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Sp(4,R) BASIS FOR THE NUCLEAR COLLECTIVE PAIRING

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1. GENERAL STRUCTURE OF THE BASIS

The pairing Hamiltonian with constant matrix elements can be written as a product of Cooper pair creation P^+ and annihilation, P, operators

$$H_P = -GP^+P, \qquad G = \text{const.}, \tag{1}$$

where

$$P^+ = \sum_{\nu>0} c_\nu^+ c_{\bar{\nu}}^+ , \qquad P = (P^+)^+ \tag{2}$$

and $c_{\bar{\nu}}^+$ creates the single-particle time-reversed nucleon state of c_ν^+. Following [1] we define a gauge transformation

$$\mathcal{G}(\phi) = e^{-i\phi\hat{A}} , \tag{3}$$

where $\hat{A} = \hat{\mathfrak{N}} - \mathfrak{N}_0$, $\hat{\mathfrak{N}}$ is the particle number operator and $\mathfrak{N}_0$=const. is the number of nucleons in an arbitrary nuclear core. The transformations (3) stand for the elements of the group SO(2) which is the symmtery group for the Hamiltonian H_p. In the work [1] it has been shown that the collective treatment of the pairing phenomena requires two collective variables. Instead of this in the present paper two kinds of boson creation (and annihilation) operators are used

$$
\begin{aligned}
s_\mu^+ &= \frac{1}{2} e^{i\mu\phi} \left[\alpha(\Lambda) - \beta(\Lambda)\frac{\partial}{\partial\Lambda} + \frac{1}{\mu\alpha(\Lambda)}\hat{A} - F(\Lambda)\right] \\
s_\mu &= \frac{1}{2} e^{-i\mu\phi} \left[\alpha(\Lambda) - \beta(\Lambda)\frac{\partial}{\partial\Lambda} + \frac{1}{\mu\alpha(\Lambda)}\hat{A} + F(\Lambda)\right] \\
\mu &= \pm 2,
\end{aligned} \tag{4}
$$

where $\Lambda \in [0, \infty)$ is a real variable which can be interpreted as the pairing gap parameter, $\alpha(\Lambda)$ is an arbitrary real function,

$$F(\Lambda) = \frac{1}{2}\left[\frac{\beta(\Lambda)}{\rho(\Lambda)}\frac{\partial\rho}{\partial\Lambda} + \frac{\partial\beta}{\partial\Lambda} - \frac{1}{\alpha}\right], \qquad \beta(\Lambda)\frac{\partial\alpha}{\partial\Lambda} = 1$$

and $\rho(\Lambda)$ is a weight function in the scalar product defined by the integral

$$(\Psi \mid \chi) = \int_0^{2\pi} d\phi \int_0^{\infty} d\Lambda\, \rho(\Lambda)\, \Psi^*(\phi,\Lambda)\, \chi(\phi,\Lambda). \tag{5}$$

In the gauge space the operators (4) transform according to the same IR of the symmtery group, SO(2), as the pairing operators P^+ and P (note that in (ϕ, Δ)-space the operator $\hat{A}$ defined by the equation (3) can be treated as the canonical conjugate variable to ϕ i.e., $\hat{A} = -i\frac{\partial}{\partial\phi}$, $\hbar$=1) namely,

$$\begin{aligned} &\tilde{s}^+_2 = e^{i2\phi} s^+_2 && \tilde{P}^+ = e^{i2\phi} P^+ \\ &\tilde{s}_{-2} = e^{i2\phi} s_{-2} && \\ &\tilde{s}^+_{-2} = e^{-i2\phi} s^+_{-2} && \tilde{P} = e^{-i2\phi} P, \\ &\tilde{s}_2 = e^{-i2\phi} s_2 && \end{aligned} \tag{6}$$

that is, these boson operators are raising and lowering number of nucleon pairs coupled to the total angular momentum J =0 operators.

The bilinear forms of (4)

$$s^+_\mu s^+_{\mu'}, \quad s_\mu s_{\mu'}, \quad s^+_\mu s_{\mu'} \qquad \mu, \mu' = \pm 2 \tag{7}$$

are the generators of the noncompact four-dimensional symplectic group Sp(4,R) [2]. In the collective space the symmetry group SO(2) of the Hamiltonian (1) is generated by the nucleon excess-deficit operator $\hat{A}$ expressed in terms of the generators (7)

$$\hat{A} = 2(s^+_2 s_2 - s^+_{-2} s_{-2}). \tag{8}$$

It can be also easily found that the largest subgroup of Sp(4,R), whose generators commute with $\hat{A}$, is a group SU (1,1) generated by the operators

$$\begin{aligned} \hat{n}_+ &= s^+_2 s^+_{-2} \\ &= \frac{1}{4}\{\beta\frac{\partial}{\partial\Delta}\beta\frac{\partial}{\partial\Delta} + (\frac{1}{\alpha} - 2\alpha + 2F)\beta\frac{\partial}{\partial\Delta} - \frac{1}{4\alpha^2}\hat{A}^2 + \alpha^2 - 2\alpha F \\ &\qquad + F^2 + \frac{F}{\alpha} + \beta\frac{\partial F}{\partial\Delta} - 2\} \end{aligned} \tag{9a}$$

$$\begin{aligned} \hat{n}_- &= s_2 s_{-2} \\ &= \frac{1}{4}\{\beta\frac{\partial}{\partial\Delta}\beta\frac{\partial}{\partial\Delta} + (\frac{1}{\alpha} + 2\alpha + 2F)\beta\frac{\partial}{\partial\Delta} - \frac{1}{4\alpha^2}\hat{A}^2 + \alpha^2 + 2\alpha F \\ &\qquad + F^2 + \frac{F}{\alpha} + \beta\frac{\partial F}{\partial\Delta} + 2\} \end{aligned} \tag{9b}$$

$$\begin{aligned} \hat{n}_0 &= \frac{1}{2}(s^+_2 s_2 + s^+_{-2} s_{-2} + 1) = \frac{1}{2}(\hat{N} + 1) \\ &= \frac{1}{4}\{-\beta\frac{\partial}{\partial\Delta}\beta\frac{\partial}{\partial\Delta} + \frac{1}{4\alpha^2}\hat{A}^2 - (2F + \frac{1}{\alpha})\beta\frac{\partial}{\partial\Delta} + \alpha^2 - \beta\frac{\partial F}{\partial\Delta} - F^2 - \frac{F}{\alpha}\} \end{aligned} \tag{9c}$$

with the commutation relations

$$[\hat{n}_+, \hat{n}_-] = -2\hat{n}_0 \quad [\hat{n}_0, \hat{n}_+] = \hat{n}_+ \quad [\hat{n}_0, \hat{n}_-] = -\hat{n}_- \tag{10}$$

and

$$[\hat{n}_0, \hat{A}] = [\hat{n}_+, \hat{A}] = [\hat{n}_-, \hat{A}] = 0,$$

In this way we obtained the following group chain for classification of the collective pairing states

$$Sp(4,R) \supset SO(2) \times SU(1,1). \tag{11}$$

By calculation of the quadratic Casimir operator C^2 of the group SU (1,1) two-one complementarity of the IR of SO (2) and SU (1,1) can be established

$$C^2 = \hat{n}_0^2 - \hat{n}_0 - \hat{n}_+\hat{n}_- = \frac{1}{4}\{\frac{1}{4}A^2 - 1\}, \tag{12}$$

i.e., one IR of SU (1.1) group corresponds to two IR of SO (2) labelled by the nucleon excess - deficit numbers $\pm$ A. This implies that only two labels are needed to fully classify the basis for IR of SO (2)x SU (1,1) group. As the quantum numbers we assume the number A and the total number of boson N, and read

$$|AN\rangle = \{\frac{(\frac{1}{2}|A|)!}{[\frac{1}{2}(N+\frac{1}{2}A)]![\frac{1}{2}(N-\frac{1}{2}A)]!}\}^{\frac{1}{2}} (\hat{n}_+)^{\frac{1}{2}(N-\frac{1}{2}|A|)} |A\rangle \tag{13}$$

$$A = 0, \pm 2, \pm 4, \ldots$$

and

$$N = \frac{1}{2}|A|, \ \frac{1}{2}|A| + 2, \ \frac{1}{2}|A| + 4, \ldots,$$

where

$$|A\rangle = \{[\frac{1}{2}|A|]!\}^{-\frac{1}{2}} (s^+_\mu)^{\frac{1}{2}|A|} |0\rangle \tag{14}$$

with μ =2 for A$\geq$0 and μ =-2 for A$\leq$0, are the states of the lowest weight for the IR $|A|$ of the group SU (1,1) defined by the conditions [3]

$$\hat{n}_- |A\rangle = 0$$

and

$$\hat{n}_0 |A\rangle = \frac{1}{2}(\frac{1}{2}|A| + 1)|A\rangle. \tag{15}$$

To find a representation of the states (13) in the (ϕ, Δ)-space, we introduce the vacuum state by the system of two equations

$$s_\mu \Omega = 0, \quad \mu = \pm 2. \tag{16}$$

A general solution of eqs. (16) can be easily recognized if we make the substitution $\Omega(\Delta) = q(\Delta)\exp(-\frac{1}{2}\alpha^2)$ and get

$$\Omega(\Delta) = a_0 \exp\{ -\frac{1}{2}\alpha^2 - \int F(\Delta)\frac{d\alpha}{d\Delta}d\Delta \}, \tag{17}$$

where a_0 is the normalization factor. Making use of equations (14) and (15) a general form of the states with the lowest weight can be written in the form

$$u_A(\phi,\Delta) = \langle \phi,\Delta | A \rangle = 2^{-|A|}\{(\frac{1}{2}|A|)!\}^{-\frac{1}{2}} e^{iA\phi} \times (2\hat{K} + \frac{|A|-2}{\alpha})(2\hat{K} + \frac{|A|-4}{\alpha})\dots(2\hat{K} + \frac{2}{\alpha})\hat{K}\Omega, \tag{18}$$

where

$$\hat{K} \equiv \alpha(\Delta) - F(\Delta) - \beta(\Delta)\frac{\partial}{\partial\Delta}.$$

In practice only A=0, ± 2 states are used. In these cases we obtain

$$u_0 = \Omega(\Delta)$$

and

$$u_{\pm 2} = \frac{1}{2} e^{\pm i 2\phi}\alpha(\Delta)\Omega(\Delta). \tag{19}$$

It is not useful to put down the explicit form of the basis (13) in (ϕ,Δ)-space in the most general form. If we have from the microscopic calculations a shape of the functions $\alpha(\Delta)$ and $\rho(\Delta)$, it is possible to find a concrete realization of the basis by the procedure given above or by solving an appropriate differential equation (see below).

2. α-VIBRATION LIMIT, F=0

Let us start from the boson form of the asymmetric two-dimensional harmonic oscillator Hamiltonian

$$H = h_0 + \epsilon_2 s_2^+ s_2 + \epsilon_{-2} s_{-2}^+ s_{-2}, \tag{20}$$

where h_0, ϵ_2 and ϵ_{-2} are constants. From the straightforward calculations in (ϕ,Δ)-space we get

$$H = h_0 + \epsilon\hat{N} + \frac{1}{2}\eta\hat{A}$$

$$= -\frac{1}{2}\epsilon\beta^2\frac{\partial^2}{\partial\Delta^2} + \frac{\epsilon}{8\alpha^2}(\hat{A} + \frac{2\eta}{\epsilon}\alpha^2)^2 - \frac{1}{2}\epsilon\beta(2F + \frac{1}{\alpha} + \frac{\partial\beta}{\partial\Delta})\frac{\partial}{\partial\Delta} + (\frac{1}{2}\epsilon - \frac{\eta^2}{2\epsilon})\alpha^2 - \epsilon + h_0 - \frac{1}{2}\epsilon\beta\frac{\partial F}{\partial\Delta} - \frac{1}{2}\epsilon F^2 - \frac{\epsilon}{2\alpha}F, \tag{21}$$

where $\epsilon = \frac{1}{2}(\epsilon_2 + \epsilon_{-2})$ is the average single-boson energy and $\eta = \frac{1}{2}(\epsilon_2 - \epsilon_{-2})$ is the asymmetry parameter.

By comparison of the Hamiltonian (21) with the quantum prescription of the kinetic energy operator /4/ we obtain the mass parameter in Δ direction B_Δ and the inertia parameter J as the functions of the collective variable Δ (because of the SO(2) symmetry of the Hamiltonian they are independent of the variable ϕ)

$$B_\Delta = \frac{1}{\epsilon}(\frac{d\alpha}{d\Delta})^2$$

and

$$J = \frac{4}{\epsilon}[\alpha(\Delta)]^2. \tag{22}$$

As usual the weight function ρ can be taken in the form

$$\rho(\Delta) = \sqrt{B_\Delta \cdot J}.$$

Now we are in a position to explain the title of the present paragraph. For F=0, with the conditions (22) the Hamiltonian (21) describes vibrations of the system around the point $\alpha = 0$. It can be easily seen from the shape of the potential extracted from (21)

$$V = h_0 - \epsilon + (\frac{\epsilon}{2} - \frac{\eta^2}{2\epsilon})\alpha^2. \tag{23}$$

Note also another characteristic of the model Hamiltonian (21). It takes into account an experimentally observed evidence as asymmetry ($\eta \neq 0$) in the rotational spectra of the collective pairing. In the work by Bès, Broglia, Perazzo and Kumar /1/ this feature wasn't examined.

For the case of F=0 the basis for IR of the group SO(2)x x SU(1,1) can be constructed by solving the eigenequation for the weight operator $\hat{n}_0$ (or $\hat{N}$)

$$\hat{n}_0\Psi_{AN}(\phi,\Delta) = \frac{1}{2}(N+1)\Psi_{AN}(\phi,\Delta), \tag{24}$$

where because of the equations (13)-(18) the eigenfunctions Ψ_{AN} can be factorized

$$\Psi_{AN}(\phi,\Delta) = \text{norm.}\, e^{iA\phi} W_{NA}(\Delta)\Omega(\Delta). \tag{25}$$

Then we get the following equation for unknown function $W_{NA}(\Delta)$

$$(-\beta\frac{\partial}{\partial\Delta}\beta\frac{\partial}{\partial\Delta} - (2\alpha - \frac{1}{\alpha})\beta\frac{\partial}{\partial\Delta} + (2N - \frac{A^2}{4\alpha^2}))W_{NA}(\Delta) = 0. \tag{26}$$

Noting that $\beta\frac{\partial}{\partial\Delta} = \frac{\partial}{\partial\alpha}$ this equation can be rewritten into a more useful for calculations from

$$W''_{NA}(\alpha) - (2\alpha - \frac{1}{\alpha})W'_{NA}(\alpha) + (2N - \frac{A^2}{4\alpha^2})W_{NA}(\alpha) = 0. \tag{27}$$

Using the standard series method the solution of the equation can be found as the following polynomial:

$$W_{NA}(\Delta) = \sum_{k=\frac{1}{2}|A|}^{N} c_k [\alpha(\Delta)]^k, \qquad (28)$$

where the coefficients c_k satisfy the simple recurrence relation

$$c_{k+2} = \frac{2(k-N)}{(k+2)^2 - \frac{1}{4}A^2} c_k \qquad (29)$$

with

$$k = \frac{1}{2}|A|, \ \frac{1}{2}|A| + 2, \dots, N$$

and

$$c_{\frac{1}{2}|A|+1} = c_{\frac{1}{2}|A|+3} = \dots = c_{\frac{1}{2}|A|+N-1} = 0$$

or in more compact form

$$c_k = \prod_{m=\frac{1}{2}|A|+2}^{k}{}^{(2)} \frac{2m-2-N}{m^2 - (\frac{1}{2}A)^2} + \delta_{k,\frac{1}{2}|A|} \qquad (30)$$

The coefficient $c_{\frac{1}{2}|A|} = 1$ is choosen to recognize in (28) the standard Laguerre polynomials $L_{\frac{1}{2}N}(\alpha^2)$, for the special case of the nucleon excess-deficit number equals zero, $A=0$.

In this way we have completed the physical and orthogonal basis for the collective pairing Hamiltonian (without isospin) for even-even nuclei.

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Гузьдзь А. $Sp(4,\mathbb{R})$ базис для коллективного ядерного спаривания   E4-82-561

Изучена коллективная трактовка парных явлений. Введены два сорта операторов рождения и уничтожения бозонов с целью получения классификации коллективных парных состояний в четно-четных ядрах /без изоспина/. Для построения физического и ортонормального базиса в бозонном пространстве использовалась следующая групповая цепочка

$$Sp(4,\mathbb{R}) \supset SO(2) \times SU(1,1).$$

С помощью простого представления бозонных операторов в коллективном пространстве получен вид базиса в этих коллективных переменных.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1982

Góźdź A. $Sp(4,\mathbb{R})$ Basis for the Nuclear Collective Pairing   E4-82-561

A collective treatment of the pairing phenomena has been investigated. Two kinds of boson creation (and annihitation) operators have been introduced to find a classification of the collective pairing states in even-even nuclei (without isospin). The following group chain

$$Sp(4,\mathbb{R}) \supset SO(2) \times SU(1,1)$$

has been used to construct the physical and orthogonal basis in the boson space. Making use of the simple representation of the boson operators in a collective space a form of the basis in the collective variables has been obtained.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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