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ON THE EQUIVALENCE OF QUADRUPOLE PHONON MODEL AND INTERACTING BOSON MODEL

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1. INTRODUCTION

The Quadrupole Phonon Model (QPM)* [1,2] and the Interacting Boson Model (IBM)** [3-5] played an important role for the progress in the theoretical description of transitional nuclei. A considerable achievement of the QPM and IBM is the construction of a mathematical formalism for a unified description of the vibrational and rotational states in even-even nuclei. Presently QPM and IBM are widely used for the interpretation of the data on transitional nuclei. In turn, the predictions of these models stimulate the experimental investigations of nuclei far from the β-stability line.

The problem of the QPM and IBM relationship has been discussed in the literature (see, e.g., refs. [3,6-8]). (When there are two models, advanced independently for the explanation of the same phenomena, the question of their comparison naturally arises). Though the QPM and IBM Hamiltonians look differently, the study of their relationship has always led to the conclusion of the equivalence (truly, in different contexts) of both models. The fact, that it should be so, is clear from a general point of view if we take into consideration that QPM and IBM are identical in the most essential point: they have one and the same underlying group of broken symmetry (see, e.g., ref. [9]), namely the six-dimensional special unitary group SU(6). (Presumably, this is the reason for the authors of paper [7] to consider QPM as two realizations of, as they call it, "phenomenological SU(6) Boson Model"***. It was noticed there, that Arima and Jachello used finite Boson expansion of "Schwinger type", while Janssen, Jolos and Dönau applied an infinite Boson expansion of "Holstein-Primakoff-type".

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* The Quadrupole Phonon Model has been developed by Janssen, Jolos and Dönau.

** The Interacting Boson Model has been proposed by Arima and Jachello.

*** Note, however, that each model has its own philosophy and, accordingly, QPM and IBM have been constructed in a different manner.

The authors of QPM and IBM were the first to point out the equivalence of their models. They showed [3,6] that the matrix elements of Hamiltonians and Quadrupole operators (in the relevant bases) in both models were identical. Touching upon the relationship of QPM and IBM, Blaizot and Marshalek proposed [7] that the investigation of the transformation, connecting the Boson expansions mentioned above, could directly lead to the substantiation of the QPM and IBM equivalence. Their detailed analysis indicated, that the quantal transformation could not be expected to be unitary in the case of SU(6)-algebra (which is of interest in establishing the QPM and IBM equivalence). The observation that QPM and IBM are essentially equivalent and their Hamiltonians look different only due to the fact that different Boson realizations were used was mentioned also by Paar [8]. He formulated the following statement (though in a little different way): QPM and IBM Hamiltonians (as operators irrespectively to the basis) coincide if definite relations between their parameters are fulfilled*. Of course, this statement is stronger, than the QPM and IBM equivalence "on the level of matrix elements".

It turns out that the statement that QPM and IBM are equivalent "on the level of operators" admits a rigorous mathematical proof. So far, no such proof seems to exist in the literature and this is the main aim of this note to present it. (The rigorous foundation of the QPM and IBM equivalence is not only of methodological interest (the proof is instructive), but it favours a deeper understanding of the essence of two models, already belonging to the arsenal of the Nuclear Structure Theory).

Clearly, to establish the QPM and IBM equivalence "on the level of operators", we should remove the source, which masks the identity of operators of both models (different bosonizations have been used). One possible way to do this is to construct explicitly the Schwinger representation (SR) of the QPM operators. Recently many papers have been dedicated to SR, mainly of Japanese school (see, e.g., refs. [11-15]). (Being finite and unitary, SR is convenient (see, however, ref. [7]) in solving such theoretical problems as quantization of the time dependent self-consistent field [7,11,12], quantum rotator and high states description [13], anharmonic and Pauli principle [15] effects, etc.). The impression is that some of the

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* Therefore, as it was emphasized in [8], QPM and IBM will give the same numerical results for the energies and transition rates.

boson representations, discussed in papers [7,11-15], were called the "Schwinger type" only because they were finite and satisfied the corresponding commutator algebra. It is not always clear to what extent these "Schwinger type" representations are consistent with the definition given by Schwinger [16]. In the present note we shall construct explicitly the SR of the SU(6) QPM generators, proceeding from the relevant definition. (Hopefully, the presented way of SR construction could be of interest not only for QPM equivalence foundation, which originally motivated this work).

Therefore, in the present paper we shall give an explicit construction of the Schwinger representation for the generators of the SU(6) algebra, obtained within QPM. IBM and QPM equivalence (identity of the Hamiltonians and the relevant operators) will be received as a corollary.

The presentation is organized as follows.

In Sec. 2 the QPM and IBM ideologies are briefly discussed and the necessary formulae and definitions are introduced. In Sec. 3 the problem is formulated. The SR construction of SU(6) generators is presented in Sec.4. QPM and IBM equivalence is proved in Sec.5.

2. COLLECTIVE HAMILTONIANS AND RELEVANT OPERATORS OF QPM AND IBM

The physical assumptions underlying the derivation of the QPM collective Hamiltonian ($\hat{H}_{QPM}$) and the philosophy which has led to the IBM Hamiltonian ($\tilde{H}_{IBM}$) are described in detail in original papers [1-6]. We regard these important questions with the purpose of making the further presentation more understandable.

To treat all the five quadrupole degrees of freedom, Janssen, Jolos and Dönau introduced generalized collective coordinates and conjugated momenta $\hat{q}^{coll}_{2\mu}$, $\hat{p}^{coll}_{2\mu}$, ($\mu = 0, \pm1, \pm2$)*. Assuming that the quadrupole states weakly interact with the other modes of excitations they showed that $\hat{q}^{coll}_{2\mu}$, $\hat{p}^{coll}_{2\mu}$ and their commutators form a closed algebra which is isomorphic to the Lie-algebra of SU(6)-group:

$$[\hat{q}_{\mu}, \hat{p}_{\mu'}] = (-1)^{\mu+\mu'} [\hat{q}_{-\mu}, \hat{p}_{-\mu'}]$$

$$[\hat{p}_{\mu}, \hat{p}_{\mu'}] = (-1)^{\mu+\mu'} [\hat{q}_{-\mu}, \hat{q}_{-\mu'}]$$

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* $\hat{A}$ means that A is an abstract operator, the absence of ^ indicates that $\hat{A}$ is in a given representation.

$$[[\hat{q}_\mu,\hat{p}_{\mu'}],\hat{q}_{\mu''}]=-2\delta_{\mu\mu'}(-1)^{\mu''}\hat{p}_{-\mu''}-\delta_{\mu''-\mu}(-1)^{\mu}p_{-\mu'}-\delta_{\mu'\mu''}(-1)^{\mu}\hat{p}_{-\mu}$$

$$[[\hat{q}_\mu,\hat{p}_{\mu'}],\hat{p}_{\mu''}]=2\delta_{\mu\mu'}(-1)^{\mu''}\hat{q}_{-\mu''}+\delta_{\mu'-\mu''}(-1)^{\mu'}\hat{q}_{\mu}+\delta_{\mu\mu''}(-1)^{\mu'}\hat{q}_{-\mu}.$$

$$[[\hat{q}_\mu,\hat{q}_{\mu'}],\hat{q}_{\mu''}]=\delta_{\mu'-\mu''}(-1)^{\mu'}\hat{q}_{\mu}-\delta_{\mu''-\mu}(-1)^{\mu}\hat{q}_{\mu'}$$

$$[[\hat{q}_\mu,\hat{q}_{\mu'}],\hat{p}_{\mu''}]=\delta_{\mu'\mu''}(-1)^{\mu}\hat{p}_{-\mu}-\delta_{\mu\mu''}(-1)^{\mu'}\hat{p}_{-\mu'}\qquad(1)$$

$$[[\hat{q}_\mu,\hat{p}_{\mu'}],[\hat{q}_{\mu''},\hat{p}_{\mu'''}]]=\delta_{\mu-\mu''}(-1)^{\mu'+\mu''+\mu'''}[\hat{q}_{-\mu'''},\hat{q}_{-\mu'}]+$$

$$+\delta_{\mu'-\mu'''}(-1)^{\mu'}[\hat{q}_{\mu''},\hat{q}_{\mu}]+\delta_{\mu\mu'''}(-1)^{\mu'''}[\hat{q}_{-\mu'''},\hat{q}_{\mu}]+\delta_{\mu\mu''}(-1)^{\mu'}[\hat{q}_{\mu'},\hat{q}_{-\mu'}]$$

$$[[\hat{q}_\mu,\hat{p}_{\mu'}],[\hat{q}_{\mu''},\hat{q}_{\mu'''}]]=\delta_{\mu-\mu''}(-1)^{\mu}[\hat{q}_{\mu'''},\hat{p}_{\mu'}]-\delta_{\mu-\mu'''}(-1)^{\mu}[\hat{q}_{\mu''},\hat{p}_{\mu'}]+$$

$$+\delta_{\mu'\mu''}(-1)^{\mu'''}[\hat{q}_{\mu},\hat{p}_{-\mu'''}]-\delta_{\mu'\mu'''}(-1)^{\mu''}[\hat{q}_{\mu},\hat{p}_{-\mu''}]$$

$$[[\hat{q}_\mu,\hat{q}_{\mu'}],[\hat{q}_{\mu''},\hat{q}_{\mu'''}]]=\delta_{\mu'-\mu''}(-1)^{\mu'}[\hat{q}_{\mu},\hat{q}_{\mu'''}]+$$

$$+\delta_{\mu'-\mu'''}(-1)^{\mu}[\hat{q}_{\mu''},\hat{q}_{\mu'}]-\delta_{\mu-\mu''}(-1)^{\mu}[\hat{q}_{\mu'},\hat{q}_{\mu'''}]-\delta_{\mu'-\mu'''}(-1)^{\mu'}[\hat{q}_{\mu},\hat{q}_{\mu''}],$$

where: $\hat{q}_\mu=\hat{q}^{coll}_{2\mu}/\sqrt{L}$, $\hat{p}_\mu=\hat{p}^{coll}_{2\mu}/\sqrt{K}$. The quantities L and K are defined in [1] and will be of no use further. Just to emphasize that this SU(6) has been obtained within QPM, throughout the paper commutator algebra (1) will be referred to as $SU(6)_{QPM}$.

The QPM collective Hamiltonian is expressed in terms of $SU(6)_{QPM}$ generators in the following way:

$$\hat{H}_{QPM}=e\sum_{\mu=-2}^{2}i[\hat{q}_\mu,\hat{p}_\mu]+u\sum_{\mu=-2}^{2}(-1)^{\mu}\hat{q}_\mu\hat{q}_{-\mu}+v\sum_{\mu=-2}^{2}(-1)^{\mu}\hat{p}_\mu\hat{p}_{-\mu}+\qquad(2)$$

$$+i\frac{W}{2}\sum_{\mu=-2}^{2}(-1)^{\mu}\hat{q}_\mu([\hat{q},\hat{p}])_{(2-\mu)}+\frac{1}{4}\sum_{\substack{L=0,2,4\\M=-L}}^{L}t_L(-1)^{M}(i[\hat{q},\hat{p}])_{(L-M)}.$$

The symbol $(\ )_{(LM)}$ denotes the usual vector coupling. The information on the average field and residual interactions is included in the parameters e, u, v, w, t_L.

The generalized Holstein-Primakoff representation of the $SU(6)_{QPM}$ generators has the form:

$$q_\mu=\sqrt{N-\sum_\nu d^+_\nu d_\nu}\,(-1)^{\mu}d_{-\mu}+d^+_\mu\sqrt{N-\sum_\nu d^+_\nu d_\nu}$$

$$p_\mu=i[(-1)^{\mu}d^+_{-\mu}\sqrt{N-\sum_\nu d^+_\nu d_\nu}-\sqrt{N-\sum_\nu d^+_\nu d_\nu}\,d_\mu]\qquad(3)$$

$$i[q_\mu,p_{\mu'}]=d^+_\mu d_{\mu'}+(-1)^{\mu+\mu'}d^+_{-\mu'}d_{-\mu}-\delta_{\mu\mu'}(N-\sum_\nu d^+_\nu d_\nu)$$

$$[q_\mu,q_{\mu'}]=(-1)^{\mu'}d^+_\mu d_{-\mu'}-(-1)^{\mu}d^+_{\mu'}d_{-\mu}.$$

where: N is the eigenvalue of the Casimir operator of SU(6); operator d^+_μ * creates quadrupole phonon with projection μ, ($\mu=0,\pm1,\pm2$). The familiar form of the QPM Hamiltonian is obtained by using eq. (3) for $\hat{H}_{QPM}$ from eq. (2):

$$H_{QPM}=h_0+h_1\sum_\mu d^+_\mu d_\mu+h_2\sum_\mu(-1)^{\mu}(d^+_\mu d^+_{-\mu}\sqrt{(N-\sum_\nu d^+_\nu d_\nu)(N-1-\sum_\nu d^+_\nu d_\nu)}+h.c)+$$

$$+h_3\sum_\mu(-1)^{\mu}(d^+_\mu(d^+d)_{(2-\mu)}\sqrt{N-\sum_\nu d^+_\nu d_\nu}+h.c.)+\qquad(4)$$

$$+\sum_{\substack{L=0,2,4\\M=-L}}^{L}h_{4L}(d^+d^+)_{(LM)}(dd)_{(L-M)}(-1)^{M}.$$

The electric quadrupole operator was defined as:

$$\hat{Q}_{QPM}=m_1\hat{q}_\mu+m_2([\hat{q},\hat{p}])_{(2\mu)},\qquad(5)$$

m_1 and m_2 being arbitrary parameters.

The expression for $\hat{Q}_{QPM}$, which has been utilized in the calculations of the E2-transitions within QPM, is obtained substituting eq. (3) into eq. (5):

$$Q_{QPM}=m_1(d^+_\mu\sqrt{N-\sum_\nu d^+_\nu d_\nu}+(-1)^{\mu}\sqrt{N-\sum_\nu d^+_\nu d_\nu}\,d_{-\mu})+m_2(d^+d)_{(2\mu)}.\qquad(6)$$

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*Thoroughout this paper + stands for Hermitian conjugation (h.c.).

IBM has been constructed in the spirit of the Theory of High Symmetries. As has been already mentioned, a fundamental role in Arima and Jachello approach is played by the SU(6) group *. Being rich enough, this group turned out to be very useful for the phenomenological parametrization of plethora of the measured nuclear spectra and electromagnetic transitions [4,5,18] (note that reacher groups, in particular, Sp(12,R) were considered for the transitional region and microscopic approach was proposed [19]). The IBM characteristic feature is that besides the quadrupole bosons d^+_μ, d_μ, Arima and Jachello introduced bosons with orbital momentum L=0, namely, S^+ and S. With the help of S^+, S, d^+_μ, d_μ, all 35 generators of SU(6) were constructed. In terms of them the expression for $\tilde H_{IBM}$ is the following [4]:

$$\tilde H_{IBM} = \epsilon_s s^+ s + \epsilon_d \sum_\mu d^+_\mu d_\mu + v_2[((d^+d^+)_{(2\mu)}(ds)_{(2-\mu)})_{(00)} + h.c.] +$$
$$+ \frac{v_0}{\sqrt{2}}[((d^+d^+)_{(00)}(ss)_{(00)} + h.c.] + u_2((d^+s^+)_{(2\mu)}(ds)_{(2-\mu)})_{(00)} + \qquad (7)$$
$$+ \frac{1}{2} u_0((s^+s^+)_{(00)}(ss)_{(00)})_{(00)} +$$
$$+ \sum_{L=0,2,4} \frac{1}{2}(2L+1)^{1/2} C_L((d^+d^+)_{(LM)}(dd)_{(L-m)})_{(00)},$$

where ϵ_s, ϵ_d, u_0, v_0, u_2, v_2, C_L are phenomenological parameters.

Thus, while $\hat H_{QPM}$ has been derived on the base of the microscopic nuclear Hamiltonian, $\hat H_{IBM}$ (in the phenomenological IBM version**) has been constructed from the outset as the most general scalar from the generators of the underlying dynamical symmetry group SU(6). (It has been proved in paper [20] that $\hat H_{IBM}$ can be expressed through the Casimir operators of three different chains of subgroups).

In the framework of IBM the following form of the electric quadrupole operator was used [4]:

$$\tilde Q_{IBM} = m_1[(d^+s)_{(2\mu)} + h.c.] + m_2(d^+d)_{(2\mu)}. \qquad (8)$$

As has been mentioned in the Introduction, QPM and IBM are equivalent on the level [4,6] of matrix elements, namely:

* From this viewpoint QPM and IBM could be considered as a further development of the Unitary Scheme Model in the Nuclear Theory [17].

** The microscopic foundation of IBM is given in paper [10].

$$\tilde H_{IBM}|s^{n_s} d^{n_d}[N]\chi LM\rangle = H_{QPM}|d^{n_d}[N]\chi LM\rangle \qquad (9)$$
$$\tilde Q_{IBM}|s^{n_s} d^{n_d}[N]\chi LM\rangle = Q_{QPM}|d^{n_d}[N]\chi LM\rangle, \qquad (10)$$

where the states $|s^{n_s} d^{n_d}[N]\chi LM\rangle$ form a complete basis in the six-dimensional oscillator Hillbert space; N is the eigenvalue of the total Boson operator

$$\tilde N = s^+s + \sum_\mu d^+_\mu d_\mu = \tilde n_s + \tilde n_d \qquad (11)$$

$n_s = N - n_d$; χ is whatever quantum number is needed to specify uniquely the states; $|d^{n_d}[N]\chi LM$ stands for $|s^{n_s} d^{n_d}[N]\chi LM\rangle$, from which the S-boson degree of freedom is "excluded".

Let us remind that eq. (9) was fulfilled under definite relations between the parameters of H_{IBM} and H_{QPM} [4].

In the Quantum Field Theory language eqs. (9) and (10) express QPM and IBM equivalence in a "weak sense". Before we pass to the formulation of the "strong equivalence", i.e., "on the level of operators", let us define the Schwinger representation of the $SU(6)_{QPM}$ generators: $\{\tilde g_\lambda\} \equiv \{\tilde q_\mu, \tilde p_\mu, i[\tilde q_\mu, \tilde p_{\mu'}], [\tilde q_\mu, \tilde q_{\mu'}]\}$.

It is known, that SU(6)-algebra has 35 generators and its fundamental representation is six-dimensional (sextuplet) (see Sec.4). As any Lie-algebra, $SU(6)_{QPM}$ can be represented in the form:

$$[\hat g_\lambda, \hat g_\rho] = \gamma^\tau_{\lambda\rho} \hat g_\tau, \qquad \text{(summation over } \tau \text{)},$$

where $(\lambda, \rho, \tau = 1, \dots, 35)$; $\gamma^\tau_{\lambda\rho}$ are the $SU(6)_{QPM}$ structure constants.

Let the set $\{g_\lambda\}$ of 6x6 matrices generate the $SU(6)_{QPM}$ fundamental matrix representation, i.e.,

$$[g_\lambda, g_\rho] = \gamma^\tau_{\lambda\rho} g_\lambda. \qquad (12)$$

The SR of this algebra is generated by the operators $\{\tilde g_\lambda\}$, defined in the following way [16]:

$$\left.\begin{array}{l} \tilde g_\lambda = x^+ g_\lambda x, \\ x^+ = (s^+ d^+_{-2} d^+_{-1} \dots d^+_2), \quad x = \begin{pmatrix} s \\ d_{-2} \\ \vdots \\ d_2 \end{pmatrix} \end{array}\right\}. \qquad (13)$$

Using eqs. (12), (13) and taking into account the boson commutation relations between S, S^+, d_μ, d^+_μ, it can be shown that: $[\tilde{g}_\lambda, \tilde{g}_\rho] = \gamma^\tau_{\lambda\rho}\tilde{g}_\tau$, i.e., $\{\tilde{g}_\lambda\}$ actually generate a $SU(6)_{QPM}$ representation.

3. THE EQUIVALENCE PROBLEM FORMULATION

From the comparison of eqs. (4), (7) and eqs. (6), (8) it is obvious, that H_{QPM}, $\tilde{H}_{IBM}$ and Q_{QPM}, $\tilde{Q}_{IBM}$ are given in different representations: $\tilde{H}_{IBM}$ and $\tilde{Q}_{IBM}$ are in SR, while for H_{QPM} and Q_{QPM} the Holstein-Primakoff bosonization has been applied (see Sec.2). The idea of QPM and IBM equivalence establishment "on the level of operators" is quite simple. Let us assume, we are able to explicitly find $\{\tilde{g}_\lambda\}$. Substituting the obtained expressions into eqs. (2) and (5), we shall get $\tilde{H}_{QPM}$ and $\tilde{Q}_{QPM}$, i.e., the SR for these operators, and now they can be compared with $\tilde{H}_{IBM}$ and $\tilde{Q}_{IBM}$, respectively (saying in advance, the results of such substitution will be as follows: all boson structures, which are present in eq. (7), will appear in eq.(2), and the resulting expression for $\tilde{Q}_{QPM}$ will have the same form, as $\tilde{Q}_{IBM}$ from eq. (8).

Provided $\hat{H}_{QPM}$ and $\tilde{H}_{IBM}$ parameters are connected by the relations:

$$\epsilon_s = -10e + (2N+3)(u+v) + \frac{t_0}{5}$$

$$\epsilon_d = 2e + u + v + \frac{1}{5}t_0 + \frac{1}{5}\sum_{L=2,4}(2L+1)t_L$$

$$v_2 = W\sqrt{5}, \qquad v_0 = (u-v)\sqrt{10} \tag{14}$$

$$v_0 = \frac{2t_0}{5} - 4(u+v), \qquad u_2 = -\frac{2}{\sqrt{5}}t_0$$

$$C_L = \frac{2}{(2L+1)^{1/2}}\sum_{L'=0,2,4}(-1)^{L'}(2L'+1)\begin{Bmatrix}2 & 2L\\ 2 & 2L'\end{Bmatrix}t_{L'}$$

we shall prove, that $\tilde{H}_{QPM} = \tilde{H}_{IBM}$. (15)

We shall also show, that:

$$\tilde{Q}_{QPM} = \tilde{Q}_{IBM}. \tag{16}$$

In order to derive rigorously eqs. (15) and (16) we have to find explicitly, using definition (13), the SR of $SU(6)_{QPM}$ generators.

Let us turn to the construction of $\{\tilde{g}_\lambda\}$.

4. THE SCHWINGER REPRESENTATION CONSTRUCTION OF $SU(6)_{QPM}$ GENERATORS

The Lie-algebra of the SU(6)-group belongs to the classical Lie-algebras. (This is actually A_5 in the Cartan's notations). The Theory of the Classical Lie-algebras is a completed mathematical theory (see, e.g., refs. /21-24/ and references therein) and it is worth to use its powerful apparatus in the SR construction of the $SU(6)_{QPM}$ generators.

It is known that SU(6) is a group of rank 5, i.e., among the 35 generators of its Lie-algebra, there are 5 generators which commute with each other. The SU(6) canonical commutation rules and the explicit form of the sextuplet's matrices are known. We give here several formulae, which will be of important use further.

The standard form of the SU(6) commutation relations is /23-26/:

$$[H_k, H_\ell] = 0, \qquad (k, \ell = 1,2,\dots,5) \tag{17}$$

$$[H_k, E_{\pm\alpha}] = r_k(\pm\alpha)E_{\pm\alpha}, \quad (\alpha = 1,2,\dots,15) \tag{18}$$

$$[E_\alpha, E_{-\alpha}] = \sum_{k=1}^{5} r_k(\alpha)H_k \tag{19}$$

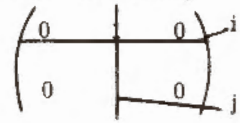
$$[E_\alpha, E_\beta] = N^\delta_{\alpha\beta}E_\delta \quad , \quad (\text{no summation over } \delta). \tag{20}$$

Let us explain the notations: the six-dimensional matrices $\{H_k\}$ are the elements of the maximal Cartan's abelian subalgebra. They are diagonal /26/:

$$H_k = [12k(k+1)]^{-1/2}\,\text{diag}(1,\dots,1,\underset{k}{-k},0,\dots,\overset{6}{0}). \tag{21}$$

The "raising" and "lowering" canonical generators have the form:

$$E_\alpha = \frac{1}{\sqrt{12}}$$



$$E_{-\alpha} = E^+_\alpha \,, \tag{22}$$

where $j > i = 1,2,\dots,6$. The one to one correspondence between $\{\alpha\}$ and $\{10i+j\}$ is given in the table:

Table

α	1	2	3	4	5	6	7	8	9	10	11	12	12	14	15
10i+j	12	13	14	15	16	23	24	25	26	34	35	36	45	46	56

The operators $\{H_k\}$ and $\{E_{\pm\alpha}\}$ are the so-called Cartan-Weyl's canonical generators. The five-dimensional root vectors $\{r(\pm\alpha)\}$ are subjected to the conditions:

$$r(-\alpha) = -r(\alpha), \quad \sum_{\pm\alpha} r_k(\alpha) r_\ell(\alpha) = \delta_{k\ell} ,$$

and their explicit form is given in appendix A.

For SU(6) the quantities $N^{\delta}_{\alpha\beta}$ are:

$$H^{\delta}_{\alpha\beta} = \begin{cases} \pm \frac{1}{\sqrt{12}}, & \text{if } r(\alpha) + r(\beta) \text{ is a nonvanishing root vector,} \\ 0, & \text{otherwise.} \end{cases}$$

Let us return to the problem of finding $\{\tilde g_\lambda\}$. It is seen from eqs. (13) that $\{\tilde g_\lambda\}$ is easy to calculate if 6x6 matrix realization $\{g_\lambda\}$ is known. Due to a rather complicated form of $SU(6)_{QPM}$ (see (1)) it is a very difficult task to directly find these 35 six-dimensional matrices. The general methods of SU(n) matrix generators construction, developed in paper [27], seem somewhat tedious for the case n=6 (which is of interest for us).

In the present paper a purely algebraic method of constructing $\{g_\lambda\}$ is applied. It is based on the general theorem due to Cartan, according to which $SU(6)_{QPM}$ (see eqs. (1)), we are interested in, can be transformed into the canonical form (17)-(20), for which the matrix realization is known (see eqs. (21), (22)). The essence of the method accepted here is the following: let us form such linear combinations from $\{g_\lambda\}$ that they satisfy relations of the type (17)-(20). The latter recommend these combinations as candidates for $\{H_k, E_{\pm\alpha}\}$, and thus the problem is reduced to a familiar one, because for $\{H_k, E_{\pm\alpha}\}$ we have matrix representation. Applying the inverse transformation, we obtain explicitly the matrices $\{g_\lambda\}$.

Let us demonstrate the method in action.

1) First we fix the basis in $SU(6)_{QPM}$:

$$\{\hat g_\lambda\} = \{ i[\hat q_\mu, \hat p_\mu], \ (\mu=0,1,2); \ [\hat q_\mu, \hat q_{-\mu}], \ (\mu=1,2); \ \hat q_\mu, \hat p_\mu (\mu=0, \pm1, \pm2);$$

$$i[\hat q_\mu, \hat p_{-\mu}], \quad (\mu = \pm1, \pm2);$$

$$i[q_{-2}, p_1], \ (i[q_0, p_\mu], \mu=1,2), \ i[q_1, p_0], \ i[q_1, p_2],$$

$$(i[q_2, p_\mu], \ \mu = 0, \pm1), \ ([q_{-2}, q_\mu], \ \mu = 0, \pm1), \tag{23}$$

$$([q_2, q_\mu], \ \mu = 0, \pm1), \ [q_0, q_{-1}], \ [q_0, q_1]\},$$

altogether 35 lineary independent generators.

II) Let us separate the Cartan's abelian subalgebra in $SU(6)_{QPM}$. Using eqs. (1), we can easily show that is it formed by the following 5 generators:

$$\{ i[q_\mu, p_\mu], \ (\mu=0,1,2) \ [q_\mu, q_{-\mu}], \ (\mu=1,2)\}.$$

They are mutually commutable and can be simultaneously expressed in a diagonal form. Let us introduce the following notation:

$$\underset{\sim}{H}' = \begin{pmatrix} [q_1, q_{-1}] \\ [q_2, q_{-2}] \\ i[q_0, p_0] \\ i[q_1, p_1] \\ i[q_2, p_2] \end{pmatrix}; \qquad \underset{\sim}{H} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{pmatrix}$$

It is clear, that $\underset{\sim}{H}'$ and $\underset{\sim}{H}$ are related by linear transformation

$$\underset{\sim}{H}' = M\underset{\sim}{H}, \tag{24}$$

where M is a 5-dimensional unknown at the moment matrix.

III) Let us look for the pretenders to $\{E_{\pm\alpha}\}$ within $SU(6)_{QPM}$. As it turns out, they are the following 30 operators:

$$\{(q_\mu \pm i p_{-\mu}, \mu=0,\pm1,\pm2), (i[q_\mu, p_{-\mu}], \mu=\pm1,\pm2), ([q_\mu, q_{\mu'}] \pm i[q_\mu, p_{-\mu'}], \tag{25}$$

being taken only such pairs (μ, μ'), which are present in the list of generators (23), there are exactly 16 of them)}.

The proof that the enumerated combinations satisfy relations analogous to eqs. (17)-(20) and, therefore, can play the role of $\{E_{\pm\alpha}\}$ is reserved for appendix B. It should be emphasized that the combinations, which are present in (25), were not guessed, but they were dictated by the commutation rules (17)-(20) and by $SU(6)_{QPM}$, eqs. (1). Therefore, up to a normalization factor the operators from (25) coincide with

$\{E_{\pm\alpha}\}$. For the complete solution of the task we have to know which value of α to assign for each of the operators from (25).

IV) Let us find the one to one correspondence $\{\mu;\mu';(\mu,\mu')\} \to \{\pm\alpha\}$. If we knew the matrix M from eq. (24), it would be easy to establish the correspondence. The guiding principle to do this is the following. Let us calculate (using, of course, eqs. (1)) the commutator of any operator from set (25) with its h.c. one, then expressions of the type (B.1) and (B.2) appear. Replacing H'_k for H_k with the help of (24), we must get sum of the type $\sum_{k=1}^{5} r_k(\alpha) H_k$. Since we know $\{r(\alpha)\}$ and $\{H_k\}$, we can assign unambiguously to the operator, we have started with, a definite value of α. The fact **M** is still unknown, somewhat complicates the situation, but a way out exists. Let us try to find simultaneously M^{-1} and the correspondence between $\{q_\mu + ip_{-\mu}, \mu=0,\pm1,\pm2\}$ and the subset of 5 canonical generators from $\{E_{\pm\alpha}\}$. In solving this subsidiary task the relations:

$$M^{-1}\underset{\sim}{\Theta}_\mu = \underset{\sim}{r}(\pm\alpha)$$
$$M^{-1}\underset{\sim}{r}(\pm\alpha) = \underset{\sim}{\beta}_\mu \qquad (26)$$

have happened to be useful. They are proved in appendix C, and the explicit form of $\underset{\sim}{\Theta}_\mu$ and $\underset{\sim}{\beta}_\mu$ is given there. As it follows from their derivation itself, the eqs. (26) are to be fulfilled on the set of the root vectors. Using eqs. (A.1) and (26), we have unambiguously found M^{-1} and the $\{q_\mu + ip_{-\mu}, (\mu=0,\pm1,\pm2)\} \to \{E_{\pm\alpha}\}$ correspondence. We give the final results of the subsidiary task solution. With the help of eqs. (26) and (A.1) matrix M^{-1} has been calculated:

$$M^{-1} = \begin{pmatrix} 0 & 1/4\sqrt{6} & 0 & 0 & -1/4\sqrt{6} \\ -1/6\sqrt{2} & 1/12\sqrt{2} & 0 & 1/6\sqrt{2} & 1/12\sqrt{2} \\ 1/24 & -1/24 & -1/8 & 1/24 & 1/24 \\ 5/8\sqrt{15} & -1/8\sqrt{15} & 1/8\sqrt{15} & -3/8\sqrt{15} & 1/8\sqrt{15} \\ 0 & -1/2\sqrt{10} & 1/12\sqrt{10} & 1/6\sqrt{10} & -1/3\sqrt{10} \end{pmatrix} \qquad (27)$$

From eq. (27) we find:

$$M = \begin{pmatrix} 0 & -2\sqrt{2} & 1\sqrt{15} & 0 \\ \sqrt{6} & -\sqrt{2} & -1-\sqrt{15}/5 & -6\sqrt{10}/5 \\ -2\sqrt{6} & -2\sqrt{2} & -80 & 0 \\ -2\sqrt{6} & -4\sqrt{2} & -1-\sqrt{15} & 0 \\ -3\sqrt{6} & -\sqrt{2} & -1-\sqrt{15}/5 & -6\sqrt{10}/5 \end{pmatrix} \qquad (28)$$

The following matrices $\{q_\mu + ip_{-\mu} (\mu=0,\pm1,\pm2)\}$ have been obtained:

$$q_2 + ip_{-2} = 2\sqrt{12}\,E_1$$
$$q_1 - ip_{-1} = -2\sqrt{12}\,E_2$$
$$q_0 + ip_0 = 2\sqrt{12}\,E_3$$
$$q_{-1} - ip_1 = -2\sqrt{12}\,E_4$$
$$q_{-2} + ip_2 = 2\sqrt{12}\,E_5$$

The rest 5 combinations we get by taking hermitian conjugation.

Having the explicit form of **M** and M^{-1}, we can now easily find the correspondence of the operators from (25) with $\{E_{\pm\alpha}\}$. Let us consider as an example the generator $i[q_1,p_{-1}]$. According to eq. (B.4) we have:

$$[i[q_1,p_{-1}],\ (i[q_1,p_{-1}])^+] = 4[q_{-1},q_1].$$

Taking into account (24) and (28), we receive:

$$[i[q_1,p_{-1}],\ (i[q_1,p_{-1}])^+] = 8\sqrt{2}H_2 - 4H_3 - 4\sqrt{15}H_4 .$$

Now, using (A.1), we see that only $\underset{\sim}{r}(-11)$ could lead to namely this r.h.s. Therefore, we conclude that $i[q_1,q_{-1}] = 2\sqrt{12}E_{-11}$. The one of the correspondence between the operators from set (25) and $\{E_{\pm\alpha}\}$ has been found.

V) Performing the inverse transformation, we express $\{g_\lambda\}$ through $\{H_k, E_{\pm\alpha}\}$ from eqs. (21) and (22), respectively. The final result for the generator matrices from (25) is the following (the same arrangement of the generators as in (23) is used here): $H' = MH$, where M is given by (28);

$$q_{-2} = \sqrt{12}(E_{-1} + E_5) \qquad p_{-2} = i\sqrt{12}(E_{-5} - E_1)$$

$$q_{-1} = \sqrt{12}\,(E_{-2} - E_4) \qquad p_{-1} = -i\sqrt{12}\,(E_{-4} + E_2)$$

$$q_0 = \sqrt{12}\,(E_3 + E_{-3}) \qquad p_0 = i\sqrt{12}\,(E_{-3} - E_3).$$

To obtain the rest matrices, we use the properties:

$$p^+_\mu = (-1)^\mu p_{-\mu}, \qquad q^+_\mu = (-1)^\mu q_{-\mu}.$$

$$i[q_1, p_{-1}] = 2\sqrt{12}\,E_{-11}$$

$$i[q_2, p_{-2}] = 2\sqrt{12}\,E_{-9}$$ and h.c. relations.

Finally:

$$i[q_{-2}, p_1] = \sqrt{12}\,(E_8 - E_{12}) \qquad i[q_1, p_2] = \sqrt{12}\,(E_{15} - E_6)$$

$$i[q_0, p_1] = \sqrt{12}\,(E_{13} - E_{10}) \qquad i[q_2, p_0] = \sqrt{12}\,(E_{-14} + E_{-7})$$

$$i[q_0, p_2] = \sqrt{12}\,(E_{14} + E_7) \qquad i[q_2, p_{-1}] = \sqrt{12}\,(E_{-12} - E_{-8})$$

$$i[q_1, p_0] = \sqrt{12}\,(E_{-13} - E_{-10}) \qquad i[q_2, p_1] = \sqrt{12}\,(E_{-15} - E_{-6}) \qquad (29)$$

$$[q_{-2}, q_0] = \sqrt{12}\,(E_7 - E_{14}) \qquad [q_2, q_{-1}] = -\sqrt{12}\,(E_{-6} + E_{-15})$$

$$[q_{-2}, q_{-1}] = -\sqrt{12}\,(E_8 + E_{12}) \qquad [q_2, q_1] = -\sqrt{12}\,(E_{-8} + E_{-12})$$

$$[q_{-2}, q_1] = -\sqrt{12}\,(E_6 + E_{15}) \qquad [q_0, q_{-1}] = -\sqrt{12}\,(E_{10} + E_{13})$$

$$[q_2, q_0] = \sqrt{12}\,(E_{-14} - E_{-7}) \qquad [q_0, q_1] = -\sqrt{12}\,(E_{-10} + E_{-13}).$$

Having in mind eqs. (22) and the table, from eqs. (29) we explicitly find, e.g., matrix $[q_2, q_{-1}]$.

$$[q_2, q_{-1}] = -\begin{pmatrix} 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 1 & & 0 & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Then with the aid of eq. (13) we get

$$[q_2, q_{-1}] = d^+_2 d_1 - d^+_{-1} d_{-2}.$$

The relations (29) give us the required matrix realization of $SU(6)_{QPM}$. The correspondent Schwinger representation can be written in the following compact form:

$$\tilde{q}_\mu = s d^+_\mu + (-1)^\mu d_{-\mu} s^+$$

$$\tilde{p}_\mu = i((-1)^\mu s d^+_{-\mu} - s^+ d_\mu)$$

$$i[\widetilde{q_\mu, p_{\mu'}}] = d^+_\mu d_{\mu'} + (-1)^{\mu+\mu'} d^+_{-\mu'} d_{-\mu} - 2\delta_{\mu\mu'} s^+ s \qquad (30)$$

$$[\widetilde{q_\mu, q_{\mu'}}] = (-1)^{\mu'} d^+_\mu d_{-\mu'} - (-1)^\mu d^+_{\mu'} d_{-\mu}.$$

Thus we have derived eqs. (30), proceeding from definition (13).

It should be emphasized that eqs. (30) have to be always considered together with eq. (11). As $[\tilde{N}, \tilde{g}_\lambda] = 0$ for all λ, $\tilde{N}$ is a c-number in each irreducible representation.

It is obvious from eqs. (3) and (30) that the Holstein-Primakoff realization of the $SU(6)_{QPM}$ generators is obtained from SR if we formally eliminate with the help of eq. (11) the S-boson degree of freedom. But the connection between the two boson expansions is by no means trivial. However, we shall not dwell on this problem as it has been scrupulously investigated in ref.[7].

5. THE PROOF OF QPM AND IBM EQUIVALENCE

We are ready now to turn to the realization of our program, outlined in Sec.3. Substituting the received expressions for $\tilde{q}_\mu$ and $i[\widetilde{q_\mu, p_{\mu'}}]$ (see eqs. (30)) in eq. (5) we easily get:

$$\tilde{Q}_{QPM} = m_1[(d^+s)_{(2\mu)} + \text{h.c.}] + m_2\,(d^+d)_{(2\mu)},$$

which identically coincide with eq. (8), therefore eq. (16) is proved. If we substitute eqs. (30) into eq. (2), the different terms of $\tilde{H}_{QPM}$ give the following Boson structures:

$$e \sum_\mu i[\tilde{q}_\mu, \tilde{q}_\mu] = -10 e s^+ s + 2e \sum_\mu d^+_\mu d_\mu = -10 e \tilde{n}_s + 2 e \tilde{n}_d \qquad (31)$$

$$u \sum_\mu (-1)^\mu \tilde{q}_\mu \tilde{q}_{-\mu} = (2N+3) u \tilde{n}_s + u \tilde{n}_d - 2u s^+ s^+ s s + u\sqrt{5}\,[((d^+d^+)_{(00)} (ss)_{(00)})_{(00)} + \text{h.c.}] \qquad (32)$$

$$v\sum_{\mu}(-1)^{\mu}\tilde{p}_{\mu}\tilde{p}_{-\mu} = (2N+3)v\tilde{n}_s + v\tilde{n}_d - 2vs^+s^+ss - v\sqrt{5}[((d^+d^+)_{(00)}(ss)_{(00)}) + h.c.] \quad (33)$$

$$\frac{w}{2}\sum_{\mu}(-1)^{\mu}\tilde{q}_{\mu}(i\widetilde{[q,p]})_{(2-\mu)} = w\sqrt{5}[((d^+d^+)_{(2\mu)}(ds)_{(2-\mu)})_{(00)} + h.c.] \quad (34)$$

$$\frac{1}{4}\sum_{\substack{L=0,2,4\\M=-L}}^{L}(-1)^M t_L (i\widetilde{[q,p]})_{(LM)}(i\widetilde{[q,p]})_{(L-M)} = \frac{t_0}{5}[\tilde{n}_s + \tilde{n}_d + s^+s^+ss - 2\sqrt{5}((s^+d^+)_{(2\mu)}(sd)_{(2-\mu)})_{(00)}] + \frac{1}{5}\sum_{L=2,4}(2L+1)t_L\tilde{n}_d + \sum_{\substack{L=0,2,4\\M=-L}}^{L}(-1)^M h_{4L}(d^+d^+)_{(LM)}(dd)_{(L-M)},\ h_{4L} = \sum_{L'=0,2,4}(-1)^{L'}(2L'+1)\begin{Bmatrix}2 & 2L\\ 2 & 2L'\end{Bmatrix} t_{L'} \quad (35)$$

In order to reduce the term $S^+S\sum_{\mu} d^+_{\mu}d_{\mu}$, which is absent in $\tilde{H}_{IBM}$ (see eq. (7)), to the terms S^+S and S^+S^+SS, which are present there, we have used eq. (11). It follows from eqs. (31)-(35) that, if we require the fulfilment of the relations (14), then eq. (15) is automatically satisfied. Thus, we have proved the QPM and IBM equivalence "on the level of operators".

6. SUMMARY

It has been rigorously proved that if the QPM Hamiltonian is bosonized after Schwinger, the obtained Hamiltonian contains all Boson structures, which are present in the IBM collective Hamiltonian. We are allowed to choose the phenomenological parameters of two Hamiltonians in such a way that they will be identical. It has been shown also that the electric quadrupole operators of the two models coincide too.

Therefore, it has been shown that QPM and IBM will give the same numerical results for the spectra and E2 -transition rates.

From the derivation of the QPM and IBM equivalence, which has been given here, it follows that QPM can also be considered as a paradigm of a finite Boson representation/7/ and namely, the Schwinger one, as it has not been guessed, but explicitly constructed on the base of the corresponding definition.

To some extent the presented proof could be regarded as a microscopic foundation of the IBM, as far as its Hamiltonian is identical to the QPM Hamiltonian, which has been derived from the microscopic one.

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APPENDIX A

SU(6) Root Vectors

The explicit form of SU(6) root vectors is the following:

$$r(1) = (\tfrac{1}{\sqrt{6}}, 0, 0, 0, 0) \qquad r(9) = (-1/2\sqrt{6}, 1/6\sqrt{2}, 1/12, 1/4\sqrt{15}, 1/\sqrt{10})$$

$$r(2) = (1/2\sqrt{6}, 1/2\sqrt{2}, 0, 0, 0) \qquad r(10) = (0, -1/3\sqrt{2}, 1/3, 0, 0)$$

$$r(3) = (1/2\sqrt{6}, 1/6\sqrt{2}, 1/3, 0, 0) \qquad r(11) = (0, -1/3\sqrt{2}, 1/12, 5/4\sqrt{15}, 0)$$

$$r(4) = (1/2\sqrt{6}, 1/6\sqrt{2}, 1/12, 5/4\sqrt{15}, 0) \qquad r(12) = (0, -1/3\sqrt{2}, 1/12, 1/4\sqrt{15}, 1/\sqrt{10})$$

$$r(5) = (1/2\sqrt{6}, 1/6\sqrt{2}, 1/12, 1/4\sqrt{5}, 1/\sqrt{10}) \qquad r(13) = (0, 0, -1/4, 5/4\sqrt{15}, 0) \quad (A.1)$$

$$r(6) = (-1/2\sqrt{6}, 1/2\sqrt{2}, 0, 0, 0) \qquad r(14) = (0, 0, -1/4, 1/4\sqrt{15}, 1/\sqrt{10})$$

$$r(7) = (-1/2\sqrt{6}, 1/6\sqrt{2}, 1/3, 0, 0) \qquad r(15) = (0, 0, 0, -1/\sqrt{15}, 1/\sqrt{10})$$

$$r(8) = (-1/2\sqrt{6}, 1/6\sqrt{2}, 1/12, 5/4\sqrt{15}, 0)$$

APPENDIX B

"Raising" and "Lowering" Operators in SU (6) QPM

With the aid of eqs. (1) we can derive the relations:

$$[q_{\mu} + ip_{-\mu}, (q_{\mu} + ip_{-\mu})^{+}] = 2(-1)^{\mu}([q_{\mu}, q_{-\mu}] - i[q_{\mu}, p_{\mu}]) \quad (B.1)$$

$$[i[q_\mu, p_\mu], q_\nu + ip_{-\nu}] = (-1)^\nu (2 - \delta_{\mu\nu} - \delta_{\mu-\nu})(q_\nu + ip_{-\nu}) \qquad (B.2)$$

$$[[q_\mu, q_{-\mu}], q_\nu + ip_{-\nu}] = (-1)^\mu (\delta_{\mu\nu} - \delta_{\mu-\nu})(q_\nu + ip_{-\nu}). \qquad (B.3)$$

Similar relations can be obtained for the generators $q_\mu - ip_{-\mu}$. Exactly in the same fashion we receive:

$$[(i[q_\mu, p_{-\mu}]), (i[q_\mu, p_{-\mu}])] = -4(-1)^\mu [q_\mu, q_{-\mu}] \qquad (B.4)$$

$$[i[q_\mu, p_\mu], i[q_\nu, p_{-\nu}]] = 0 \qquad (B.5)$$

$$[[q_\mu, q_{-\mu}], i[q_\nu, p_{-\nu}]] = -2(-1)^\nu (\delta_{\nu-\mu} - \delta_{\mu\nu}) i[q_\nu, p_{-\nu}] \qquad (B.6)$$

From eqs. (B.1)-(B.6) and (17)-(19) it follows, that 10 operators $q_\mu \pm ip_{-\mu} (\mu=0,\pm1,\pm2)$ and 4 operators $i[q_\mu, p_{-\mu}], (\mu=\pm1,\pm2)$ can play the role of "raising" and "lowering" operators.

The most general linear expressions, we can form from the rest of 16 generators (see (23)) are:

$$C_1 [q_\mu, q_{\mu'}] + C_2 i[q_{\mu''}, p_{\mu'''}].$$

If we require that they obey relations like (17)-(19), then as a consequence we get $C_1 = C_2$, $\mu = \mu''$, $\mu' = -\mu'''$. But the rest 16 generators from set (23) are exactly of this type.

With the aid of eqs. (1) we get:

$$[([q_\mu, q_{\mu'}] + i[q_\mu, p_{-\mu'}]), ([q_\mu, q_{\mu'}] + i[q_\mu, p_{-\mu'}])^+] = \qquad (B.7)$$

$$= 2\{(-1)^{\mu'}[q_{\mu'}, q_{-\mu'}] + (-1)^\mu [q_\mu, q_{-\mu}] + i(-1)^{\mu'} [q_\mu, p_\mu] - (-1)^{\mu'} i[q_{\mu'}, p_{\mu'}]\}$$

(to find the correspondence between operators from set (25) and $\{E_{\pm\alpha}\}$ we need only these commutators of $[q_\mu, q_{\mu'}] + i[q_\mu, p_{-\mu'}]$).

We can check, using again eqs. (1) that operators from set (25) satisfy relations, which are consistent with relations (20).

APPENDIX C

The Subsidiary Task

Let us solve the subsidiary task, formulated in IV of Sec.4. For convenience we consider only $q_\mu + ip_{-\mu}$. From (B.1)-(B.3) and eqs. (17)-(20) it follows that $E_\alpha = K_\mu (q_\mu + ip_{-\mu})$. Let us substitute this expression for E_α into eq. (18), taking into account eq. (24):

$$K_\mu \sum_{\ell=1}^{5} (M^{-1})_{k\ell} [H'_\ell, (q_\mu + ip_{-\mu})] = r_k (\pm\alpha) K_\mu (q_\mu + ip_{-\mu}).$$

From here using (B.2) and (B.3) we have:

$$M^{-1} \underset{\sim}{\Theta}_\mu = \underset{\sim}{r}(\pm\alpha) \qquad (C.1)$$

Θ_μ are connected with the integer coefficients in (B.2) and (B.3). The explicit form is:

μ \ ℓ	1	2	3	4	5
-2	(0,	-1,	-2,	-2,	-3)
-1	(1,	0,	2,	3,	2)
0	(0,	0,	-4,	-2,	-2)
1	(-1,	0,	2,	3,	2)
2	(0,	1,	-2,	-2,	-3)

Similarly, proceeding from the commutation relations (19), we obtain:

$$M^{-1} \underset{\sim}{r}(\alpha) = \underset{\sim}{\beta}_\mu . \qquad (C.2)$$

where β_μ are connected with the coefficients in (B.1). Their explicit form is:

μ \ ℓ	1	2	3	4	5
-2	(0,	-1/24,	0,	0,	-1/24)
-1	(1/24,	0,	0,	1/24,	0)
0	(0,	0,	-1/24,	0,	0)
1	(-1/24,	0,	0,	-1/24,	0)
2	(0,	1/24,	0,	0,	-1/24)

Eqs. (C.1) and (C.2) give eqs. (26).

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