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исследований
дубна

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E.B.Balbutsev, I.N.Mikhailov, Z.Vaishvila

MACROSCOPIC DESCRIPTION OF NORMAL QUADRUPOLE OSCILLATIONS AND SHAPE OF ROTATING NUCLEI (SPHEROIDS)

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1. INTRODUCTION

The dynamics of the processes in the nuclear systems with very high angular momenta is a subject of a considerable interest because of its relation to the heavy-ion reactions. Here we study the simplest questions concerning it: (i) the description of the "secular equilibrium" of rotating nuclei or, in the language of nuclear physics, description of the nuclear yrast states and (ii) the analyses of small oscillations around the equilibrium configuration (i.e., of collective nuclear modes).

Both these questions have been studied earlier: the shape of rotating nuclei is studied, e.g., in refs.[1-3] using in some or another way the nuclear liquid drop model (LDM); the collective modes in rotating nuclei are treated in the random phase approximation (RPA) in refs.[4,5]. In contrast to earlier papers on the subject both parts of the problem are treated here on the basis of the same model which has some semblance of RPA and of LDM. It was developed for studying the nuclear structure at low spins in refs.[6-8] and given a name "distorted-Fermi-surface model". Its formulation starts with the introduction of a distribution function $f(\vec{r},\vec{v},t)$ giving the probability of finding a nucleon inside the nucleus at time t near the point $\vec{r}$ with the velocity close to $\vec{v}$. Then the kinetic equation of a Vlasov type for the distribution function is formulated. Then the moments in nuclear velocities are taken from the kinetic equation to generate equations for the nuclear density $n(\vec{r},t) = \int f(\vec{r},\vec{v},t)\,d\vec{v}$, and current $\vec{j}(\vec{r},t) = \int f(\vec{r},\vec{v},t)\,\vec{v}\,d\vec{v}$. To solve the equations which ensue we use the mathematical methods developed in the theory of rotating self-gravitating masses[9].

2. THE MOMENTS OF KINETIC EQUATION

We generalize the methods[7,8] and start from the Vlasov equation in the reference frame rotating with angular velocity $\vec{\Omega}$:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i} - \sum_{i=1}^{3} \frac{\partial f}{\partial v_i}\left\{\frac{1}{m}\cdot\frac{\partial U}{\partial x_i} + [\dot{\vec{\Omega}},\vec{r}]_i + [\vec{\Omega},[\vec{\Omega},\vec{r}]]_i + 2[\vec{\Omega},\vec{v}]_i\right\} = 0. \qquad (1)$$

Here $\mathcal{U}(\vec{r})$ is the potential of the Coulomb and nuclear forces; $[\vec{a},\vec{b}]$ is the vector product, m is the nucleonic mass, $\dot{a} \equiv \frac{\partial a}{\partial t}$. The multiplication of Eq.(1) by 1, v_i, $v_i v_j$,... and integration over $\vec{v}$ leads, respectively, to the first, second, third,... moment.

The first moment coincides with the equation of continuity

$$\frac{\partial n}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(n u_i) = 0,$$

where $u_i(\vec{r},t)$ is an i-th component of the mean nucleonic velocity: $u_i(\vec{r},t) = \frac{1}{n}\int v_i f(\vec{r},\vec{v},t)d\vec{v}$.

The second moment is known as the equation of motion:

$$\frac{\partial}{\partial t}(nu_i) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\frac{1}{m}P_{ij} + n u_i u_j) +$$
$$+ n\{\frac{1}{m}\frac{\partial \mathcal{U}}{\partial x_i} + [\dot{\vec{\Omega}},\vec{r}]_i + [\vec{\Omega},[\vec{\Omega},\vec{r}]]_i + 2[\vec{\Omega},\vec{u}]_i\} = 0, \quad (2)$$

where

$$P_{ij}(\vec{r},t) = m\int w_i w_j f(\vec{r},\vec{v},t)d\vec{v}$$

is the pressure tensor, $w_i = v_i - u_i$.

The third moment is the equation of energy-mass transfer:

$$\frac{dP_{ij}}{dt} + \sum_{k=1}^{3}(P_{ij}\frac{\partial u_k}{\partial x_k} + P_{jk}\frac{\partial u_i}{\partial x_k} + P_{ik}\frac{\partial u_j}{\partial x_k}) +$$
$$+ 2\sum_{k,\ell=1}^{3}\Omega_\ell(\epsilon_{i\ell k}P_{kj} + \epsilon_{j\ell k}P_{ki}) + m\sum_{k=1}^{3}\frac{\partial}{\partial x_k}\int w_i w_j w_k f(\vec{r},\vec{v},t)d\vec{v} = 0. \quad (3)$$

Here $\epsilon_{i\ell k}$ is the Levi-Civita symbol; $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \sum_{k=1}^{3} u_k \frac{\partial}{\partial x_k}$.

The Eqs. (2), (3) can be simplified after some natural physical assumptions. We shall neglect the integral $\int w_i w_j w_k f d\vec{v}$ as it was done in ref. /8/. Also let us consider the nucleus as a drop of the ideal incompressible liquid. Then it follows from the equation of continuity, that $\sum_{k=1}^{3}\frac{\partial u_k}{\partial x_k} = 0$. Introduce the tensor

$$\kappa_{ij} = P_{ij} - p_0\,\delta_{ij},$$

where p_0 is the pressure in the infinite nuclear matter, $p_0 = \frac{1}{3} m n_0 \langle v^2 \rangle_0$ /8/. We consider the case $p_0 \gg \kappa_{ij}$, when the distortions in the velocity distribution of nucleons introduced by the surface and Coulomb forces and by oscillations are small.

So we have simplified Eqs. (2), (3):

$$\rho \frac{du_i}{dt} + \rho \Omega^2 x_i(\delta_{i3}-1) + \sum_{j=1}^{3}(2\rho\Omega\epsilon_{i3j}u_j + \frac{\partial \kappa_{ij}}{\partial x_j}) + \frac{\rho}{m}\frac{\partial \mathfrak{U}}{\partial x_i} = 0, \qquad (4)$$

$$\frac{d\kappa_{ij}}{dt} + p_0(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + 2\Omega K_{ij} = 0, \qquad (5)$$

where $\rho = m \cdot n$; $\vec{\Omega}$ lies along x_3, and is constant; $K_{11} = -2\kappa_{12}$, $K_{12} = K_{21} = \kappa_{11} - \kappa_{22}$, $K_{13} = K_{31} = -\kappa_{23}$, $K_{22} = 2\kappa_{12}$, $K_{23} = K_{32} = \kappa_{13}$, $K_{33} = 0$.

When $\Omega = 0$, Eqs. (4) and (5) coincide with the corresponding equations of [8]. We use the method of the tensor virial[9] to solve Eqs. (4),(5).

Eq. (4) will be replaced by the virial equation of the second order, i.e., the second coordinate moment of the equation of motion:

$$\frac{d}{dt}\int \rho\, u_i x_j d\vec{r} = 2T_{ij} + \Pi_{ij} - 2\sigma_{ij} - 2\Omega\sum_{k=1}^{3}\epsilon_{i3k}\int \rho u_k x_j d\vec{r} - \Omega^2 I_{ij}(\delta_{i3}-1) + Q_{ij} + U\delta_{ij} \qquad (6)$$

where

$$T_{ij} = \frac{1}{2}\int \rho u_i u_j d\vec{r}, \quad I_{ij} = \int \rho x_i x_j d\vec{r}, \quad \Pi_{ij} = \int P_{ij}\, d\vec{r};$$

$\sigma_{ij} = \frac{1}{2}\sum_{k \neq \ell \neq m}^{3}\iint x_j P_{ik}\, dx_\ell\, dx_m$ is the surface-energy tensor;

$Q_{ij} = \frac{1}{2}\iint q(\vec{r})\, q(\vec{r}\,')\frac{(x_i - x_i')(x_j - x_j')}{|\vec{r} - \vec{r}\,'|^3} d\vec{r} d\vec{r}\,'$ is the Coulomb energy tensor, $q(\vec{r})$ is the charge density in a nucleus; $U = \frac{1}{3}\int U(\vec{r})d\vec{r}$. $U(\vec{r})$ is the average field of a nucleus. Its form follows the form of the mass distribution, so we assume $U(\vec{r})$ to be a square well. Unlike[8], we do not neglect the average field, the surface and Coulomb forces. The virial method enables us to take them exactly into account.

3. SHAPE OF ROTATING NUCLEUS

Following[10], we consider the nucleus to be an oblate spheroid with semiaxes:

$$a_1^2 = a_2^2 = a_0^2(1 + \frac{2}{3}\delta), \qquad a_3^2 = a_0^2(1 - \frac{4}{3}\delta),$$

where δ is the deformation parameter and a_0 is fixed by the volume conservation condition: $a_1 a_2 a_3 = R^3 = r_0^3 A$. $r_0 = 1.18$ fm (as in ref.[8]), A is the nucleus mass number.

The shape of a rotating nucleus in the state of secular equilibrium can be determined from Eq. (6), which under these conditions looks as

$$\Pi_{ij} - 2\sigma_{ij} + \Omega^2(1-\delta_{i3})I_{ij} + Q_{ij} + U\delta_{ij} = 0. \tag{7}$$

It is the balance of the surface, centrifugal, Coulomb and nuclear forces and of pressure. Eq. (7) gives a unique relation of semiaxes a_i to Ω.

Tensor Π_{ij} is diagonal. Really, if $\vec{u}=0$, tensor $P_{ij} = m\int v_i v_j f\, d\vec{v}$, and the function f is even in $\vec{v}$ as it follows from the definition of u_i. Hence, $P_{ij} = P_i\,\delta_{ij}$. $P_i(\vec{r}) = m\int v_i^2 f\, d\vec{v}$ is the pressure along an i-th axis, and the distribution function does not depend on time in the state of secular equilibrium. We deal with spherical nuclei - they deform due to the rotation only. There is no any distinguished direction in such a nucleus, so assume the pressure to be isotropic: $P_1 = P_2 = P_3 \equiv P$. Hence, $\Pi_{ij} = \delta_{ij}\int P(\vec{r})\,d\vec{r} \equiv \delta_{ij}\Pi$.

The pressure in the interior, immediately adjacent to the surface S is given by Laplace's formula: $P = T\,\mathrm{div}\,\vec{s}$, where $\vec{s}$ is the unit outward normal on S and T is the surface tension coefficient, which is proportional to the parameter $b \simeq 17$ MeV of the Weizsäcker formula: $T = \frac{b}{4\pi r_0^2}$. Under these conditions the surface energy tensor looks like: $\sigma_{ij} = \frac{1}{2}T\iint(\delta_{ij} - s_i s_j)\,dS$.

If we use ellipsoidal configurations, all the off-diagonal components of Eq. (7) vanish identically. The three diagonal components are:

$$\Omega^2 I_{11} = 2\sigma_{11} - Q_{11} - (\Pi + U),$$

$$\Omega^2 I_{22} = 2\sigma_{22} - Q_{22} - (\Pi + U),$$

$$0 = 2\sigma_{33} - Q_{33} - (\Pi + U).$$

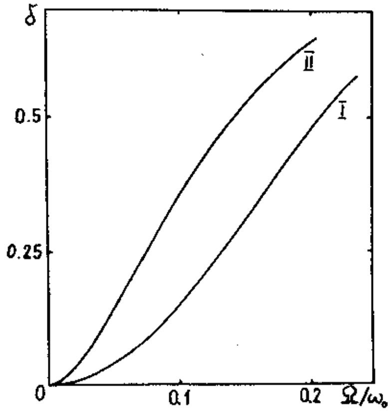
The first two are identical, as $a_1 = a_2$. The expressions for Q_{ii} and σ_{ii} can be found in the Appendix. Using the last equation to eliminate $\Pi + U$ we have the equilibrium condition

$$\Omega^2 I_{11} = 2(\sigma_{11} - \sigma_{33}) - (Q_{11} - Q_{33}), \tag{8}$$

which is identical to that for the liquid drop[10]. The substitution of σ_{ii}, Q_{ii} and $I_{ii} = \frac{4\pi R^3}{15}\rho\, a_i^2$ into Eq. (8) leads to

$$\Omega^2 = \frac{2R^3}{\rho} e^2 \left(\frac{15}{4} T\,\mathcal{A}_{13} - \pi q^2 B_{13}\right), \tag{9}$$

where $e = (1 - a_3^2/a_1^2)^{1/2}$ is an eccentricity; the expressions for $\mathcal{A}_{ij}$, B_{ij} can be found in the Appendix; $q^2 = 0.0665\,\frac{Z^2}{A^2}\frac{\mathrm{MeV}}{r_0^5}$; Z is



the number of protons in the nucleus. The typical dependence $\delta(\Omega)$ following from Eq. (9) is shown in Fig.1 for two nuclei from the β-stability line with different fissility parameter X.

Fig.1. The dependence of the deformation parameter δ on the angular velocity Ω : I - ^{168}Er(X = 0.56667); II - Z = 114, A = 300 (X = 0.89472); $\hbar\omega_0 = 41A^{-1/3}$MeV.

4. QUADRUPOLE-OSCILLATION EQUATIONS

To study normal quadrupole oscillations, let us analyse the reaction of a nucleus upon the infinitesimal perturbation. The necessary information can be found from the Lagrangian variation of Eq. (6):

$$\frac{d^2V_{i,j}}{dt^2} + \frac{d}{dt}\int\rho(u_i\xi_j - u_j\xi_i)d\vec{r} = \int\rho\left(u_j\frac{d\xi_i}{dt} + u_i\frac{d\xi_j}{dt}\right)d\vec{r} + \int\Delta\kappa_{ij}\,d\vec{r} + \tag{10}$$
$$+\Omega^2(1-\delta_{i3})V_{ij} - 2\Omega\sum_{k=1}^{3}\epsilon_{i3k}\int\rho\left(\frac{d\xi_k}{dt}x_j + u_k\xi_j\right)d\vec{r} - 2\Delta\sigma_{ij} + \Delta Q_{ij}\ .$$

Here Δ means the Lagrangian variation; $\xi_i(\vec{r},t) = \Delta x_i$, $V_{i,j}(t) = \int\rho\,\xi_i x_j d\vec{r}$, $V_{ij} = V_{i,j} + V_{j,i}$, $\Delta U(\vec{r}) = 0$.

We are interested in small oscillations about the equilibrium shape. So, $\vec{u} = 0$ and only mean velocities corresponding to small displacements ξ_i, i.e., $\dot{\xi}_i$ remain nonvanishing:

$$\frac{\partial^2 V_{i,j}}{\partial t^2} = \int\Delta\kappa_{ij}(\vec{u}=0)\,d\vec{r} + \Omega^2V_{ij}(1-\delta_{i3}) - 2\Omega\sum_{k=1}^{3}\epsilon_{i3k}\int\rho\,\dot{\xi}_k x_j d\vec{r} - 2\Delta\sigma_{ij} + \Delta Q_{ij}\ . \tag{11}$$

$\Delta\kappa_{ij}(\vec{u}=0)$ can be found from the Lagrangian variation of Eq.(5):

$$\frac{\partial}{\partial t}\Delta\kappa_{ij} = -p_0\frac{\partial}{\partial t}\left(\frac{\partial\xi_i}{\partial x_j} + \frac{\partial\xi_j}{\partial x_i}\right) - 2\Omega\Delta K_{ij}\ .$$

The Laplace transformation is a convenient method to solve this system:

$$\lambda^2\tilde{V}_{i,j} = \int\Delta\tilde{\kappa}_{ij}\,d\vec{r} + \Omega^2(1-\delta_{i3})\tilde{V}_{ij} - 2\lambda\Omega\sum_{k=1}^{3}\epsilon_{i3k}\int\rho\tilde{\xi}_k x_j\,d\vec{r} -$$
$$-2\Delta\tilde{\sigma}_{ij} + \Delta\tilde{Q}_{ij} + 2\Omega\sum_{k=1}^{3}\epsilon_{i3k}V_{k,j}(0) + \lambda V_{i,j}(0) + \dot{V}_{i,j}(0), \tag{12}$$

$$\lambda\tilde{\Delta\kappa}_{ij} - \Delta\kappa_{ij}(\vec{r},0) = -p_0\lambda\left(\frac{\partial\tilde{\xi}_i}{\partial x_j} - \frac{\partial\tilde{\xi}_j}{\partial x_i}\right) + p_0\left(\frac{\partial\xi_i(\vec{r},0)}{\partial x_j} + \frac{\partial\xi_j(\vec{r},0)}{\partial x_i}\right) - 2\Omega\Delta\tilde{K}_{ij}, \quad (13)$$

where $\tilde{F}(x,\lambda) = \int_0^\infty F(x,t)\, e^{-\lambda t}\, dt$. The solution of system (13) gives the formulas for $\tilde{\Delta\kappa}_{ij}$ in terms of $\tilde{\xi}_k$. The substitution of them into Eq. (12) leads to the system of nine equations for $\tilde{V}_{ij}$.

The frequencies of normal modes of oscillations do not depend upon the initial conditions, so we suppose $\xi_i(\vec{r},0) = \dot{\xi}_i(\vec{r},0) = 0$ and, hence, $V_{i,j}(0) = \dot{V}_{i,j}(0) = 0$. $\Delta\kappa_{ij}(\vec{r},0)$ requires a special analysis. It was shown that $P_{ij}(\vec{r},t) = \delta_{ij} P(\vec{r})$ if $\vec{u} = 0$. So, we have by definition:

$$\Delta\kappa_{ij}(\vec{r},0) = \Delta P_{ij}(\vec{r},0) = P'_{ij}(\vec{r}+\vec{\xi},0) - P_{ij}(\vec{r},0) = \delta_{ij}[P'(\vec{r}+\vec{\xi}) - P(\vec{r})] = \delta_{ij}\Delta P(\vec{r}).$$

Here $P'_{ij}(\vec{r},0)$ is the pressure tensor, corresponding to the perturbed distribution function $f'(\vec{r},\vec{v},0)$, which is even due to the condition $\dot{\xi}_i(\vec{r},0) = 0$. The specification of $\Delta P(\vec{r})$ in terms of $\vec{\xi}$ requires some supplementary assumtpion concerning the physical nature of the oscillations. For the incompressible fluid $\vec{\xi}$ is required to be solenoidal in order to preserve the total volume: $\mathrm{div}\vec{\xi} = 0$. If we supplement the system (12) with this equation, we may dispense with the evaluation of $\Delta P(\vec{r})$ and eliminate it from the system.

We restrict ourselves to the quadrupole oscillations, so the Lagrangian displacement can be parametrized [10]:

$$\xi_i = \sum_{j=1}^{3} L_{i,j}\, x_j \,, \quad (14)$$

where $L_{i,j}$ are nine unknown functions of time (their number is equal to the number of equations in system (12)). It is easy to show that $L_{i,j} = \frac{5}{M a_j^2} V_{i,j}$ for ellipsoids, where $M = mA$.

With all simplifications, system (13) becomes:

$$\lambda\tilde{\Delta\kappa}_{ij}(\vec{r},\lambda) = -\lambda\mu_{ij} - 2\Omega\,\Delta\tilde{K}_{ij}(\vec{r},\lambda), \quad (15)$$

where $\mu_{ij} = \frac{5p_0}{M}\left(\frac{\tilde{V}_{i,j}}{a_j^2} + \frac{\tilde{V}_{j,i}}{a_i^2}\right)$. The right-hand side of the equation is symmetric in the indices i,j , hence, $\tilde{\Delta\kappa}_{ij}$ has the same symmetry:

$$\tilde{\Delta\kappa}_{12} = \tilde{\Delta\kappa}_{21} = \frac{2\Omega\lambda(\mu_{11}-\mu_{22}) - \lambda^2\mu_{12}}{\lambda^2 + 16\Omega^2}, \qquad \lambda^2 \neq -16\Omega^2$$

$$\tilde{\Delta\kappa}_{11} - \tilde{\Delta\kappa}_{22} = -\frac{8\Omega\lambda\mu_{12} + \lambda^2(\mu_{11}-\mu_{22})}{\lambda^2 + 16\Omega^2}, \qquad \lambda^2 \neq -16\Omega^2$$

$$\Delta\tilde{\kappa}_{11}+\Delta\tilde{\kappa}_{22} = 2\Delta P(\vec{r})/\lambda-\mu_{11}-\mu_{22}, \qquad \lambda\neq 0$$

$$\Delta\tilde{\kappa}_{31}=\Delta\tilde{\kappa}_{13}=-\frac{2\Omega\lambda\mu_{23}+\lambda^2\mu_{13}}{\lambda^2+4\Omega^2}, \qquad \lambda^2\neq -4\Omega^2$$

$$\Delta\tilde{\kappa}_{32}=\Delta\tilde{\kappa}_{23}=\frac{2\Omega\lambda\mu_{13}-\lambda^2\mu_{23}}{\lambda^2+4\Omega^2}, \qquad \lambda^2\neq -4\Omega^2$$

$$\Delta\tilde{\kappa}_{33}=\Delta P(\vec{r})/\lambda-\mu_{33}, \qquad \lambda\neq 0$$

By taking appropriate linear combinations of equations (12) three noncombining systems can be formed. The first two contain the functions $V_{i,j}$, which are even with respect to the 180° rotation of the reference frame about axis x_3. They determine the positive signature[11] modes of oscillations:

$$\left(\frac{\lambda^2}{2}-\Omega^2\right)(\tilde{V}_{11}-\tilde{V}_{22})-\int(\Delta\tilde{\kappa}_{11}-\Delta\tilde{\kappa}_{22})d\vec{r}-2\Omega\lambda\tilde{V}_{12}+2(\Delta\tilde{\sigma}_{11}-\Delta\tilde{\sigma}_{22})-(\Delta\tilde{Q}_{11}-\Delta\tilde{Q}_{22})=0,$$

$$\lambda^2\tilde{V}_{12}-2\int\Delta\tilde{\kappa}_{12}\,d\vec{r}+\Omega\lambda(\tilde{V}_{11}-\tilde{V}_{22})-2\Omega^2\tilde{V}_{12}+4\Delta\tilde{\sigma}_{12}-2\Delta\tilde{Q}_{12}=0, \qquad (16)$$

$$\frac{\lambda^2}{2}(\tilde{V}_{11}+\tilde{V}_{22}-2\tilde{V}_{33})-\int(\Delta\tilde{\kappa}_{11}+\Delta\tilde{\kappa}_{22}-2\Delta\tilde{\kappa}_{33})d\vec{r}+2\Omega\lambda(\tilde{V}_{1,2}-\tilde{V}_{2,1})-$$

$$-\Omega^2(\tilde{V}_{11}+\tilde{V}_{22})+2(\Delta\tilde{\sigma}_{11}+\Delta\tilde{\sigma}_{22}-2\Delta\tilde{\sigma}_{33})-(\Delta\tilde{Q}_{11}+\Delta\tilde{Q}_{22}-2\Delta\tilde{Q}_{33})=0 \qquad (17)$$

$$\lambda^2(\tilde{V}_{1,2}-\tilde{V}_{2,1})-\Omega\lambda(\tilde{V}_{11}+\tilde{V}_{22})=0,$$

$$\frac{\lambda^2}{2}\tilde{V}_{33}-\int\Delta\tilde{\kappa}_{33}d\vec{r}+2\Delta\tilde{\sigma}_{33}-\Delta\tilde{Q}_{33}=0.$$

And the last system determines the negative signature modes:

$$\lambda^2\tilde{V}_{1,3}-\int\Delta\tilde{\kappa}_{13}d\vec{r}-2\Omega\lambda\tilde{V}_{2,3}-\Omega^2\tilde{V}_{13}+2\Delta\tilde{\sigma}_{13}-\Delta\tilde{Q}_{13}=0,$$

$$\lambda^2\tilde{V}_{2,3}-\int\Delta\tilde{\kappa}_{23}d\vec{r}+2\Omega\lambda\tilde{V}_{1,3}-\Omega^2\tilde{V}_{23}+2\Delta\tilde{\sigma}_{23}-\Delta\tilde{Q}_{23}=0,$$

$$\lambda^2\tilde{V}_{3,1}-\int\Delta\tilde{\kappa}_{31}d\vec{r}+2\Delta\tilde{\sigma}_{31}-\Delta\tilde{Q}_{31}=0, \qquad (18)$$

$$\lambda^2 V_{3,2}-\int\Delta\tilde{\kappa}_{32}d\vec{r}+2\Delta\tilde{\sigma}_{32}-\Delta\tilde{Q}_{32}=0.$$

Expressions for $\Delta\tilde{\sigma}_{ij}$ and $\Delta\tilde{Q}_{ij}$ are shown in the Appendix.

It is useful to note that Eqs. (16)-(18) can be used to describe the quadrupole oscillations of classical charged liquid drop, if one supposes $p_0 = 0$.

In the theory of self-gravitating masses [9] the normal modes of oscillations determined by the systems (16), (17), (18) are known as the toroidal, pulsation and transverse-shear modes, respectively. In nuclear physics the first two are known as the γ- and β-modes; we shall name the third one the a-mode.

Insertion of $\Delta\tilde{\kappa}_{ij}$, $\Delta\tilde{\sigma}_{ij}$ and $\Delta\tilde{Q}_{ij}$ into system (16) gives:

$$(\tilde{V}_{11}-\tilde{V}_{22})\cdot\left(\frac{\lambda^2}{2}+\frac{\lambda^2\mu}{\lambda^2+16\Omega^2}-\Omega^2+D\right)+\tilde{V}_{12}\left(\frac{8\Omega\lambda\mu}{\lambda^2+16\Omega^2}-2\Omega\lambda\right)=0,$$

$$(\tilde{V}_{11}-\tilde{V}_{22})\left(\Omega\lambda-\frac{4\Omega\lambda\mu}{\lambda^2+16\Omega^2}\right)+\tilde{V}_{12}\left(\lambda^2+\frac{2\lambda^2\mu}{\lambda^2+16\Omega^2}-2\Omega^2+2D\right)=0.$$

Here $\mu=\frac{5p_0}{\rho a_1^2}=\left(\frac{\hbar}{2mr_0^2}\right)^2\left(\frac{9\pi}{A}\right)^{2/3}$, $D\equiv\frac{2R^3}{\rho}\left(\frac{15}{4}T\mathcal{A}_{11}-\pi q^2B_{11}\right)$.

The characteristic equation of this system is

$$\left(\frac{\lambda^2}{2}+\frac{\lambda^2\mu}{\lambda^2+16\Omega^2}-\Omega^2+D\right)^2+\Omega^2\lambda^2\left(\frac{4\mu}{\lambda^2+16\Omega^2}-1\right)=0.$$

If we replace λ^2 by $-\omega^2$ (the real ω is a necessary and sufficient condition for the stability of modes), two equations follow for the frequencies of γ-modes:

$$\omega^3\pm\omega^2\cdot 6\Omega+2\omega(5\Omega^2-D-\mu)\mp 8\Omega(D-\Omega^2)=0. \qquad (19)$$

The roots of these two equations differ only in sign and are physically identical. So, we deal with positive ω only. The dependence of three real roots of Eq. (19) is shown in Fig.2 (γ-curves). Two roots of Eq. (19) become complex at $\Omega=\Omega_{cr}$, which is the least root of the equation

$$2(\Omega^2+\mu+D)^3=27\Omega^2(\Omega^2+\mu-D)^2. \qquad (20)$$

Hence, every infinitesimal perturbation of the configuration will increase exponentially in time if $\Omega>\Omega_{cr}$, i.e., the motion becomes unstable. This is the reason we break off all the curves at $\Omega=\Omega_{cr}$. One of γ-modes becomes neutral ($\omega=0$) at $\Omega=D^{1/2}$ (see a part of Fig.2a in a larger scale). It is a point of bifurcation. The continuous transition from the sequence of spheroids to that of ellipsoids is possible here [10]. The location of the point of bifurcation does not depend on the Fermi-momentum and is determined by the Coulomb and surface forces as in the liquid-drop model.

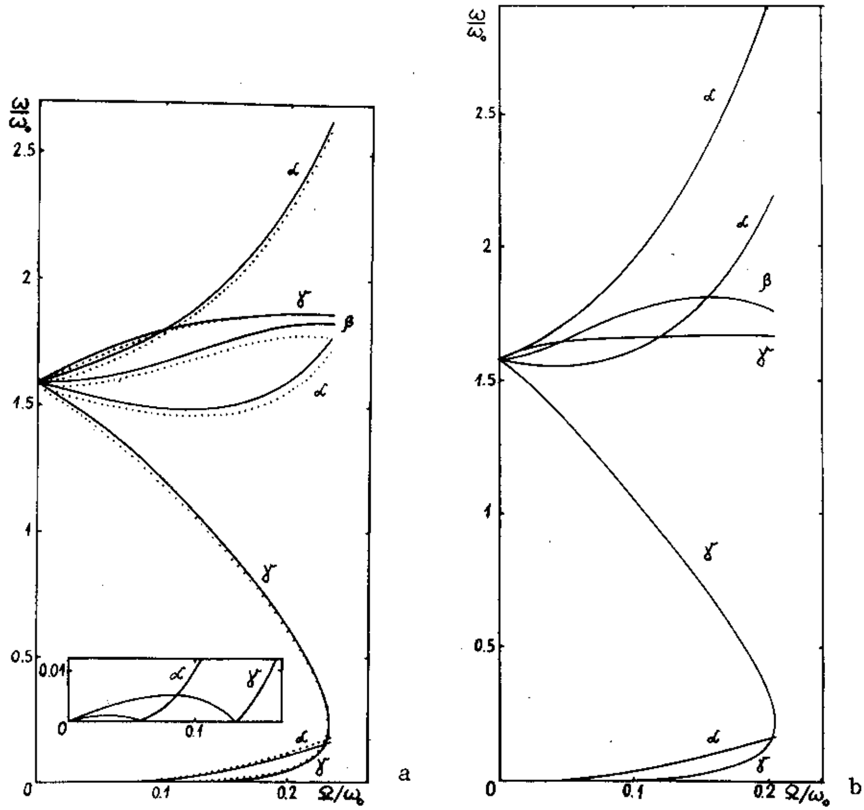


Fig.2. The dependence of the α-, β-, γ -oscillation spectrum of nuclei on the angular velocity Ω : a - ^{168}Er ($X = 0.56667$), b - $Z \simeq 114$, $A \approx 300$ ($X = 0.89472$); $\hbar\omega_0 = 41A^{-1/3}$ MeV.

Let us substitute $\Delta\tilde{\kappa}_{ij}$, $\Delta\tilde{\sigma}_{ij}$ and $\Delta\tilde{Q}_{ij}$ into system (17). The last equation involves $\Delta\tilde{P}(\vec{r})$. We omit it to supplement the system with the solenoidal requirement (see page 6), which looks as $\sum_{j=1}^{3} V_{jj}/a_j^2 = 0$ for ellipsoids. So, we have

$$(\tilde{V}_{11}+\tilde{V}_{22})\left(\frac{\lambda^2}{2}+\mu-\Omega^2+E\right)-\tilde{V}_{33}(\lambda^2+2\mu\nu+G)+(\tilde{V}_{1,2}-\tilde{V}_{2,1})2\Omega\lambda=0,$$

$$(\tilde{V}_{11}+\tilde{V}_{22})\cdot\Omega\lambda-(\tilde{V}_{1,2}-\tilde{V}_{2,1})\cdot\lambda^2=0,$$

$$(\tilde{V}_{11}+\tilde{V}_{22})+\nu\tilde{V}_{33}=0.$$

Here $\nu \equiv a_1^2/a_3^2$, $E \equiv \frac{2R^3}{\rho}\left[\frac{15T}{4a_1^2}(\mathcal{A}_1+\mathcal{B}_{13}-2\mathcal{B}_{11})-\pi q^2 B_{11}\right]$,

$G \equiv \frac{2R^3}{\rho}[\frac{15T}{4a_3^2}(2\mathcal{A}_3 - 3\mathcal{B}_{33} + \mathcal{B}_{31}) - 2\pi q^2 B_{33}]$. The characteristic equation of the system gives the formula for the frequencies of β-modes:

$$\omega^2 = -\lambda^2 = 2(3\mu + \Omega^2 + E + G/\nu)/(1+2/\nu) \qquad (21)$$

(β -curve in Fig.2).

Insertion of $\tilde{\Delta\kappa}_{ij}$, $\tilde{\Delta\sigma}_{ij}$ and $\Delta\tilde{Q}_{ij}$ into system (18) yields:

$$\lambda^2\tilde{V}_{1,3} + \frac{\lambda\mu}{\lambda^2+4\Omega^2}[\lambda(\nu\tilde{V}_{1,3}+\tilde{V}_{3,1})+2\Omega(\nu\tilde{V}_{2,3}+\tilde{V}_{3,2})] - 2\Omega\lambda\tilde{V}_{2,3} + (H-\Omega^2)\tilde{V}_{13} = 0$$

$$\lambda^2\tilde{V}_{2,3} + \frac{\lambda\mu}{\lambda^2+4\Omega^2}[\lambda(\nu\tilde{V}_{2,3}+\tilde{V}_{3,2})-2\Omega(\nu\tilde{V}_{1,3}+\tilde{V}_{3,1})] + 2\Omega\lambda\tilde{V}_{1,3} + (H-\Omega^2)\tilde{V}_{23} = 0$$

$$\lambda^2\tilde{V}_{3,1} + \frac{\lambda\mu}{\lambda^2+4\Omega^2}[\lambda(\nu\tilde{V}_{1,3}+\tilde{V}_{3,1})+2\Omega(\nu\tilde{V}_{2,3}+\tilde{V}_{3,2})] + H\tilde{V}_{13} = 0$$

$$\lambda^2\tilde{V}_{3,2} + \frac{\lambda\mu}{\lambda^2+4\Omega^2}[\lambda(\nu\tilde{V}_{2,3}+\tilde{V}_{3,2})-2\Omega(\nu\tilde{V}_{1,3}+\tilde{V}_{3,1})] + H\tilde{V}_{23} = 0,$$

where $H \equiv \frac{2R^3}{\rho}(\frac{15}{4}T\mathcal{A}_{13} - \pi q^2 B_{13})$. The characteristic equation of this system is:

$$\{\omega^6 - \omega^4[5\Omega^2 + 4H + 2\mu(1+\nu)] + \omega^2[(\mu(1+\nu)+2H)^2 + 2\Omega^2(2\Omega^2 + 8H + \mu - 5\mu\nu)] - \Omega^2[\mu(1-\nu)+4H]^2\}(\omega^2-\Omega^2)\omega^2 = 0. \qquad (22)$$

The roots $\omega^2=0$ and $\omega^2=\Omega^2$ have been analysed in ref.[10] – they do not indicate instability. The Ω dependence of the rest of pisitive roots is shown in Fig.2 (α-curves). One α-mode becomes neutral ($\omega=0$), when $4H=\mu(\nu-1)$ (see the fragment of Fig.2a in a larger scale). The form of Eq. (22) implies, that ω^2 does not change sign, when the function $4H+\mu(1-\nu)$ changes sign. Therefore, no instability occurs at this point.

5. GIANT QUADRUPOLE RESONANCE (GQR)

In a nonrotating nucleus all nonzero roots of Eqs. (20)-(22) are equal to

$$\omega_{sph} = \sqrt{2(\mu + D)} = \sqrt{2}\left(\mu + \frac{12T}{a_0^3 \rho} - \frac{8\pi q^2}{15\rho}\right)^{1/2}. \qquad (23)$$

If one neglects the surface and Coulomb forces, the energy of quadrupole oscillations

$$E_2 = \hbar\omega_{sph} = \frac{\hbar^2}{\sqrt{2} m r_0^2}\left(\frac{9\pi}{A}\right)^{1/3} = \frac{64.7}{A^{1/3}}\ \mathrm{MeV}$$

is the same as in works /7,8/ and is in fine agreement with the GQR experiment (Fig.3). One can see that the Coulomb and surface forces give rather small effect (the agreement with experiment can be improved by increasing r_0 slightly). Fig.2a shows, that these forces do not effect very much the GQR energy in rotating nuclei also.

The increase of Ω splits the GQR (see Fig.2). This phenomenon is analogous with the well known fact of GQR splitting in deformed nuclei. But here the deformation is not static and appears because of the rotation. The static quadrupole deformation does not eliminate the degeneration of the GQR state completely: the modes with the projection of angular momentum K and $-K$ ($|K| = 1;2$) on the axis of symmetry have the same energy. The presence of Coriolis forces eliminates this degeneration in rotating nuclei. Hence, in our model there are five GQR states every one with its own quantum number K.

Fig.4 shows the spectrum of quadrupole oscillations of the liquid drop ($p_0 = 0$ in all equations). One knows that the liquid drop model is not capable to describe quantitatively the collective states of nuclei at $\Omega = 0$ (see the dashed curve in Fig.3). But the qualitative changes of normal modes in the region of small Ω are the same as for the GQR (cf. Fig.4 and Fig.2).

6. SOFT MODES

There are two low-lying collective modes in our model. They are different in sign of signature. It is interesting to estimate these frequencies for small Ω. For the positive signature mode (γ-mode) we have:

$$\omega_\gamma \simeq 4\Omega\left(\frac{\mathring{\omega}_{sph}}{\omega_{sph}}\right)^2,$$

where ω_{sph} is the frequency of GQR in a nonrotating nucleus (Eq. (23)), and $\mathring{\omega}_{sph}$ is the quadrupole oscillation frequency of a nonrotating liquid drop.

For negative signature mode we have:

$$\omega_\alpha \simeq \Omega(1 - \nu + 4H/\mu)/(1 + \nu + 2H/\mu).$$

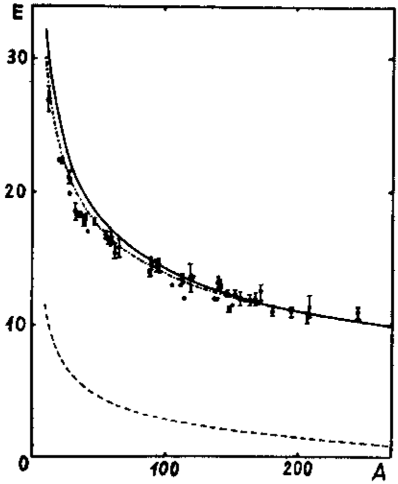
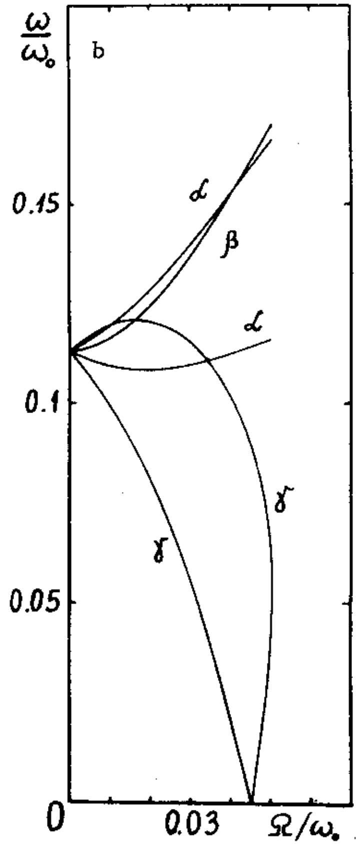


Fig.3. The GQR energy of non-rotating nuclei from the β-stability line: the solid curve - the Coulomb and surface forces are taken into account, the dott-dashed curve - without these forces. The dashed curve - the energy of quadrupole oscillations of the liquid drop. Experimental values are borrowed from [8].



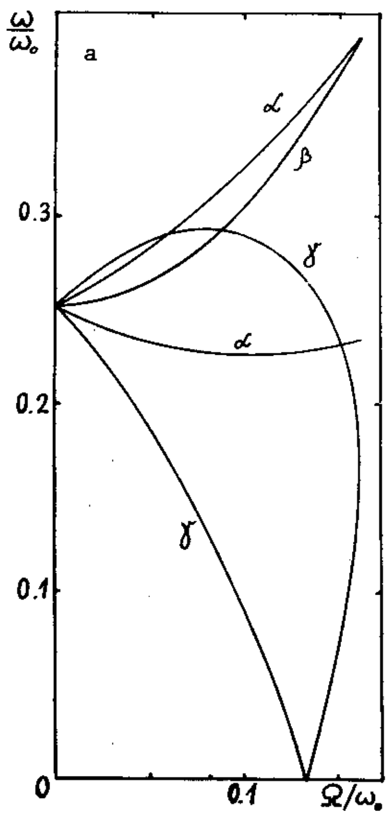


Fig.4. The dependence of the α-, β-, γ-oscillation spectrum of the liquid drop on the angular velocity Ω: a - Z=68, A=168 (X=0.56667, b - Z=114, A=300 (X=0.89472); $\hbar\omega_0 = 41\ A^{-1/3}$MeV.

The effect of the Coulomb and surface forces that determine the value of H is rather small (Fig.2a), so $H/\mu \ll 1$. If the more strict condition

$$H/\mu \ll 1-\nu \qquad (24)$$

is fulfilled, then

$$\omega_{\alpha} \simeq \Omega(1-a_3^2/a_1^2)/(1+a_3^2/a_1^2) \equiv \omega_W ,$$

where

$$\omega_W = \Omega[(J_1-J_3)(J_2-J_3)/(J_1J_2)]^{1/2}$$

is the frequency of wobbling oscillations of the rigid-body rotator with moments of inertia $J_1 \leq J_2 < J_3$. Hence, the distortions in the velocity distribution of nucleons (the Fermi-surface deformation[8]) lead to the appearance of the oscillation mode, that is typical for elastic bodies. Inequality (24) becomes wrong, when $\Omega \to 0$ because of $1-\nu \sim \Omega^2$. So, the difference between ω_{α} and ω_W can be arbitrary large for these Ω.

7. CRITICAL ANGULAR MOMENTA

The critical angular momenta, which lead to the instability of nuclei, are the object of great interest in the nuclear fission theory. The limiting values Ω_{cr} are determined in our model by Eq. (20). The analogous condition follows for the liquid drop if one puts $\mu = 0$ in this equation. For a nucleus we have $\mu \gg D$, so the region of stability with respect to the quadrupole oscillations is much larger as compared with the liquid drop. One can see in Fig.5 the angular momenta corresponding to these Ω_{cr} calculated for nuclei from the β-stability line (curve I) and for the liquid drop (curve II). The results of Cohen, Plasil and Swiatecki[1] for the axial configurations (without restriction to spheroids) of the liquid drop are shown too (dotted curve). This curve is rather close to the curve I for $A < 25$. And our curve moves rapidly away from the dotted curve to the region of big angular momenta for $A > 25$. But one must have in mind, that this region of I is the region of very large eccentricities e. It has been proved[10], that the spheroids are not expected to be adequate approximations to the exact figures of equilibrium, when $e \gtrsim 0.8745$. The dashed curve in Fig.5 corresponds to the angular momenta, at which $e = 0.8745$. It seems, we must conclude, that the instability of nuclei with respect to the fission is not connected with the instability with respect to the quadrupole oscillations.

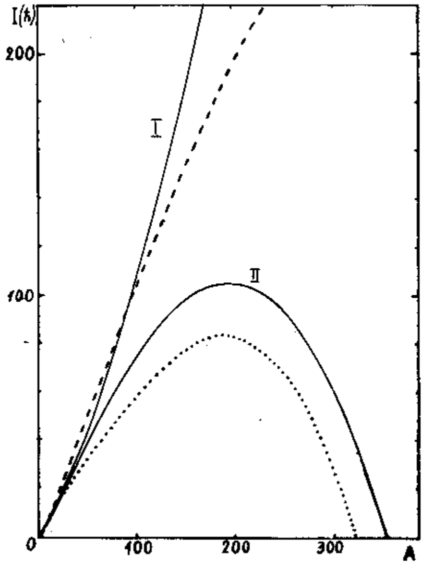


Fig.5. Critical angular momenta, where the instability with respect to the γ-oscillations appears in the nuclei from the β-stability line (curve I) and in the liquid drop with the same A, Z (curve II). The dashed curve - angular momenta, when e= 0.8745. The dotted curve - the results of [1] for the axial liquid drops.

8. SUMMARY

The results we have obtained are useful in two aspects. First of all the model enables us to predict the character of the collective modes in rotating nuclei, and second, it can be used as a basis for new approaches in describing the dynamics of the nuclear systems with high excitation energy and high angular momentum.

The model of nucleus we use is rather simple and describes perfectly the well-known experimental GQR data. Its results on the deformation of rotating nuclei are in agreement with the results of the liquid-drop model. They are reliable for heated nuclei and allow the specification with respect to the shell effects [3].

This model predicts the splitting of GQR modes in rotating nuclei, when spins are moderate, and sharp changes in the spectrum of these states, when $\Omega \to \Omega_{cr}$. The similar conclusions about GQR in rotating nuclei have been done earlier on the basis of the schematic microscopic model [4,5]. Characteristic values of Ω, which give rise to sharp changes in the GQR spectrum, are close in both the models, but the functional Ω-dependence of energy in separate branches of GQR is different.

The model predicts the appearance of two soft modes in the spectrum of quadrupole oscillations of rotating nuclei. One of them is similar to the wobbling mode of the rigid-body rotator. The second mode is close in energy to the first one; and its symmetry type is the same as for β- and γ-modes. Note, that in the model of refs. /4,5/ only the first soft mode is presented.

It is necessary to mark great potential possibilities of the virial method we have used to solve the equations of the model. The use of it requires rather simple computating means. However, the spheroids are not very good approximations for the shape of rotating nuclei and are not adequate for studying many actual problems of nuclear physics. It shows the advisability of working out the virial method with more realistic assumptions on the shape of rotating nuclei.

APPENDIX

The expressions for tensors of the Coulomb and surface energy and their Lagrangian variations in terms of semiaxes of the ellipsoid can be found in refs. /9,10/:

$$Q_{ii} = \pi^2 q^2 R^6 a_i^2 A_i \, 8/15; \quad \sigma_{ii} = \pi R^6 T(\mathcal{A}_j + \mathcal{A}_k), \quad (i \neq j \neq k)$$

$$\Delta Q_{ij} = [2 B_{ij} V_{ij} + \delta_{ij} \sum_{\ell=1}^{3} V_{\ell\ell}(B_{i\ell} - A_\ell)] \cdot \pi q^2 R^3 / \rho$$

$$\Delta\sigma_{ij} = [2 \mathcal{A}_{ij} V_{ij} - \delta_{ij} \sum_{\ell=1}^{3} V_{\ell\ell}(\mathcal{A}_\ell + \mathcal{B}_{i\ell})/a_\ell^2] \cdot 15 R^3 T/(4\rho)$$

$$B_{ij} = A_i - a_j^2 A_{ji} = A_j - a_i^2 A_{ij}, \quad A_{ij}(a_i^2 - a_j^2) = A_j - A_i,$$

$$\mathcal{B}_{ij} = \mathcal{A}_i - a_j^2 \mathcal{A}_{ji} = \mathcal{A}_j - a_i^2 \mathcal{A}_{ij}, \quad \mathcal{A}_{ij}(a_i^2 - a_j^2) = \mathcal{A}_j - \mathcal{A}_i .$$

For oblate spheroids ($a_1 = a_2 > a_3$) one has:

$$4 a_1^2 \mathcal{A}_{11} = 4\mathcal{A}_1 - a_3^2 \mathcal{A}_{13}, \qquad 3 a_3^2 \mathcal{A}_{33} = 4\mathcal{A}_3 - 2 a_1^2 \mathcal{A}_{13},$$

$$4 a_1^2 A_{11} = 3 A_1 - a_3^2 A_{13}, \qquad 3 a_3^2 A_{33} = 3 A_3 - 2 a_1^2 A_{13},$$

$$A_1 = A_2 = (a_1 e)^{-3} \, (\arcsin e - e\sqrt{1 - e^2}),$$

$$A_3 = 2(a_1 e)^{-3} \cdot [\, e(1-e^2)^{-1/2} - \arcsin e\,] ,$$

$$\mathcal{A}_1 = \mathcal{A}_2 = [\,(1+e^2)\,\mathrm{arth}\, e - e\,] \cdot (2a_1^4 e^3)^{-1} ,$$

$$\mathcal{A}_3 = [(1-e^2)^{-1} - e^{-1}\ \mathrm{arth}\, e] \cdot (a_1^4 e^2)^{-1}; \quad e^2 = 1 - a_3^2/a_1^2 \ .$$

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