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1. Introduction

In two recent works [1, 2] it has been reported on a method proposed for the calculation of the so-called Weinberg functions. These functions are eigenfunctions of the integral kernel of the Lippmann-Schwinger equation and have been first thoroughly examined by Weinberg[3]. They are the solutions of the following integral equation

$$\lambda_n \psi_n = - G(E) V \psi_n , \tag{1.1}$$

where $G(E)$ is the two-body Green function and $-V$ stands for the interaction potential, which will be assumed in what follows to be attractive for all r. In terms of the supplementary function, which is defined by

$$\phi_n = V^{\frac{1}{2}} \psi_n , \tag{1.2}$$

equation (1) reads

$$\lambda_n \phi_n = - V^{\frac{1}{2}} G(E) V^{\frac{1}{2}} \phi_n \tag{1.3}$$

and the orthonormality conditions are

$$(\phi_n, \phi_{n'}) = (\psi_n, V \psi_{n'}) = \delta_{nn'} . \tag{1.4}$$

For positive energies $E > 0$ the integral kernel of eq. (1.3) is a complex and nonhermitian one, however, instead of eqs. (1.1) and (1.3) we can consider those which have real kernels and are defined by

$$\lambda_n \psi_n = -\bar{G}(E)\, V \psi_n \,, \qquad (1.5)$$

$$\lambda_n \phi_n = -V^{1/2}\, \bar{G}(E)\, V^{1/2}\, \phi_n \,, \qquad (1.6)$$

where $\bar{G}(E)$ denotes the principal value of the Green function. The latter equations have real eigenvalues and real eigenfunctions, satisfying the orthonormality condition (1.4). From the fact that the kernel of eq. (1.6) at $E > 0$ is not of one sign it follows that in any expansions in terms of the real Weinberg functions the terms corresponding to positive and negative eigenvalues must be taken into account. This has been shown in previous works [1,2] using the square well, the Hulthen and the Yukawa potentials and calculating the K-matrix.

Potentials of the Saxon-Woods form have been widely employed especially in nuclear structural calculations. Therefore, it seems to be very useful to have a manageable method in calculating the Weinberg functions for such types of potentials. Then, for example these functions can be applied to describe the continuous part of the ordinary shell model spectrum and, e.g., form factors for stripping reactions to unbound states can be calculated in a rather simple manner [4].

As an application of the real Weinberg function in sec. 4 we will consider the low-energy scattering of neutrons and calculate the neutron strength function. Before we will present our method in sec. 2 and perform methodical calculations in sec. 3. Concluding remarks can be found in sec. 5.

2. The Saxon-Woods Potential and the Calculation of the Weinberg Function

The method proposed for the calculation of the Weinberg functions using an ordinary Saxon-Woods potential of the form

$$-V(r) = \frac{2m}{\hbar^2}\, U_0 \left(1 + \exp\left(\frac{r-R}{a}\right)\right)^{-1} \qquad (2.1)$$

is close to that employed in the previous works [1,2]. The Fourier transform for the ℓ-th partial wave is given by

$$V_\ell(k,k') = \int_0^\infty r^2 dr\, V(r)\, j_\ell(kr)\, j_\ell(k'r)\,. \qquad (2.2)$$

Our aim is to find for (2.2) a separable representation of the same kind as in refs. [1,2]. This means that we envisage for (2.2) an expression of the form

$$V_\ell(k,k') = \sum_{i=1}^{N} V_i\, j_\ell(k x_i)\, j_\ell(k' x_i)\,, \qquad (2.3)$$

where the quantities V_i and X_i are respectively the weights and knot points of some quadrature polynomials. In so doing we decompose the integral in (2.2) into two parts

$$V_\ell(k,k') = \frac{2m}{\hbar^2} U_0 R^3 \int_0^1 \left[1 + \exp\left(\frac{xR-R}{a}\right)\right]^{-1} j_\ell(k x R) \times$$

$$\times j_\ell(k' x R)\, x^2\, dx + \frac{2m}{\hbar^2} U_0\, a \int_0^\infty (ax + R)^2 \exp(-x)\,(1 + \exp(-x))^{-1} \times$$

$$\times j_\ell(k(ax + R))\, j_\ell(k'(ax + R))\, dx \qquad (2.4)$$

and can approximate each integral with high accuracy making use again of the Gauss quadrature polynomials. The weights for both intervals $[0,1]$ and $[0,\infty]$ can be expressed analytically through Legendre or, respectively, Laguerre polynomials. On the accuracy of the expansion (2.3) it will be reported in the next section.

Inserting (2.3) into (1.5) one gets for the Weinberg functions

$$\psi_n(k,E) = -\frac{1}{\lambda_n(E)}\, \frac{1}{E-k^2} \sum_{i=1}^{N} V_i^{1/2}\, j_\ell(k r_i)\, c_i^{(n)}(E)\,. \qquad (2.5)$$

The coefficients $c_i^{(n)}$ are defined as follows

$$c_i^{(n)} = \frac{2 V_i^{1/2}}{\pi} \int_0^\infty k^2 dk \, j_\ell(kr_i) \, \psi_n(k,E) = V_i^{1/2} \psi_n(r_i, E) \quad (2.6)$$

and satisfy the homogeneous system of linear equations

$$\sum_{j=1}^N [\lambda_n(E) \delta_{ij} + (V_i V_j)^{1/2} \bar{G}(r_i, r_j; E)] c_j^{(n)} = 0, \quad (2.7)$$

where the Green function is defined by

$$\bar{G}(r, r'; E) = \frac{2}{\pi} \int_0^\infty k^2 dk \, \frac{j_\ell(kr) j_\ell(k'r)}{E - k^2} \quad (2.8)$$

and can be evaluated analytically.

From eqs. (2.7) it becomes evident why our method is so profitable. Namely, the approximation (2.3) makes it possible to obtain the eigenvalues and eigenfunctions after diagonalizing the real symmetrix matrix $(V_i V_j)^{1/2} \bar{G}(r_i, r_j, E)$ and no problems arise evaluating the Green function.

Another advantage of the method proposed is connected with the fact that for higher partial waves the approximation (2.3) converges not worse than for the s -waves, because the factor $r^{2\ell}$ can be taken away from the Bessel functions and included into the weights [1,2]. Therefore, it is sufficient in the following to investigate the accuracy of the approximation (2.3) for the case $\ell = 0$, only. This fact facilitates the subsequent discussions because, for instance, for s -waves the K -matrix can be calculated in an exact way and compared with the approximated one. In so doing the accuracy of the expansion (2.3) can be checked rather well.

3. The K -Matrix

The on-shell K -matrix is defined by

$$\frac{1}{k} \tan \delta_\ell \equiv K(E) = \sum_n \frac{1}{1-\lambda_n} (j_\ell(kr) V(r) \psi_n(r,E))^2. \quad (3.1)$$

Replacing in (3.1) the Weinberg function by the approximated ones

$$\psi_n(r,E) = -\frac{1}{\lambda_n(E)} \sum_{i=1}^N V_i^{1/2} \bar{G}(r, r_i; E) \, c_i^{(n)}(E) \quad (3.2)$$

and taking into account eq. (2.6) one gets for the approximated on-shell $\tilde{K}$ -matrix a rather simple expression

$$\tilde{K}_\ell(E) = \sum_n (\sum_{i=1}^N j_\ell(kr_i) V_i^{1/2} c_i^{(n)})^2 / (1 - \lambda_n). \quad (3.3)$$

The exact result for the K -matrix is given by [5]

$$K_0(E) = \frac{1}{k} \tan[\arg F(ia(\kappa+k), -ia(\kappa-k), 1+2iak, -e^{-R/a})],$$
$$(3.4)$$

where F is the hypergeometric function, $\kappa^2 = k^2 + \frac{2m}{h^2} U_0$ and m is the reduced mass.

To criticize the approximation (2.3) it seems to be rather instructive to investigate the case, when the diffuseness parameter of the potential, a, is equal to zero. This means that we get an ordinary square well potential and can again exactly determine the K -matrix and the Weinberg functions (see f.i. refs. [1,2]). In table 1 the corresponding results are shown. We see that the method proposed yields eigenvalues and scattering phases, which for various energies differ from the exact ones only by about few percents at the worst. This result demonstrates the accuracy of the above introduced approximation rather evidently. Now we turn to the calculation of the K -matrix elements employing the Saxon-Woods potential. In table 2 the values of the approximated and exact K -matrix are compared to each other for different energies. To obtain a good convergence after a few terms is crucial in practical computations. From the comparison of the exactly and approximately calcula-

Table 2

The values of (3.3) summed up to the n-th eigenvalues for the ordinary Saxon-Woods potential (2.1) with the parameters $U_0 = 48$ MeV, $R = 1.27A^{1/3}$ and $a = 0.65$ fm. The mass number is $A = 16$. The exact values are due to eq. (3.4). The number of knot points is $N_1 = 15$ and $N_2 = 20$.

n	1	2	3	4	exact
E=.0	-3.660	-7.129	-6.634	-6.506	-6.491
E=.0001	-3.660	-7.129	-6.634	-6.506	-6.492
E=.001	-3.660	-7.133	-6.637	-6.509	-6.495
E=.01	-3.664	-7.166	-6.668	-6.539	-6.524
E=.1	-3.704	-7.516	-6.992	-6.854	-6.840
E=1.0	-4.074	-14.878	-14.084	-13.831	-14.175
E=10.0	1.454	-0.489	0.166	0.271	0.359

(handwritten annotations: $\frac{1}{k}\tan\tilde\delta$)

Table 1

The values of (3.3) summed up to the n-th eigenvalue for the square well potential with the parameters $V_0 = 1\,fm^{-2}$ and $R = 1\,fm$. The last two columns show the corresponding exact values. The number of knot points is $N_1 = 15$.

n	$\frac{1}{\pi^2}\tilde\lambda_n^2$	$\frac{1}{\pi}\tan\tilde\delta$	$\frac{1}{\pi^2}\tilde\lambda_n^2$	$\frac{1}{\pi}\tan\tilde\delta$
$\tilde k = .1$ [fm^{-1}]				
1	.5501	.4042	.5497	.4036
2	.5544	.0456	.5539	.0450
3	.5550	.0168	.5545	.0162
4	.5551	.0086	.5546	.0083
5	.5552	.0056	.5547	.0050
exact			.5547	.5547
$\tilde k = 1.0$ [fm^{-1}]				
1	.3535	.2643	.3533	.2537
2	.3641	.0420	.3639	.0414
3	.3657	.0163	.3655	.0157
4	.3662	.0087	.3660	.0081
5	.3664	.0056	.3661	.0049
exact			.3663	.3663
$\tilde k = 2.0$ [fm^{-1}]				
1	.0536	.0921	.0534	.0914
2	.1443	-.0430	.1443	-.0433
3	.1482	.0208	.1481	.0202
4	.1491	.0098	.1490	.0092
5	.1492	.0060	.1493	.0053
exact			.1497	.1497

ted K_0-matrix elements it becomes evident that the convergence of the expansion (3.3) is not so fast as observed earlier, when using the Yukawa, Hulthen or square well potentials. Such an effect could be expected because the effective range of the Saxon-Woods potential is relatively large in comparison with that of the Yukawa potential used earlier. However, for practical applications the convergence stated in table 2 seems to be satisfactory.

Note that without any difficulties the case $E=0$ can be considered replacing the Green function in (2.8) by

$$\bar{G}(r, r'; 0) = \begin{cases} \dfrac{1}{r'} & r < r' \\ \dfrac{1}{r} & r > r' \end{cases} \qquad (3.5)$$

For the on-shell K-matrix (s-waves) one gets

$$K(E \to 0) = \lim_{k \to 0} \frac{1}{k \cot \delta_0} = [-\frac{1}{a_s} + \frac{1}{2} r_0 k^2 + 0(k^4)]^{-1} = \qquad (3.6)$$

$$= \sum_{n}^{N} \frac{1}{1 - \lambda_n} (\sum_i V_i^{1/2} c_i^{(n)})^2 .$$

In this way we have found an expression to determine the scattering lengths a_s and effective radius r_0 for relatively complicated potentials.

4. The Neutron Strength Function

In order to exemplify the applicability of the formalism developed in the first sections, we calculate the s-wave neutron strength function. The latter is determined by introducing a complex optical potential, calculating the corresponding compound cross section for $\ell = 0$ and comparing with the low-energy neutron average compound cross section given by

$$\sigma_c^o = \frac{2\pi^2}{k^2} \sqrt{E_{eV}} <\Gamma^o>/D (1 - \pi \frac{<\Gamma_n^o>}{}/D) . \qquad (4.1)$$

The quantity $<\Gamma_n^o>/D$ is called the strength function, where $<\Gamma_n^o>$ is the average neutron width at 1 eV and D is average level spacing. It is rather simple to introduce the complex optical potential having a volume absorption term into our method replacing only U_0 in the expression for the K-matrix by the complex quantity $U_0 + iW_0$. The Weinberg function remains then the same as before, while the K-matrix becomes a complex quantity. Such a representation of the K-matrix is rather favourable investigating, for instance, its poles. From the expression

$$\delta = \arctan(k \cdot K(E)) = -\frac{i}{2} \ln \frac{1 + ik K(E)}{1 - ik K(E)} \qquad (4.2)$$

the complex scattering phases can be determined.

Figure 1 shows the neutron s-wave strength function calculated for the energy $E = 1$ eV by means of (4.1). There can be seen the usual shape of the neutron s-wave strength function depending on the mass number.

Note, that also for larger mass numbers the exact solution coincides with the approximated one nearly completely. The differences are less than 1% at the worst including 4 terms in the expansion (3.3).

Note also, that in calculating the strength function it is a problem numerically to integrate the corresponding Schrödinger equation at small energies ($E \ll 1$ MeV), because the results depend rather sensitively on the integration mass size. Moreover, we must numerically integrate up to a very large matching radius R_m in order to satisfy the condition $kR_m \gg 1$. Therefore, usually first σ_c^c is evaluated using the optical model at 1 keV neutron energy and the strength function is deduced using the above equation (4.1). From table 2 it can be seen that
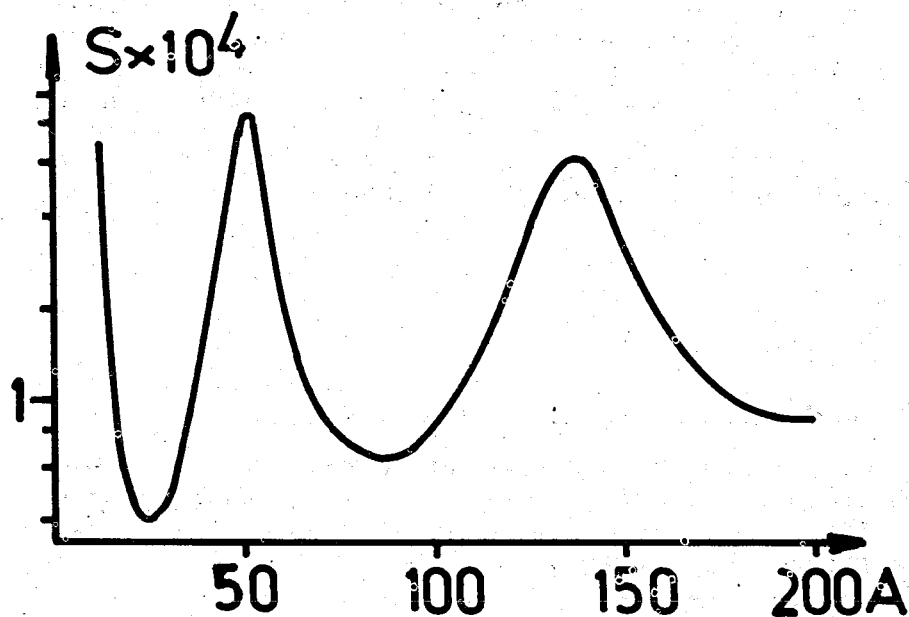
$S \times 10^4$

50   100   150   200A

Fig. 1. The neutron strength function calculated at $E = 1$eV as depending on the mass number A. The optical parameters are $U_0 = 52$ MeV, $W_0 = 3.12$ MeV, $r_0 = 1.25$ fm and $a = 0.52$ fm. For the number of knot points see table 2.

the incident energy of 1 keV is quite satisfactory to evaluate the strength function with high accuracy.

5. Conclusion

The method proposed for the calculation of the real Weinberg function at positive energies using the Saxon-Woods potential seems to be rather convenient for practical applications, when calculating scattering phases or resonance states belonging to not very high energies ($E \leq 50$ MeV). Because the K -matrix expanded in terms of the Weinberg functions takes a rather simple form it is possible to study, for instance, resonance phenomena arising by using the complicated complex optical potential in a rather convenient manner.

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