E to their purely numerical approximations. The MCF method lies somewhere in between these two extremes and represents a reasonably flexible formalism comprising many specific

approaches as special cases. In the present context, we may explain its structure by recalling the inspiration of our paper, namely, the solution of the one-body AHO problem in the one spacial dimension as given by Graffi and Grecchi[1]. In this case, the use of the HO basis (Hermite polynomials) implies the band structure of the Hamiltonian H. In full analogy with the classical treatment of the three-diagonal (Jacobi) matrices[8] the authors arrive at the MCF representation of the Green function and identify its numerically determined poles with the AHO energy levels.

In detail, the formalism is worked out in Ref.[2] and it looks as follows: To the partitioned HO basis $|X_k^m\rangle$, $m=1,2,\ldots M_k$, $k=1,2,\ldots$, and to the related Hamiltonian matrix in the k-partitioned block-three-diagonal (BTD) form

$$H = \begin{pmatrix} A_1 & B_1 & 0 & \ldots \\ B_1^+ & A_2 & B_2 & 0 \ldots \\ 0 & B_2^+ & \ldots & \\ \ldots & & & \end{pmatrix}, \qquad \begin{aligned} &A_k = A_k^{ij}\ ,\ B_k = B_k^{in}\ , \\ &i,j = 1,2,\ldots,M_k,\ \ n = 1,2,\ldots,M_{k+1} \\ &k = 1,2\ldots \end{aligned} \tag{2.1}$$

we may assign the auxiliary sequence $F_k(E)$, $k=1,2,\ldots$ satisfying the recurrences

$$F_k(E) = [EI - A_k - B_k F_{k+1}(E) B_k^+]^{-1}\ . \tag{2.2}$$

The finite truncation of the matrix H is equivalent to the initialization $F_{N+1}(E)=0$ of Eq. (2.2). It is assumed that the limit $N \to \infty$ exists and defines each MCF $F_k(E)$ in the same way as the classical continued fractions are defined as limits of their finite approximants - they coincide with the one-dimensional ($M_k=1$) case of the present MCF's.

Assuming the existence of the MCF sequence $F_k(E)$ in the vicinity of the AHO energy levels $E=E_0$, we may identify $\mathcal{D}(E) = \det F_1(E)$ with the Green function of the Schrödinger Eq. (1.1). Moreover, the Schrödinger equation becomes reduced to the finite-dimensional model-space form

$$\sum_{i=1}^{M_1} (E\delta_{ij} - \mathcal{H}_{ij}^{(eff)}) D_1^i = 0, \quad j = 1,2,\ldots,M_1, \tag{2.3}$$

where the effective Hamiltonian is defined explicitly by the exact MCF expression $\mathcal{H}^{(eff)} = A_1 + B_1 F_2(E) B_1^+$. Once the energies $E = E_0$ are determined numerically as the poles of $\mathcal{D}(E)$, the projections $D_1^i = \langle X_1^i | \psi \rangle$ of the exact solution on the model-space bases may be found easily from the $M_1 \times M_1$-dimensional linear algebraic Eq. (2.3). As a consequence, the complete solution of Eq. (1.1) with any H of the BTD form (2.1) may be written in the compact form

$$\psi = a \sum_{k=1}^{\infty} \sum_{i=1}^{M_k} |X_k^i\rangle D_k^i , \quad D_{k+1} = D_k B_k F_{k+1}(E_0), \; k \geq 1. \qquad (2.4)$$

The normalization formula determining a,

$$||\psi|| = a^2 \sum_{k=1}^{\infty} \sum_{i=1}^{M_k} D_k^i D_k^i \qquad (2.5)$$

follows from the orthonormality of the HO basis.

The proofs of convergence are an important ingredient to the MCF formalism converting the formal solution into the analytical one. In the present contex, they are still missing even when A=1, expecially for the infinite series (2.4). At the same time, their numerical A=1 tests[1] have inspired our expectation that the favourable numerical properties might survive the transition to the many-body AHO cases.

III. DISTINGUISHABLE PARTICLES

The A=1→ A=2-body or the one → three-dimensional extensions of Ref.[1] are trivial and need not be explained in detail. In a way, their further A=2 → A=3 →... generalization is a matter of mere technicalities as well - they will be worked out in what follows.

A. Anharmonic Forces in the Jacobi Coordinates

The first technical question we must resolve when considering the A≥2 systems is the removal of the center-of-mass (cms) degrees of freedom. This step is entirely standard - assuming for simplicity that all the particle masses are equal ($\hbar = 2m_i = 1$, $i = 1, 2, ..., A$), we may define the Jacobi coordinates

$$\bar{\xi}_i = (\xi_i \sin\theta_i \cos\phi_i, \; \xi_i \sin\theta_i \sin\phi_i, \; \xi_i \cos\phi_i) = \qquad (3.1)$$

$$= \frac{1}{[i(i+1)]^{1/2}} (-i\bar{r}_{i+1} + \sum_{j=1}^{i} \bar{r}_j), \quad i = 1, 2, ..., A-1, A,$$

where $\bar{r}_{A+1} = 0$. Since the kinetic-energy operator remains proportional to the sum of Laplacians in the new variables (3.1), we may put

$$H = -\Delta_{\bar{\xi}_A} + H_0 , \quad \psi = \psi_0 \, e^{i\bar{k}_A \bar{\xi}_A} , \quad E = E_0 + \bar{k}_A^2 \qquad (3.2)$$

and obtain the AHO Schrödinger Eq. (1.1) in the translationally invariant form

$$H_0 \psi_0 = E_0 \psi_0 , \quad H_0 = -\sum_{i=1}^{A-1} \Delta_{\bar{\xi}_i} + V_{AHO}^{(A,p)} ,$$

$$V_{AHO}^{(A,p)} = \sum_{j=2}^{A} \sum_{i=1}^{j-1} \sum_{m=1}^{p} g_m [(\bar{r}_i - \bar{r}_j)^2]^m = \sum_{m=1}^{p} g_m v_m^{(A)}(\bar{\xi}_1, ..., \bar{\xi}_{A-1}) ,$$

$$v_m^{(A)} = v_m^{(A)}(\xi_1, ..., \xi_{A-1}, \cos\omega_{ij}), \quad \cos\omega_{ij} = \cos\theta_i \cos\theta_j + \qquad (3.3)$$

$$+ \sin\theta_i \sin\theta_j \cos(\phi_i - \phi_j), \quad 1 \leq i < j \leq A-1.$$

Here, ψ_0 is independent of $\bar{\xi}_A$ and $V_{AHO}^{(A,p)}$ is a genuine many-body operator.

The exceptionality of the HO interaction (p =1) lies essentially in the removal of this many-body character of V since

$$V_{AHO}^{(A,1)} = g_1 A \sum_{i=1}^{A-1} \xi_i^2 .$$

Starting from the first nondegenerate (quartic, p =2) case, the angular dependence of $V_{AHO}^{(A,p)}$ does not drop out and we get

$$v_2^{(3)} = \frac{9}{2}(\xi_1^2 + \xi_2^2)^2 - 6\xi_1^2\xi_2^2 \sin^2\omega_{12} ,$$

$$v_2^{(4)} = 5\xi_1^4 + 5\xi_2^4 + \frac{16}{3}\xi_3^4 + \frac{10}{3}\xi_1^2\xi_2^2 + \frac{8}{3}\xi_1^2\xi_3^2 + \frac{8}{3}\xi_2^2\xi_3^2 +$$

$$+ \frac{8}{3}\xi_1\xi_2\xi_2\cos^2\omega_{12} + \frac{16}{3}\xi_1^2\xi_3^2\cos^2\omega_{13} + \frac{16}{3}\xi_2^2\xi_3^2\cos^2\omega_{23} +$$

$$+ \frac{16}{3\sqrt{2}}\xi_1^2\xi_2\xi_3\cos\omega_{12}\cos\omega_{13} + \frac{8}{3\sqrt{2}}(\xi_1^2 - \xi_2^2)\xi_2\xi_3\cos\omega_{23} ,$$

etc. The derivation and structure of the general formula is rather lengthy but straightforward - we obtain

$$v_m^{(A)} = \sum_{B=2}^{A} w_m^{(B)} , \quad w_m^{(B)} = \sum_{j=1}^{B-1} [(\bar{r}_B - \bar{r}_j)^2]^m =$$

$$= \sum_{j=1}^{B-1} \{ B^2 T_{B-1} + \sum_{n=j}^{B-2} T_n + (j-1)^2 T_{j-1} + 2B \sum_{n=j}^{B-2} S_{n\,B-1} -$$

$$- 2B(j-1) S_{j-1\,B-1} + 2 \sum_{n=j}^{B-2} \sum_{p=j}^{n-1} S_{pn} - 2 \sum_{n=j}^{B-2} (j-1) S_{j-1\,n} \}^m ,$$

$$T_n = \xi_n^2/[n(n+1)], \quad n = 1, ..., A-1,$$

$$S_{ij} = \xi_i \xi_j \cos\omega_{ij} / [ij(i+1)(j+1)]^{1/2}, \quad 1 \leq i < j \leq A-1, \; m = 1, 2, ..., p$$

and observe that the complexity of the explicit $V_{AHO}^{(A,p)}$ prescription increases rather quickly with p. Nevertheless, the

corresponding lengthy formulas may comfortably be generated for any fixed A and p by an appropriate symbolic-manipulation language algorithm on the computer.

We note that the general structure of $V_{AHO}^{(A,p)}$ characterized by the explicit presence of the angular variables resembles strongly the case of the noncentral anharmonic potential solved by the MCF technique in Ref.[8] for A=1. This is the main inspiration of the following paragraph.

B. Unsymmetrized Oscillator Bases and the MCF Solvability

Preserving the full analogy with the noncentral modification[8] the one-body AHO, we may get rid of the angular variables in the next, still entirely standard step using the "multipolar" partial-wave decomposition of ψ_0. Of course, when A >2, the orthonormalized and complete set of the spherical "A-1 - polar" harmonics $|\{\ell\}>$ is not unique and may be defined with the different angular-momentum couplings[9]. In the simplest arrangement

$$\bar{\ell}_{A-1} + \bar{\ell}_{A-2} = \bar{\lambda}_{A-2}, \quad \bar{\lambda}_{i+1} + \bar{\ell}_i = \bar{\lambda}_i \ , \ i = A-3, \ldots, 2, \ \bar{\lambda}_2 + \bar{\ell}_1 = \bar{L} \tag{3.4}$$

of the vector-addition scheme corresponding to the composite index (quantum numbers)

$$\{\ell\} = \{\ell\}_{A-2} = (\ell_{A-1}\ell_{A-2}(\lambda_{A-2})\ell_{A-3}(\lambda_{A-3}) \ldots \ell_1 LM)$$

we obtain the harmonics ($\Omega_i = (\theta_i, \phi_i)$)

$$\langle\Omega_1 \Omega_2 \ldots \Omega_{A-1}|\{\ell\}_{A-2}\rangle =$$

$$= \sum_{\substack{m_1 \ldots m_{A-1} \\ \mu_2 \ldots \mu_{A-2}}} C^{LM}_{\ell_1 m_1 \lambda_2 \mu_2} C^{\lambda_2 \mu_2}_{\ell_2 m_2 \lambda_3 \mu_3} \ldots C^{\lambda_{A-2}\mu_{A-2}}_{\ell_{A-2} m_{A-2} \ell_{A-1} m_{A-1}} \times \tag{3.5}$$

$$\times Y_{\ell_1 m_1}(\theta_1, \phi_1) \ldots Y_{\ell_{A-1} m_{A-1}}(\theta_{A-1}, \phi_{A-1}) \ ,$$

where $C^{LM}_{\ell m \lambda \mu}$ and $Y_{\ell m}(\theta, \phi)$ denote the standard Clebsch-Gordan coefficients and the spherical harmonics, respectively. Another coupling pattern we shall need below,

$$\{\ell\} = \{\ell\}_m = \ell_{A-1} \ldots \ell_{m+2}(\lambda_{m+2})\ell_{m+1}\ell_m(\lambda)(\lambda_m)\ell_{m-1} \ldots LM$$

may be obtained when we replace the m+1-th and m-th items in Eq. (3.4) by the vector compositions

$$\bar{\ell}_{m+1} + \bar{\ell}_m = \bar{\lambda} \ , \quad \bar{\lambda}_{m+2} + \bar{\lambda} = \bar{\lambda}_m \ .$$

The overlap with the original states coincides with the so-called Racah coefficients

$$\langle\{\ell\}_{A-2} | \{\ell\}_m\rangle = (-1)^{\lambda_{m+2} + \ell_{m+1} + \ell_m + \lambda_m} \times$$

$$\times (2\lambda_{m+1} + 1)^{1/2} (2\lambda + 1)^{1/2} \begin{Bmatrix} \lambda_{m+2} & \ell_{m+1} & \lambda_{m+1} \\ \ell_m & \lambda_m & \lambda \end{Bmatrix} \tag{3.6}$$

and is proportional to the 6-j symbol $\begin{Bmatrix} \ldots \\ \ldots \end{Bmatrix}$.

For any coupling scheme, we introduce the partial waves $\phi_{\{\ell\}} = \langle\{\ell\}|\psi_0\rangle$ and derive the radial form of the Schrödinger Eq. (3.3) in a usual way. Since the action of the kinetic-energy operator on the harmonics $|\{\ell\}>$ is well known, the detailed form of the radial equation depends on the action of the angular variables S_{ij}. When we identify $\cos\omega_{ij}$ with the bipolar function $-4\pi\langle\Omega_i \Omega_j|1\,1\,0\,0\rangle/\sqrt{3}$ and employ Clebsch-Gordan series

$$\langle\Omega_1\Omega_2|\,1\,1\,0\,0\rangle\langle\Omega_1\Omega_2|\ell_1\ell_2\lambda\mu\rangle =$$

$$= \frac{\sqrt{3}}{4\pi} \sum_{\mu,\nu=0}^{1} (-1)^{\mu+\nu} (\ell_1 + 1 - \mu)^{1/2} (\ell_2 + 1 - \nu)^{1/2} \times \tag{3.7}$$

$$\times \begin{Bmatrix} \ell_2 + 1 - 2\nu & \ell_2 & 1 \\ \ell_i & \ell_{i+1-2\nu} & \lambda \end{Bmatrix} \langle\Omega_1\Omega_2|\,\ell_1 + 1 - 2\mu, \ \ell_2 + 1 - 2\nu, \ \lambda\mu\rangle$$

it is not difficult to specify the decomposition of S_{ij} into the finite number of $|\{\ell'\}>$'s in accord with the triangular inequalities, $\ell'_{i,j} = \ell_{i,j}$, $\ell_{i,j} \pm 1$.

Concerning the partial-wave representation of the operator H_0, we may therefore infer that the multipolar basis may be ordered in such a way that H_0 acquires the block-three-diagonal operator form resembling Eq. (2.1) - an example is given in Sec. III C below.

Completing the analogy with Ref.[8], we introduce therefore the A-body HO basis $|\langle n\rangle\{\ell\}>$, $\langle n\rangle = (n_1, n_2, \ldots, n_{A-1})$ as the multipolar harmonics Eq. (3.5) multiplied by (A-1)-tuple products of the radial A=1 HO states

$$\langle\xi|n\ell\rangle = (-1)^n \left[\frac{2\Gamma(n+1)}{\Gamma(n+\ell+3/2)}\right]^{1/2} e^{-\xi^2/2} \xi^\ell \, L^{\ell+1/2}(\xi^2) \ , \tag{3.8}$$

$$L_n^\alpha(x) = e^x x^{-\alpha} \frac{d^n}{dx^n} e^{-x} x^{\alpha+n}/n! \ ,$$

where $L_n^\alpha(x)$ are the classical Laguerre polynomials. Since they satisfy the fundamental identities[10]

$$L_n^{a-1}(x) = L_n^a(x) - L_{n-1}^a(x)\,, \qquad (3.9)$$

$$L_n^{a+1}(x) = (n+a+1)\,L_n^a(x) - (n+1)\,L_{n+1}^a(x)\,,$$

the action of the radial variables T_i on any HO state Eq. (3.8) is similar to the action of the kinetic-energy operator - it generates two other new states with $n_i' = n_i \pm 1$ only. At the same time, the action of the ξ-linear variables S_{ij} is accompanied by the ℓ-shift - hence, Eqs. (3.9) remain to be applicable. This is of fundamental importance here - the action of the full operator H_0 on any state $|<n>\{\ell\}>$ generates always the finite number of the similar states, i.e., the matrix representation of H_0 coincides with BTD Eq. (2.1) due to the orthogonality of the HO basis. This completes the proof of our main statement, i.e., of applicability of the MCF technique of Sec. II to any $A \geq 1$ AHO problem. In detail, we define the Green's function $G(E) = \det F_1(E)$, its poles $E = E_0$ and the HO projections $D_k(E_0)$ of ψ_0, i.e., the exact solution of the AHO many-body Schrödinger equation, provided that all the corresponding $N \to \infty$ limits exist.

In accord with Ref. [2] we may specify the optimal ordering of the basis states. This generalized Lanczos (GL) construction leads to the minimal dimensions of the blocks M_k and proceeds as follows:

(a) We choose any finite "model-space" subset of the HO $|<n>\{\ell\}>$ states and denote its elements by the kets $|X_1^m>$, $m=1,2,\ldots M_1$,

(b) The action of the Hamiltonian H_0 on this model space generates the finite superpositions of the new "doorway" states $|<n'>\{\ell'\}>$ to be denoted as $|X_2^m>$, $m=1,2,\ldots M_2$,

(c) Repeatedly, we re-numerate the full HO basis in such a way that each group $|X_{k+1}^m>$, $m=1,2,\ldots M_{k+1}$ of the k-th "hallway" states contains precisely the new states $|<n''>\{\ell''\}>$ generated from the old group $|X_k^m>$, $m=1,2,\ldots M_k$ by the action of H_0.

C. The Three-Body Illustration

In the simplest case with with $A=3$, $p=2$ and $L=0$, where

$$<\Omega_1\Omega_2|\ell_1\ell_2 00> = \delta_{\ell_1\ell_2}\,\frac{(-1)^\ell}{4\pi}\,(2\ell_1+1)^{1/2}\,P_{\ell_1}(\cos\omega_{12})$$

and $P_\ell(x)$ are the Legendre polynomicals [9,10], the partial-wave expansion of ψ_0 becomes extremely simple and reads

$$\psi_0 = \sum_\ell \frac{(\ell+1/2)^{1/2}}{\xi_1\xi_2}\,\phi_\ell(\xi_1,\xi_2)\,P_\ell(\cos\omega_{12}). \qquad (3.10)$$

We shall further require the $\vec{r}_1 \to \vec{r}_2$ symmetry of the wave function which is equivalent to the even parity of the summation index ℓ in Eq. (3.10). Owing to simplicity of this example, the radial Schrödinger equation

$$\begin{pmatrix} H_{00}-E & \beta_0\xi_1^2\xi_2^2 & 0 & \ldots \\ \beta_0\xi_1^2\xi_2^2 & H_{02}-E & \beta_2\xi_1^2\xi_2^2 & \ldots \\ 0 & \beta_2\xi_1^2\xi_2^2 & H_{04}-E & \ldots \\ \ldots & & & \end{pmatrix} \begin{pmatrix} \phi_0(\xi_1,\xi_2) \\ \phi_2(\xi_1,\xi_2) \\ \phi_4(\xi_1,\xi_2) \\ \ldots \end{pmatrix} = 0\,, \qquad (3.11)$$

$$H_{0\ell} = a_\ell\xi_1^2\xi_2^2 + \sum_{i=1}^{2}\Big[-\frac{\partial^2}{\partial\xi_i^2} + \frac{\ell(\ell+1)}{\xi_i^2} + 3g_1\xi_i^2 + \frac{9}{2}g_2\xi_i^4\Big],$$

$$a_\ell = 6g_2(1+1/(4\ell-2)(4\ell+6))\,,\quad \beta_\ell = 6g_2\,\frac{(\ell+1)(\ell+2)}{(2\ell+3)(2\ell+1)^{1/2}(2\ell+5)^{1/2}}$$

resembles strongly that of Ref. [8] - the abbreviations a_ℓ and β_ℓ denote also here the normalized matrix elements

$$<\ell|\,3g_2(1+2\sin^2\omega_{12})\,|\ell> \quad \text{and} \quad <\ell\,|6g_2\cos^2\omega_{12}|\,\ell+2>,$$

respectively.

Let us admit that the third particle is distinguishable from the remaining two bosons so that the Pauli principle is satisfied. In a formal way, we may formulate the following PROPOSITION.

Assuming that the auxiliary MCF quantities are convergent, Eqs. (2.3) and (2.4) with $M_k = k(2k-1)$ represent the exact solution of our three-body quartic AHO example.

PROOF

Denoting the basis states by $|n_1, n_2, \ell>$, the matrix H_0 is three-diagonal in ℓ and its infinite submatrices

$H_{\ell\ell n_2 n_2+2}$, $H_{\ell\ell n_2 n_2+1}$, $H_{\ell\ell+2 n_2 n_2}$ and $H_{\ell\ell n_2 n_2}$

contain one, three, three lower and five nonzero diagonals, respectively. Hence, each HO state is coupled to at most 5+2x3+2x1+2x3x3=31 other HO states. We may choose the one-dimensional model space with $|X_1^1> = |000>$ and generate the GL ordering with $M_1=1$, $M_2=6$, $M_3=15,\ldots$, in accord with Sec. III B.

IV. IDENTICAL PARTICLES

A. Action of the Pauli Principle

The complete set of the commuting operators H_0 (Hamiltonian), L^2 (square of the full angular momentum) and L_z (its projection) characterized an intrinsic state ψ_0 of the A-body system provided that the particles are distinguishable. The more interesting cases (identical bosons or fermions, without spin for simplicity) necessitate an addition of the projectors $\sigma_A^{(\pm)}$ (symmetrizer or antisymmetrizer, respectively). It is a matter of simple algebra to verify that the symmetrization antisymmetrization operator $\sigma_{(A)}^{(+/-)}$ may be defined by the recurrent formula

$$\sigma_{(A)}^{(\pm)} = w_{(A)}^{(\pm)} w_{(A-1)}^{(\pm)} \cdots w_{(2)}^{(\pm)} / A! \qquad (4.1)$$

$$w_{(A)}^{(\pm)} = 1 \pm w_{(A-1)}^{(\pm)} P_{(A)} \quad , \quad w_1^{(\pm)} = 1 \pm P_{(2)}$$

in the bosonic/fermionic case. Here, the operator $P_{(i)}$ corresponds to the interchange of the neighbouring particles $\vec{r}_{i-1}$ and $\vec{r}_i$.

The most important property of factorization (4.1) is the simplicity of its Jacobi-coordinate representation. First, the trivial algebra implies that $P_{(m+2)}$ will be represented by the (pseudo) orthogonal and real symmetric transformation

$$\begin{pmatrix} \vec{\xi}'_m \\ \xi'_{m+1} \end{pmatrix} = \begin{pmatrix} \cos\phi_m & \sin\phi_m \\ \sin\phi_m & -\cos\phi_m \end{pmatrix} \begin{pmatrix} \vec{\xi}_m \\ \vec{\xi}_{m+1} \end{pmatrix}, \quad \cos\phi_m = 1/(m+1) \qquad (4.2)$$

involving just the two coordinates ($\vec{\xi}_m = \vec{\xi}'_m = 0$ for m =0). The quantities ϕ_m in Eq. (4.2) are the "Euler" angles in the general ratation

$$R(\vec{r}_i \to \vec{r}_{i+j}) = P_{(i+1)} P_{(i+2)} \cdots P_{(i+j)} P_{(i+j-1)} \cdots P_{(i+1)}.$$

Next, the elementary rotation Eq. (4.2) may be re-interpreted as the "unequal-mass" transition from the "cms" coordinates $\vec{\xi}_m = \vec{R}$, $\vec{\xi}_{m+1} = \vec{r}$ to the "one-particle" variables $\vec{r}_1 = \vec{\xi}'_m$ and $\vec{r}_2 = \vec{\xi}'_{m+1}$. If we change also the coupling of the angular momenta in accord with Eq. (3.6), then Eq. (4.2) degenerates to a transformation of a certain two-particle subset of the full HO basis only. Of course, the $P_{(m+2)}$ matrix may be represented by the Moshinsky brackets /11/ $\langle \ldots / \ldots \rangle_D$ with the mass ratio $D = \tan^2\phi_m$

$$\langle \xi'_m \,|\, n'_m \ell'_m \rangle \langle \xi'_{m+1} \,|\, n'_{m+1} \ell'_{m+1} \rangle \langle \ldots \Omega'_m \Omega'_{m+1} \ldots |\{\ell'\}_m \rangle =$$

$$= \sum_{n_m \ell_m n_{m+1} \ell_{m+1}} \langle n_{m+1} \ell_{m+1} n_m \ell_m; \lambda | n'_m \ell'_m n'_{m+1} \ell'_{m+1}; \lambda \rangle_{1/m(m+2)} \times$$

$$\times \langle \xi_m \,|\, n_m \ell_m \rangle \langle \xi_{m+1} \,|\, n_{m+1} \ell_{m+1} \rangle \langle \ldots \Omega_m \Omega_{m+1} \ldots |\{\ell\}_m \rangle \qquad (4.3)$$

so that the complete symmetrization/antisymmetrization matrix $\sigma_{(A)}^{(\pm)}$ must remain diagonal with respect to the energy quantum number $N = \sum_{i=1}^{A-1} (2n_i + \ell_i)$ and the parity of $\sum_{i=1}^{A-1} \ell_i$. As a consequence, the BTD structure of H_0 survives its symmetrization/antisymmetrization $\sigma_{(A)}^{(+/-)} H_0 \sigma_{(A)}^{(+/-)} = H_{os/a}$.

B. Symmetrized Oscillator Basis

Let us start this paragraph with a short summary. To the many-body Schrödinger equation with the AHO two-particle interaction we may assign the MCF solution as described in Sec.II. This is a consequence of the BTD structure of the Hamiltonian H_o or $H_{os/a}$ in the unsymmetrized GL-ordered HO basis.

Unfortunately, the Eq. (3.5)xEq.(3.8) - product construction of this basis is unable to reflect the singular character of the projectors $\sigma_{(A)}$. In other words, the action of $H_{os/a}$ on $|\langle n \rangle \{\ell\}\rangle$ generates a few independent new states only which must be represented as superpositions of a large number of the unsymmetrized products $|\langle n \rangle \{\ell\}\rangle$. In this way, $H_{os/a}$ in the form of the original BTD matrix H_0 multiplied by the BTD matrix $\sigma_{(A)}$ will be characterized by an inadequate increase of the block-dimensional M_k for higher k's.

The key to the problem lies in the symmetrization of the basis itself, $|X_k^i\rangle \sim |\langle n \rangle \{\ell\}\rangle \to |X_k^i\rangle \sim \sigma_{(A)} |\langle n \rangle \{\ell\}\rangle$. In the more general setting, we must therefore construct the basis states as such superpositions of $|\langle n \rangle \{\ell\}\rangle$'s, which possess, besides the fixed total energy N and angular momentum L also the fixed quantum number [f] (Young tableau) characterizing the irreduciable representations of the permutation group. We may emphasize that the fixed symmetry-pattern [f] is the most important ingredient in the modification of the basis. It has two aspects:

(a) We may simplify the evaluation of the matrix elements of H_0 in the way which is standard /12,13/ and considers each particular component $V(r_i - r_j)$ of the potential $V_{AHO}^{(A,p)}$ separately. Formally, the commutation of H_0 and $\sigma_{(A)}$ is taken fully into account.

(b) We may simply extend our discussion to the particles with spin.

Of course, H_0 is diagonal with respect to N, L and [f] so that the new symmetrized basis will be more adequate for our purposes. There arise some new technical problems with the complete classification and algebraic construction of this basis. This was discussed by Kramer and Moshinsky [7] introducing further quantum numbers $(\lambda\mu)$ and Λ numbering the representations of the groups SU(3) and O_{A-1}, respectively. Concerning the general case, we omit the details here.

C. The Three-Body Example

For p =2 and the three identical spinless bosons in the s -state, each old group $|X_k^m\rangle$, $m=1,2,\ldots M_k$ as specified in Sec. IIIC contains k different fixed-energy subgroups with $N = 2k-2, 2k, \ldots, 4k-4$. The partition dimensions of the symmetrized operator $\sigma_{(3)}^{(+)} H_0 \sigma_{(3)}^{(+)}$, $M_k = 2^k(2^{k+1}-1)(7.2^{k-1}-5)/12$ - grow extremely quickly even for the low cutoffs ($M_1=1$, $M_2 = 21$, $M_3 = 230$, ...), due to the non-diagonality of $\sigma_{(3)}^{(+)}$. Even the fixed-energy re-partitioning with $M_k = k(2k-1)(4k-1)/3$ or, alternatively, $M_k = k(2k+1)(4k+1)/3$ is rather inefficient (M_3=140 or 91, respectively). At the same time, the Kramer-Moshinsky [7] A=3 classification

$$|N(\lambda\mu)LM\Lambda[f]\rangle = \sum_{n_1\ell_1 n_2\ell_2} |n_1\ell_1 n_2\ell_2 LM\rangle \begin{pmatrix} (n_1 0) & (n_2 0) & | & (\lambda\mu) \\ \ell_1 & \ell_2 & | & L \end{pmatrix} \times d^{\lambda/2}_{(n_1-n_2)/2,\Lambda/2}(\pi/2) \qquad (4.4)$$

with the Wigner function d and the SU(3) Clebsh-Gordan coefficients $\begin{pmatrix} \cdot & \cdot & | \\ \cdot & \cdot & | \end{pmatrix}$ appears to be complete and sufficiently simple for the practical purposes [12]. Its introduction reduces the block-dimensions M_k significantly below the values of Sec. III (cf. Table 1).

The nontrivial optimalization of the new basis is still possible since, rather surprisingly, H_0 becomes diagonal with respect to the rotational quantum number Λ, which was originally introduced for the purely classification purposes. In this way, we obtain the form of the symmetrized HO basis

$$|X_k^i\rangle = |N(\lambda\mu)LM\Lambda[f]\rangle$$

with fixed L =0, [f] = [3], Λ =0 and with k =integer part to $(N+6)/4$. In this basis, the numerical test of the MCF convergence of the MCF representation of the Green's function $\det F_1(E)$ was performed.

Table 1

Block-dimensions M_k (k =1,2,...) of H_{os} in the symmetrized oscillator basis

maxN	L=0		[f] =3		Λ =0	Λ =6
	p =2	p =3	p =2	p =3	p =2	p =2
0	1	1	1	1	1	
2						
4	6		3		3	
6		13		6		1
8	17		7		5	
10						3
12	33	43	12	16	7	
14						5
16	54		18		9	
18		91		30		7
20	81		26		11	

For the monotonous potentials ($g_1 > 0$ and $g_2 > 0$), the convergence proved to be extremely rapid. From the physical point of view, similar potentials are, however, less interesting than the more realistic models of the nucleonic forces with the repulsive core. A priori [13] our choice of the core-simulating values of $g_1 = -g_2 = -10$ will worsen the convergence and is therefore well suited also for the illustration purposes. The sample results is presented in Table 2 and the Figure and shows that the energy calculations remain to be easily managenable even on the small computers. In Tab-

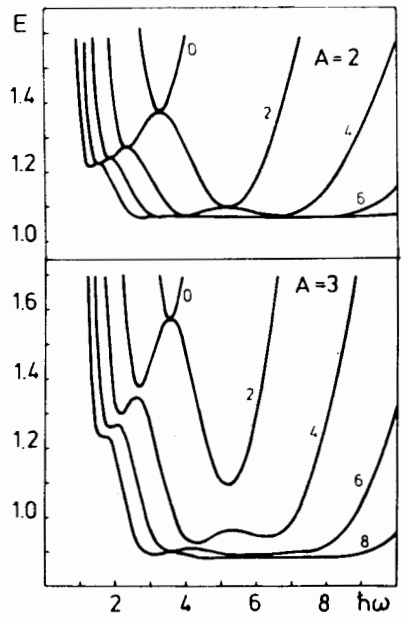

Fig. Scaling behaviour of the ground-state energies for the two- and three-body quartic oscillator and different maxN.

Table 2

Sample of convergence of the ground-state energies for three bosons and quartic forces

$\hbar\omega$ [MeV]	2.4	6.5	7.0	7.5	9.0
$\max M_k$					
1	27.355	5.489	6.376	7.250	9.889
2	8.345	1.617	2.015	2.484	4.100
3	3.286	0.945	0.993	1.104	1.797
4	1.712	0.896	0.895	0.899	1.045
5	1.287	0.883	0.853	0.884	0.897
6	1.217	0.882	0.882	0.882	0.883
7	1.189	0.882	0.882	0.882	0.882

le 2, we demonstrate the sufficiency of the small-matrix algorithms to reach the convergent results. The Figure illustrates in more detail the typical core-induced oscillatory dependence of energies on the variation of scale ("spring constant") of the HO basis and its smoothing for higher cutoffs, and also the similarity of this feature in the two- and three-body systems.

V. CONCLUSIONS

In the one-particle quantum mechanism and its applications, the AHO problem is an old and traditional subject testing the practical algorithms as well as the various theoretical ideas. The MCF solution of Graffi and Grecchi is one of the promising approaches to its A=1 (or A=2) form. In the many-body context, the exceptional character of the HO force is even more pronounced and the AHO corrections spoil the Schrödinger equation more profoundly. Nevertheless, we have shown that for $A>2$, the full formal analogy with A=2 case may be preserved. In particular, we have found that both the technical (cf. slow increase of dimensions M_k) and numerical (rate of convergence, etc.) aspects of both the two- and three-body AHO systems proved to be comparable from the practical point of view. We may therefore expect that also some more realistic models (with $A>3$, $p>2$, including spin, isospin, tensor forces, etc.) will remain to be exactly solvable by the present MCF technique.

Of course, for very high A's the numerical performance of the MCF formalism becomes less efficient since the BTD blocks grow too quickly. Nevertheless, preserving even there

the AHO-model description of the microscopic interaction, the iterative character of the MCF representation of the hallway effective Hamiltonians and, in particular, of the Green function $F_1(E)$ seems to remain at least a useful guide for making approximations. Preliminarily, their character might resemble either the fixed-point techniques of Ref.[5] or some sort of averaging of the type employed in the reaction-theory context [3].

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Received by Publishing Department
on February 1 1982.

Знойил М., Майлинг Л. Система многих частиц с ангармоническим взаимодействием E4-82-76

Рассмотрена возможность получения быстро сходящихся решений /собственных значений и собственных функций/ многочастичного уравнения Шредингера. Показано, что взаимодействие ангармоничного осциллятора $V(r) = g_1 r^2 + g_2 r^4$ допускает запись решения уравнения Шредингера в виде непрерывной дроби в матричном представлении. Показано, что использование трансляционного-инвариантного базиса многочастичных волновых функций, характеризуемых квантовыми числами неприводимых представлений групп SU(3) и O(A-1) /вращения в A-1 -мерном пространстве/ резко сокращает размерности матриц и существенно упрощает конкретные расчеты. Эффективность предлагаемого общего метода решения многочастичного уравнения Шредингера продемонстрирована на примере A=3 и $V(r) = -r^2 + r^4$.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1982

Znojil M., Majling L. The Many-Body Anharmonic Oscillators and the Matrix Continued Fractions E4-82-76

For the class of the anharmonic two-body forces, we recollect and show in detail how the A-body Hamiltonian may be converted into an infinite block-three-diagonal matrix in the properly arranged translationally invariant oscillator basis. This generalizes the recent reformulation of the anharmonic one-dimensional A=1 problem by Graffi and Grecchi. As a consequence, the exact Green function and all the projections of eigenstates of the microscopic Schrödinger are expressible in terms of the matrix continued fractions, the convergence of which is demonstrated by the simple three-body example. In this way, the nonperturbative method of solving the many-body bound-state problem is established. Its iterative alteration may be based on the systematic polynomial approximations to arbitrary realistic potentials and is applicable to both the identical and distinguishable particles.

The investigation has been performed at the Laboratory of the Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1982