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дубна

E4-82-525

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ON EXACT SOLUTIONS OF SCATTERING PROBLEMS

Submitted to "ЯФ"

1982

Exact solutions of integral equations for the inverse scattering problem were obtained for a wide class of interactions - Bargmann potentials (see the review article[1]).

The Bargmann potential chosen from these model solutions for the best approximation of the scattering data in some particular case can be considered as an approximate solution of the corresponding inverse problem.

Such calculations have been already performed for numerous nuclear systems (potentials for quarks N-N , N-α , α-α , N - nucleus, nucleus-nucleus, and even for heavy ions have been reconstructed, see the review[1]).

It is impossible to judge about the quality of this interaction reconstruction procedure without its verification in the cases with a priori known potentials (how are these potentials deformed after successive solution of the direct and inverse problems). However, very little of such necessary verifications have been fulfilled until now. And these are absent at all for some classes of Bargmann potentials.

We present results of such calculations with an approximation of scattering data for the known potentials by choosing singularities of the trial S-matrix in the complex k-plane.

A multichannel (matrix) generalization of Bargmann potentials ($V_{\alpha\beta}(r)$) was originally considered by Newton and Fulton[2,3]. They restricted themselves, however, to the case when the thresholds of excitation for all the channels coincide (kinetic energies of the free asymptotic motion in different channels are equal). This formalism was used for reconstruction of the tensor and spin-orbital nucleon-nucleon forces [3,4].

Systems with different channel excitation thresholds have been investigated by Cox[5].

Recently it has been found[6] that new solutions, more simple than the exact solutions of Cox, exist in the case of separable dependence with respect to channel indices, of the kernels of the inverse problem integral equations. In ref.[6] interaction matrices $V_{\alpha\beta}$, for which the kernels of Gelfand-Levitan equations contain the contribution from bound states only, have been constructed (with kernels not necessarily completely degenerated as in the Cox paper[5].).

In the present paper exact solutions of multichannel systems of equations (for direct and inverse problems) corresponding to scattering matrices with a finite number of resonance singularities are derived. The resonance states have also not necessarily to be completely degenerated, as it was demanded in ref./5/.

1. EXAMPLES OF APPROXIMATE SOLUTIONS OF SINGLE-CHANNEL INVERSE PROBLEM

A remarkable demonstration of the quality of the approximate potential reconstruction was given in papers /7,8/ by collaborators of Fermilab. (Batavia). They restricted themselves, however, to infinitely deep potential wells (with the discrete spectrum only) developing the algorithm of reconstruction of interquark confinement forces. A specific technique /7,8/ does not permit the applications of their results for estimating the errors of the potential reconstruction for systems with the continuous spectrum.

The scattering function $S(k)$ in our calculations was chosen in the factorized form

$$S^B = \prod_j^N \frac{(k-a_j)(k+a_j^*)(k+b_j)(k-b_j^*)}{(k-b_j)(k+b_j^*)(k+a_j)(k-a_j^*)}, \qquad (1)$$

to which there corresponds the Bargmann potential

$$V^B = 4i\frac{d}{dr}\left\{\sum_j^{2N} b_j \frac{a_j-b_j}{a_j+b_j} f(b_j,r)\, e^{i b_j r}\right\}, \qquad (2)$$

where

$$f(b_j r) = \sum_n^{2N} (I-2P(r))^{-1}_{jn}\, e^{i b_n r},$$

$$P_{jn} = \frac{-b_j}{b_j+b_n} \cdot \frac{a_j-b_j}{a_j+b_j}\, e^{i(b_j+b_n)r}.$$

Parameters a_j, b_j were taken as to provide the best possible approximation of $S(k)$ (real $k>0$) corresponding to the potential to be reconstructed.

As typical examples the following potentials will be considered here A) Wood-Saxon potential which is often used in nuclear physics and B) $V = V_0\, e^{-\alpha r^2}$ for which many calculations have been already done in the inverse scattering problem in the framework of a formalism with fixed energy*, C) potential $V = 2e^{-10r} - e^{-5r}$ with a repulsive core and an attractive tail.

*There are two main approaches in the inverse scattering problem: the reconstruction of interaction from partial phase shifts δ_ℓ with different values of the angular momentum ℓ for a given value of E (successes along this line are discussed in review/1/) and formalism with fixed ℓ considered in this paper.

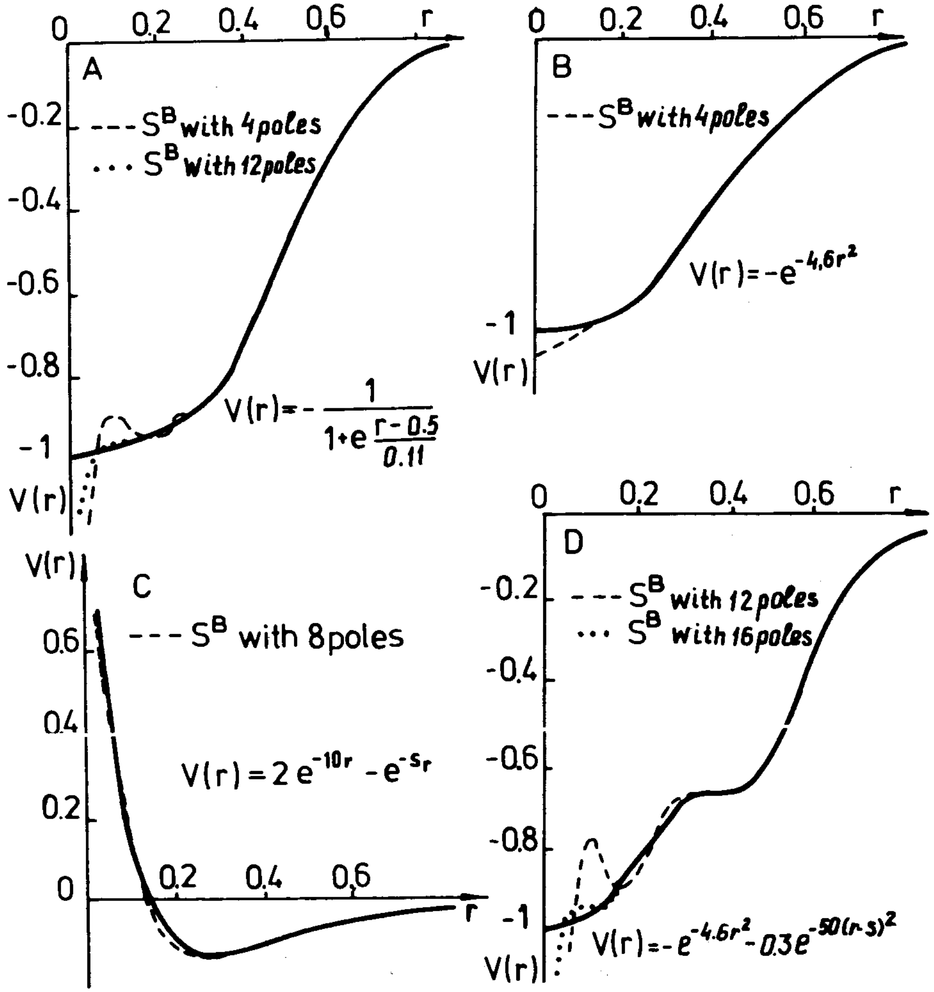

Fig.1. Approximate solution of the inverse problem using Bargmann potentials. Original potentials are drawn by solid lines, and the ones reconstructed from phase shifts $\delta(E)$ - by dashed lines.

Initial V_i and reconstructed V_r potentials (upon solving the direct and inverse problems: $V_i \to S \to V_r$) are compared in fig.1. The position of poles of S^B in the complex k-plane corresponding to approximate solutions of the inverse problems in case D are shown in fig.2. It is remarkable that trial potentials V^B appear to be rather close to the original V for essentially different sets of parameters a_j, b_j.

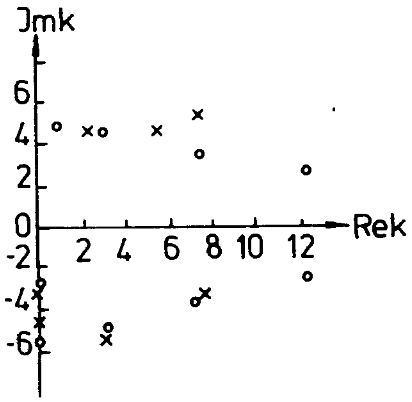

Fig.2. The displacement of poles of scattering functions in the complex k -plane derived for reconstruction of the potential shown on the figure ID: (x) - in 12-pole, and (o) - 16-pole approximation.

2. MULTICHANNEL EQUATIONS

The motion in systems with N coupled channels will be described here by equations ($h^2 = 2m = 1$):

$$-\Psi''_{\alpha\beta}(\mathcal{K},r)+\sum_{\alpha}^{N} V_{\alpha\alpha'}(r)\,\Psi_{\alpha'\beta}(\mathcal{K},r)=E_\alpha\,\Psi_{\alpha\beta}(\mathcal{K},r)\,, \tag{3}$$

where $V_{\alpha\alpha'}(r)$ are elements of an interaction matrix $V(r)$, $\Psi_{\alpha\beta}(\mathcal{K},r)$ is the wave function in a channel "α", corresponding to the incident wave in a channel "β":

$$\Psi_{\alpha\beta}(\mathcal{K},0)=0\,;\quad \Psi_{\alpha\beta}\underset{r\to\infty}{\sim} e^{-ik_\alpha r}\,\delta_{\alpha\beta} - S_{\alpha\beta}(\mathcal{K})\,e^{ik_\alpha r}\,, \tag{4}$$

$S_{\alpha\beta}$ are elements of the scattering matrix S, $\mathcal{K}$ is a diagonal matrix of channel wave numbers k_α: $\mathcal{K} \equiv \{ \mathcal{K}_{\alpha\beta} \equiv k_\alpha \delta_{\alpha\beta} \}$; $E_\alpha = k_\alpha^2 = k_1^2 - \Delta_\alpha^2$ is a kinetic energy of the asymptotic motion in the channel "α"; Δ_α^2 is the threshold energy at which the channel "α" opens.

Besides $\Psi_{\alpha\beta}$, we will use the matrix of Jost solutions $F(\mathcal{K},r)$ corresponding to asymptotic conditions

$$\lim_{r\to\infty} e^{-ik_\alpha r} F_{\alpha\beta}(\mathcal{K},r) = \delta_{\alpha\beta}\,. \tag{5}$$

The values of Jost solutions at $r=0$ determine the Jost matrix $F_{\alpha\beta}(\mathcal{K}) = F_{\alpha\beta}(\mathcal{K},0)$ connected with the scattering matrix:

$$S(\mathcal{K}) = \mathcal{K}^{1/2} F(-\mathcal{K})F(\mathcal{K})\mathcal{K}^{-1/2}\,. \tag{6}$$

Solutions $F(\mathcal{K},r)$ corresponding to a potential matrix $V(r)\neq 0$ can be obtained from solutions $\overset{\circ}{F}_{\alpha\beta}(\mathcal{K},r) = e^{ik_\alpha r}\,\delta_{\alpha\beta}$ of the free motion by the integral transformation (a generalized shift):

$$F_{\alpha\beta}(\mathcal{K},r) = e^{ik_\alpha r}\,\delta_{\alpha\beta} + \int_r^\infty K_{\alpha\beta}(r,r')\,e^{ik_\beta r'}dr'\,, \tag{7}$$

where kernels $K_{\alpha\beta}(r,r')$ of transformation (7) are determined by multichannel integral equations of the inverse problem (Marchenko):

$$K_{\alpha\beta}(r,r') + Q_{\alpha\beta}(r,r') + \sum_\gamma \int_r^\infty K_{\alpha\gamma}(r,r'')\,Q_{\gamma\beta}(r'',r')\,dr'' = 0\,, \tag{8}$$

and kernels $Q_{\alpha\beta}$ of equations (8) can be determined by the scattering matrix $S_{\alpha\beta}$, bound state energies $E_\alpha^\lambda = -(\kappa_\alpha^\lambda)^2$ and by the corresponding normalization constants $M_{\alpha\beta}^\lambda$:

$$Q_{\alpha\beta}(r,r') = \frac{1}{2\pi}\int_{-\infty}^{\infty} k_\alpha^{-1/2}\,(\delta_{\alpha\beta}-S_{\alpha\beta})\,e^{i(k_\alpha r + k_\beta r')}\,k_\beta^{1/2}\,k_1\,dk_1 + \sum_\lambda e^{-\kappa_\alpha^\lambda r}\,M_{\alpha\beta}^\lambda\,e^{-\kappa_\beta^\lambda r'}\,. \tag{9}$$

The cases when (8), (9) have exact solutions, which are especially interesting for practical applications, are considered in the next section.

3. INTERACTION MATRICES OF THE BARGMANN TYPE

The simplest exactly solvable models were derived in the Gelfand-Levitan approach when the continuous spectrum does not contribute to kernels Q of inverse problem equations /1,6/*. But in the Marchenko approach it turned out to be more easy to construct the solutions for which only resonance singularities of S contribute to kernels Q

Let a system have no bound states and kernels $Q_{\alpha\beta}$ in (9) have the factorized dependence on coordinates r, r' and channel indices α, β

$$Q_{\alpha\beta}(r,r') = \overset{\circ}{F}_{\alpha\alpha}(\mathcal{K}^\nu,r)\,\Gamma_\alpha\,\Gamma_\beta\,\overset{\circ}{F}_{\beta\beta}(\mathcal{K}^\nu,r')\,, \tag{10}$$

where $\mathcal{K}^\nu$ is the value of $\mathcal{K}$ corresponding to a pole of the S - matrix on the imaginary axis in the complex k_1-plane.

The solution of Marchenko equations with such kernels can be constructed in the form

$$K_{\alpha\alpha'} = \sum_\beta^N F_{\alpha\beta}(\mathcal{K}^\nu,r)\,\Gamma_\beta\,\Gamma_{\alpha'}\,\overset{\circ}{F}_{\alpha'\alpha'}(\mathcal{K}^\nu,r')\,. \tag{11}$$

We find $\sum_\beta F_{\alpha\beta}(\mathcal{K}^\nu,r)\,\Gamma_\beta$, substituting K and Q from (11) and (10) into (8)

$$\sum_\rho^N F_{\alpha\beta}(\mathcal{K}^\nu,r)\Gamma_\beta = -\frac{\overset{\circ}{F}_{\alpha\alpha}(\mathcal{K}^\nu,r)\,\Gamma_\alpha}{1+\sum_\gamma \int_r^\infty \Gamma_\gamma^2\,\overset{\circ}{F}{}^2_{\gamma\gamma}(\mathcal{K}^\nu,r')dr'} = -\frac{e^{-\kappa_\alpha^\nu r}\,\Gamma_\alpha}{1+\sum_\gamma \Gamma_\gamma^2\,\dfrac{e^{-2\kappa_\gamma^\nu r}}{2\kappa_\gamma^\nu}}\,, \tag{12}$$

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*The integral term in (9) cannot be omitted restricting the contribution into Q by discrete states only, because this corresponds to the choice $S_{\alpha\beta}=1$ and violates the Levinson theorem.

So $K_{\alpha\alpha'}$ is also determined. The potential matrix can be expressed using these $K_{\alpha\alpha'}$

$$V_{\alpha\alpha'}(r) = 2\frac{d}{dr}\left\{\frac{e^{-\kappa_\alpha^\nu r}\,\Gamma_\alpha\,\Gamma_{\alpha'}\,e^{-\kappa_{\alpha'}^\nu r}}{1+\sum_\gamma \Gamma_\gamma^2\, e^{-2\kappa_\gamma^\nu r}/2\kappa_\gamma^\nu}\right\} \qquad (13)$$

and according to (7) we have for solution $F_{\alpha\beta}(K,r)$:

$$F_{\alpha\beta}(K,r) = e^{ik_\alpha r}\,\delta_{\alpha\beta} + \frac{e^{-\kappa_\alpha^\nu r}\,\Gamma_\alpha\,\Gamma_\beta\, e^{(-\kappa_\beta^\nu + ik_\beta)r}}{(1+\sum_\gamma \Gamma_\gamma^2\, e^{-2\kappa_\gamma^\nu r}/2\kappa_\gamma^\nu)\,(ik_\beta - \kappa_\beta^\nu)}. \qquad (14)$$

The validity of formulae (13), (14) can be verified by direct substitution of $V_{\alpha\alpha'}$ and $F_{\alpha\beta}$ into the system (3).

Using these $F_{\alpha\beta}(K,r)$ we can find the Jost funtion

$$F_{\alpha\beta}(K) = \delta_{\alpha\beta} - \frac{\Gamma_\alpha\,\Gamma_\beta}{(\kappa_\beta^\nu - ik_\beta)(1+\sum(\Gamma_\gamma^2/2\kappa_\gamma^\nu))}, \qquad (15)$$

and we get the scattering matrix, substituting (15) into (6).

In a more general case of M resonance singularities, the expressions for Q and K in the right-hand sides of eqs. (10), (11) have to be summarised over ν. Then a system of M algebraic equations has to be solved. So, we get K, which determines $V_{\alpha\beta}(r)$ and $F_{\alpha\beta}(K,r)$, from (13), (14), we get $F_{\alpha\beta}(K)$ from $F_{\alpha\beta}(K,r)$ and $S_{\alpha\beta}(K)$ is determined from (6);

$$F_{\alpha\beta}(K,r) = \breve{F}_{\alpha\alpha}(K,r)\,\delta_{\alpha\beta} + \sum_{\nu\nu'} P^{-1}_{\nu\nu'}(r)\,\breve{F}_{\alpha\alpha}(K^\nu,r)\,\Gamma_\alpha^{\nu'}\,\Gamma_\beta^\nu \int_r \breve{F}_{\beta\beta}(K^\nu,r')\times$$

$$\times \mathring{F}_{\beta\beta}(K,r')\,dr' = e^{ikr}\,\delta_{\alpha\beta} - \sum P^{-1}_{\nu\nu'}(r)\,e^{-\kappa_\alpha^{\nu'} r}\,\Gamma_\alpha^{\nu'}\Gamma_\beta^\nu\,\frac{e^{(ik_\beta - \kappa_\beta^\nu)r}}{ik_\beta - \kappa_\beta^\nu}; \qquad (16)$$

$$V_{\alpha\beta}(r) = -2\frac{d}{dr}\left\{\sum_{\nu\nu'}^{\mu} P^{-1}_{\nu\nu'}(r)\,e^{-(\kappa_\alpha^{\nu'}+\kappa_\beta^\nu)r}\right\};$$

$$F_{\alpha\beta}(K) = \delta_{\alpha\beta} - \sum_{\nu\nu'}^{\mu} P^{-1}_{\nu\nu'}(0)\,F_\alpha^{\nu'}\,\Gamma_\beta^\nu\,\frac{1}{ik - \kappa_\beta^\nu}.$$

where

$$P_{\nu\nu'}(r) = \delta_{\nu\nu'} + \frac{\Gamma_\gamma^\nu\, e^{-(\kappa_\gamma^\nu + \kappa_\gamma^{\nu'})r}}{\kappa_\gamma^\nu + \kappa_\gamma^{\nu'}}.$$

4. CONCLUSION

Results of investigations on the theory of Bargmann potentials and applications of the corresponding technique can be classified according to the following scheme

Systems / Solutions of inverse problem		Single channel: l=const	Single channel: E=const	Multichannel
exact				+
approximate	for known potentials	+		?
approximate	for particular nuclear systems			

Crosses signify the domains to which there correspond the results of the present article. The completely shaded cells represent the areas where a lot of investigations were performed (see review [1]). Until now no results exist concerning the estimation of errors of solutions of multichannel inverse problems (the cell with a question-mark).

The reconstruction of forces for particular nuclear systems with a coupling of channels (the lowest right corner of the scheme) was made only in the case when thresholds of all the channels are equal to each other (tensor and ℓ-s interactions of nucleons [3,9]). About the fitting of parameters of S^B for approximation of S see papers [9,10].

The authors are grateful to I.V.Amirkhanov, V.N.Pivovarchik and A.A.Suzko for useful discussions concerning the subject of this paper.

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Received by Publishing Department
on July 7 1982.

Никишов П.Ю., Плеханов Е.Б., Захарьев Б.Н. E4-82-525
О точных решениях задач рассеяния

Даны примеры, характеризующие качество восстановления потенциалов по данным одноканального рассеяния с помощью точно решаемых моделей. Найдены простые точные решения для многоканальных систем с невырожденными резонансными особенностями матрицы рассеяния.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1982

Nikishov P.Yu., Plekhanov E.B., Zakhariev B.N. E4-82-525
On Exact Solutions of Scattering Problems

Examples illustrating the quality of the reconstruction of potentials from single-channel scattering data by using exactly solvable models are given. Simple exact solutions for multi-channel systems with non-degenerated resonance singularities of the scattering matrix are derived.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1982