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ON THE METHODS FOR CALCULATING ALPHA DECAY WIDTHS. THE SHELL-MODEL AND THE RESONANCE APPROACHES

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Методы расчета α-ширин.
Оболочечный и резонансный подходы

Проанализировано соотношение между оценками ширины α-распада, основанными на оболочечном подходе к проблеме и на методе резонансных состояний. Использована формулировка теории α-распада в терминах общей теории реакций. Получены новые формулы для α-ширин, которые могут быть сопоставлены с выражениями различных моделей. Приведены расчеты ширин α-распада для ядер Bi, Po и At в разных приближениях. Различные варианты расчета привели к близким оценкам α-ширин.

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On the Methods for Calculating Alpha Decay Widths.
The Shell-Model and the Resonance Approaches

In a unified formulation of reaction theory, new ex-

1. Introduction

In the recent paper[7] an attempt has been made to describe the α decay process in the framework of a unified formulation of reaction theory. This theory unifies the advantages of the shell-model description of the mother and daughter nuclei and the optical model describing the motion of the emitted α particle.

The present paper is aimed at clarifying relations between the conventional theories and the interpretation of the α decay process as a special case of the nuclear resonance reaction.

In Sec.2, we give the basic formulas which describe the α-decay process.

In Sec.3 the integral α width formula is reduced to the surface formula with the help of usual approximations.

The relation between the resonance states and α form factors is discussed in Sec. 4. In Sec. 5 we analyse the approximations leading to the one body formulas of the α decay width in the conventional theory [1-4]. Two approximative α width formulas are proposed in Sec. 6. Some conclusions are given in Sec. 7.

2. The α decay width

As is well known (see, e.g., refs.[1-7]) the α decay width is given by the matrix elements of the total Hamiltonian between the decaying state $|\Phi_k\rangle$ (quasibound) and scattering states $|\chi_E^c\rangle$ (final states):

$$\Gamma_k = 2\pi \sum_c |\langle\Phi_k|H|\chi_E^c\rangle|^2 . \tag{1}$$

In eq. (1) we denote by index c all the discrete quantum numbers for the daughter nucleus and the α - particle. The continuous number E gives the total energy of the system (i.e.,the sum of the ground states and the excitation energies of the daughter nucleus and the α particle and also the kinetic energy ϵ of their relative motion. The normalization of the basic states $|\Phi_k\rangle$ and $|\chi_E^c\rangle$ is given by

$$\langle\Phi_k \mid \Phi_{k'}\rangle = \delta_{kk'}, \tag{2}$$

$$\langle\chi_E^c \mid \chi_{E'}^{c'}\rangle = \delta_{cc'}\,\delta(E-E') . \tag{3}$$

The scattering state $|\chi_E^c\rangle$ satisfies the integrodifferential equation

$$(E-H)|\chi_E^c\rangle = -\sum_k \langle\Phi_k|H|\chi_E^c\rangle\,|\Phi_k\rangle . \tag{4}$$

The solution to eq. (4) may be written in the form:

$$|\chi_E^c\rangle = |\chi_{E;o}^c\rangle - \sum_k \langle\Phi_k|\,|\chi_E^c\rangle\,|\chi^c_{E;k}\rangle , \tag{5}$$

where $|\chi^c_{E;o}\rangle$ is a solution to the homogeneous equation

$$(E-H)\,|\chi^c_{E;o}\rangle = 0 \tag{6}$$

satisfying the usual boundary conditions for the scattering states (i.e.,containing an incoming wave in the channel c and outgoing waves in all the open channels). Further, $|\chi^c_{E;k}\rangle$ is a particular solution to the inhomogeneous equation

$$(E-H)\,|\chi^c_{E;k}\rangle = |\Phi_k\rangle \tag{7}$$

which fulfils the usual boundary conditions for the bound states (i.e.,vanishes at large distance between the fragments).

The inhomogeneous equation (7) describes the relative motion of the products of the α decay which is disturbed by the coupling of the scattering states to quasibound states.

If rearrangement collisions (in which the quantum number of the scattering state c is changed) are absent and the nondiagonal matrix elements between open-channel wave functions vanish, the scattering states must be orthogonal to the quasibound states

$$\langle\chi_E^c \mid \Phi_k\rangle = 0 . \tag{8}$$

Now, using eq. (5) and the orthogonality condition (8) one obtains the system of algebraic equations

$$\sum_{k'} \langle\Phi_{k'}|H|\chi_E^c\rangle\langle\Phi_k|\chi^c_{E;k}\rangle = \langle\Phi_k|\chi^c_{E;o}\rangle \tag{9}$$

from which the unknown matrix elements $\langle\Phi_k|H|\chi_E^c\rangle$ can be determined. In an important case of an isolated quasiresonance state k, the sum in eq. (9) contains only one term $k' = k$ and the corresponding expression for the α decay width becomes

$$\Gamma_k = 2\pi\sum_c \left|\frac{\langle\Phi_k|\chi^c_{E;o}\rangle}{\langle\Phi_k|\chi^c_{E;k}\rangle}\right|^2 . \tag{10}$$

The scattering states can be written as products of intrinsic states of the fragments and the radial $|\phi_{cE}\rangle$ and angular part $|Y_\ell\rangle$ of the function describing the relative motion

$$|\chi_E^c\rangle = |\phi_{c,E}\rangle\,\big[|\phi_\alpha(\xi)\rangle|\phi_\Delta(\eta)\rangle\,|Y_\ell(\Omega)\rangle\big]_c = |\phi_{c,E}\rangle\,|c\rangle . \tag{11}$$

The Hamiltonian H is the sum of the Hamiltonians for the separated fragments and the Hamiltonian describing the relative motion of the fragments:

$$H = H_\alpha + H_D + T_{\alpha D} + V_{\alpha D} . \tag{12}$$

In eq. (12) $T_{\alpha D}$ and $V_{\alpha D}$ stand for the kinetic and potential energy, respectively. We assume that the potential energy (integrated over the internal variables of the fragments ξ and η) can be approximated as a sum of the nuclear and Coulomb potentials which depend only on the relative distance between the centers of mass of the fragments.

Multiplying eqs. (6), (7) from the left by the channel wave function $|c>$ and integrating over ξ,η and Ω we obtain the following differential equations for the radial parts $\Phi_{c,E}$ (r)

$$(L_r+\varepsilon_c)\begin{bmatrix}\Phi_c^o(r)\\ \Phi_c^k(r)\end{bmatrix} = \begin{bmatrix}o\\ I_{ck}(r)\end{bmatrix} . \qquad (13)$$

The definitions are

$$\Phi_c^o(r) = r<c|X_{E:o}^c> ;\Phi_c^k(r) = r<c\ |X_{E:k}^c> \qquad (14)$$

$$I_c(r) = r <c\ |\Phi_k> \qquad \text{(overlap integral)} \qquad (15)$$

$$L_r = -\frac{\hbar}{2\mu}\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right) - V_{\alpha D}(r) \quad \text{(differential operator)} \qquad (16)$$

$$\mu = \frac{m_\alpha \cdot m_D}{m_\alpha + m_D} \qquad \text{(reduced mass)}$$

$$\varepsilon_c = E - E_\alpha - E_D \quad (E_\alpha \text{ and } E_D \text{ are the eigenenergies of the fragments}). \qquad (17)$$

The procedure of calculating the overlap integral for the Gaussian form[2)] of the intrinsic α particle wave function is given in refs.[10,11)]. The solutions to eqs. (13) must satisfy the following asymptotic conditions:

$$\Phi_c^o(r=o) = o \qquad \Phi_c^o(r\to\infty) = \sim U_c^{(-)} - \eta_c U_c^{(+)} \qquad (18)$$

$$\Phi_c^k(r=o) = o \qquad \Phi_c^k(r\to\infty) = o, \qquad (19)$$

where $U_c^{(\pm)}$ are the outgoing and incoming Coulomb waves, and η_c is the scattering amplitude.

Finally, using in eq.(1o) the definitions from eqs. (14,15), one comes to the following expression for the α width

$$\Gamma_k = 2\pi\sum_c\left|\frac{<I_{ck}(r)|\ \Phi_c^o(r)>}{<I_{ck}(r)|\ \Phi_c^k(r)>}\right|^2 . \qquad (2o)$$

Equation (2o) is free from the nuclear radius parameter. No boundary conditions at the channel radius was used in deriving eq. (2o). Thus, the present approach is free from the uncertainties of the R-matrix theory of the α decay.

3. Relation to the shell-model description and R-matrix expressions

In the microscopic description one may try to approximate the quasiresonant state $|\Phi_k>$ by the nuclear shell-model wave functions $|\Phi_k^{SM}>$ satisfaying the equation

$$(E - H^{SM})\ |\Phi_k^{SM}> = 0 . \qquad (21)$$

The solution to eq. (21) is the Slater determinant

$$|\Phi_k^{SM}> = \frac{1}{\sqrt{A!}}\ \det\ \Psi_{n_i l_i j_i m_i}(r_i)\ ,i=1,2,\dots A. \qquad (22)$$

The shell model overlap integral (see eq.(15))

$$I_{ck}^{SM}(r) = \left[\binom{Z}{2}\binom{N}{2}\right]^{1/2}<rc|\Phi_k^{SM}> \qquad (23)$$

approximated in this way is proportional to the expression for the reduced width $\gamma_{ck}(r)$ introduced by Mang[2)]

$$I^{SM}_{ck}(r) = \frac{\sqrt{2\mu r}}{\hbar}\ \gamma_{ck}(r) . \qquad (24)$$

Assuming that the largest contribution to the matrix element $< I^{SM}_{ck}(r)|\Phi^{o}_{c}(r)>$ in eq. (2o) is coming from the surface region of the nucleus[15)], one obtains the well-known formula of the R-matrix theory

$$\Gamma_k = \sum_c P_c(r_c)\ \gamma^2_{ck}(r_c) , \qquad (25)$$

where the penetrability is

$$P_c(r_c) = \frac{4\pi\mu}{\hbar^2}\ \frac{r_c\ |\Phi^{o}_{c}(r_c)|^2}{|<I^{S.M.}_{ck}(r)|\Phi^{k}_{c}(r)>|^2} . \qquad (26)$$

Table 1. Calculated alpha decay widths for Po^{210} and Bi^{212} with eqs.(25.26). The α optical potential is taken from ref.[8-9)]

r_c	Γ_k	Γ_{Mang}	$\Gamma^{exp.}$	Γ_k	Γ_{Mang}	$\Gamma^{exp.}$
fm	MeV	MeV	MeV	MeV	MeV	MeV
7.0	$.40\ 10^{-27}$		$.38\ 10^{-28}$	$.39\ 10^{-30}$		$.55\ 10^{-29}$
7.5	$.19\ 10^{-29}$			$.18\ 10^{-31}$		
8.0	$.16\ 10^{-27}$			$.13\ 10^{-29}$		
8.5	$.13\ 10^{-28}$			$.11\ 10^{-30}$		
9.0	$.37\ 10^{-32}$			$.33\ 10^{-34}$		
9.5	$.33\ 10^{-31}$	$.42\ 10^{-31}$		$.29\ 10^{-33}$	$.47\ 10^{-33}$	
10.	$.90\ 10^{-32}$			$.73\ 10^{-37}$		

In Table 1 it is shown that the α widths obtained from eq. (25) using the classical[2)] and the present (approximative) penetrabilities (26) are in very good agreement.

4. The resonance states

The wave function of the parent nucleus can be written in the general case as

$$|\Phi^{res}_{k}> = \sum_c \frac{U^{res}_{ck}(r)}{r}\ |c> , \qquad (27)$$

where $U^{res}_{ck}(r)$ is the resonance wave functions of the α particle plus the daughter nucleus system. The index k labels the decaying state of the parent nucleus.

The resonance wave functions are usually obtained as solutions to the Sturm-Liouville problem associated with the radial part of the Schrödinger equation for the α particle in the field of the daughter nucleus.

$$\frac{d^2}{dr^2}\ U^{res}_{ck}(r) + \left[K^2 + \frac{2\mu}{\hbar^2}\left(\lambda_c V_{nucl}(r) + V_{coul}(r)\right) - \frac{\ell(\ell+1)}{r^2}\right] U^{res}_{ck}(r) = 0, \qquad (28)$$

where $K^2\hbar^2 = 2\mu\varepsilon_c$ and

$$\begin{bmatrix} \lambda_c V_{nucl}(r) \\ V_{coul}(r) \end{bmatrix} = <c|\begin{bmatrix} V^{nucl}_{\alpha D} \\ V^{coul}_{\alpha D} \end{bmatrix}|c> \qquad (29)$$

are the nuclear and the Coulomb potentials averaged over the channel wave function $|c>$. In eq. (28) the depth of the nuclear potential λ_c is chosen as a resonance parameter.

The solution to eq. (28) must satisfy the following conditions:

$$U^{res}_{ck}(r=o) = 0 \tag{30}$$

$$U^{res}_{ck}(r\to\infty) \sim G(r),$$

where G(r) is irregular Coulomb function[13]. In order to compute the resonance functions, one may use the procedure given in ref.[12] based on numerical solution of the integral form of the Schrödinger equation.

The number of modes of resonance functions is chosen according to the harmonic oscillator constraint:

$$2(N-1) + L = \sum_{i=1,4} (2(n_i-1)+l_i) . \tag{31}$$

The overlap integral in eq.(15) corresponding to $|\phi^{res}_k>$ is equal to

$$\begin{aligned} I^{res}_{ck}(r) &= r <c|\phi^{res}_k(r)> \\ &= U^{res}_{ck}(r). \end{aligned} \tag{32}$$

The same parameters of the optical potential which determine the resonance functions are used in eq. (13) for the scattering wave functions.

The phenomenological depth of the nuclear potential may also be used as an input by searching for the resonant depth.

5. One body α width formulas

Inserting in eq. (20) the resonant overlap integral (32) one body α width formula can be obtained

$$\Gamma^{o.b.}_k = 2\pi \sum_c \left| \frac{<I^{res}_{ck}(r)|\phi^o_c(r)>}{<I^{res}_{ck}(r)|\phi^k_c(r)>} \right|^2 . \tag{33}$$

In table 2 the numerical values obtained by this formula (one level k one channel c) are compared with those given by the Breit[3] and Feshbach[4] one body formulas

$$\Gamma_B = \frac{\hbar v}{<U^{res}_{ck}(r) \,|\, U^{res}_{ck}(r)>} , \quad v = \sqrt{\frac{2\varepsilon_c}{\mu}} , \tag{34}$$

$$\Gamma_F = \frac{\hbar^2}{\mu K} \frac{|<F_{coul}(r)\,\lambda_c\, V_{nucl}(r) \,|\, U^{res}_{ck}(r)>|^2}{<U^{res}_{ck}(r) \,|\, U^{res}_{ck}(r)>} . \tag{35}$$

Table 2. The one body α widths of the polonium isotopes

Nucleus	ε_c (MeV)	Γ_B(MeV)	Γ_F(MeV)	$\Gamma^{o.b.}_k$ (MeV)
Po^{204}	5.370	.3318 10^{-25}	.3317 10^{-25}	.3318 10^{-25}
Po^{206}	5.218	.5329 10^{-26}	.5329 10^{-26}	.5327 10^{-26}
Po^{208}	5.108	.1374 10^{-26}	.1374 10^{-26}	.1374 10^{-26}
Po^{210}	5.229	.1766 10^{-25}	.1768 10^{-25}	.1767 10^{-25}
Po^{214}	7.680	.1432 10^{-15}	.1431 10^{-15}	.1433 10^{-15}

Analysing these values we have concluded that all the three formulas (33), (34) and (35) give with high accuracy the same values. It is interesting to mention that inserting the many body shell model overlap integral in the Mang formula (25) for the α width by the one body resonant overlap integral (32), the Breit formula can be easily obtained.

6. Two approximative α width formulas

For small values of the differential operator L_r (i.e., the difference between the kinetic and potential energies of the

Table 3. Alpha widths calculated with eqs. (20) and (37)

Nucleus	ε_c MeV	λ_c^{*} MeV	Γ_k^{SM} MeV	$\tilde{\Gamma}_k^{SM}$ MeV	Γ_{exp} MeV
Bi^{210}	4.973	109.581	$.63\ 10^{-30}$	$.17\ 10^{-30}$	$.55\ 10^{-29}$
Bi^{211}	6.553	107.035	$.26\ 10^{-22}$	$.95\ 10^{-23}$	$.35\ 10^{-23}$
Po^{204}	5.401	111.510	$.27\ 10^{-27}$	$.15\ 10^{-26}$	$.33\ 10^{-27}$
Po^{206}	5.249	111.060	$.10\ 10^{-28}$	$.36\ 10^{-28}$	$.29\ 10^{-28}$
Po^{208}	5.139	110.565	$.15\ 10^{-19}$	$.16\ 10^{-29}$	$.52\ 10^{-29}$
Po^{210}	5.530	109.682	$.22\ 10^{-28}$	$.74\ 10^{-29}$	$.38\ 10^{-28}$
At^{207}	5.792	110.597	$.10\ 10^{-26}$	$.30\ 10^{-26}$	$.63\ 10^{-26}$
At^{209}	5.680	110.108	$.43\ 10^{-27}$	$.26\ 10^{-27}$	$.10\ 10^{-26}$
At^{210}	5.557	109.956	$.50\ 10^{-28}$	$.55\ 10^{-28}$	$.26\ 10^{-28}$

* the other parameters of the optical potential are taken from ref.[13] (set A).

particle vanishes) the solution $\phi_c^k(r)$ to eq. (13) becomes proportional to the overlap integral $I_{ck}(r)$

$$\phi_c^k(r) \approx \varepsilon_c^{-1}\, I_{ck}(r) . \tag{36}$$

Of course, equality (36) is satisfied exactly for the classical turning points and approximately for the points situated around them. Inserting ϕ_c^k in eq. (20) we obtain a simple α width formula

$$\tilde{\Gamma}_k = 2\pi \sum_c \varepsilon_c^2 \left| \frac{\langle I_{ck}(r) | \phi_c^o(r) \rangle}{\langle I_{ck}(r) | I_{ck}(r) \rangle} \right|^2 \tag{37}$$

in which the overlap integral I_{ck} can be chosen from eq. (23) or eq. (32).

Table 3 presents the α width values obtained from eq.(37) with the shell-model overlap integrals (oscillator basis).

We may observe that the exact (Γ) and approximative ($\tilde{\Gamma}$) values of the α widths are slightly different. We note that in the last case the computer time is two times shorter than in the former.

7. Conclusions

In the framework of unified formulation of reaction theory the α width has been determined by using the preformation factors and the α optical potential. Finally, the α width problem has been reduced to solve (numerically) an uncomplicated system of differential equations.

Therefore the calculation does not become more difficult as is usual. Also it is shown that it is preferable to use an approximative expressions for α width which requires a short computational time. The calculated α widths in the many body approach are in good agreement with the experimental data, if the optical potential with the parameters fitted by the low energy α scattering is used. From the present analysis of the many body and one body aspects of the α decay it is possible to obtain directly the α spectroscopic factors.

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