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SOLUTION OF THREE-DIMENSIONAL LIPPMAN-SCHWINGER EQUATION WITHOUT PARTIAL-WAVE EXPANSION

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A series of problems do exist in nuclear physics with physical systems having no central symmetry. In these cases finding solution to Schrödinger equation for the wave function ψ or the Lippmann-Schwinger equation for the t-matrix as a series over eigenfunctions of angular momentum may appear to be noneffective. Examples are: motion of a particle in the noncentral potential as well as the scattering characterized not by orbital momentum but definite momentum value, i.e., the scattering at rather high energies. In the latter case at a fixed energy the convergence of partial wave expansion is defined in general by range of forces. At low energies and for short-range interaction potential we may restrict ourselves, as is well known, to a small amount of partial waves and so deal with solving small number of the one-dimensional equations. At very high energies and small scattering angles, when the eikonal approximation works, the partial wave expansion may be avoided. However the region of intermediate angles and energies does exist where none of the above approximations does work.

In this note we propose a method for solving three-dimensional Lippmann-Schwinger equation, without the partial wave expansion and therefore suitable for the situations indicated above.

Consider the scattering of two spinless particles interacting via a potential $V(r)$. The t-matrix is the solution of the Lippmann-Schwinger equation which in momentum representation has the form:

$$\langle\vec{k}|t(z)|\vec{k}'\rangle = \langle\vec{k}|V|\vec{k}'\rangle - \int d\vec{q}\,\langle\vec{k}|V|\vec{q}\rangle G_0(q^2;z)\langle\vec{q}|t(z)|\vec{k}'\rangle\,, \qquad (1)$$

where z is energy of the system, $G_0(q^2;z)$ the Green function of noninteracting particles.

Let us now introduce a set of N linearly independent functions $\chi_i(\vec{r},\alpha_i)$, $i=1,\dots,N$,

α_i – a set of parameters which will be fixed below.

We shall consider the following matrix elements of the potential $V(\vec{r})$:

$$\langle\vec{k}|V|\chi_i\rangle = (2\pi)^{-3/2}\int d\vec{r}\,e^{i\vec{k}\vec{r}}V(\vec{r})\chi_i(\vec{r},\alpha_i) \equiv \eta_i(\vec{k}). \qquad (2)$$

By means of these quantities we construct an approximate potential[1] with the following matrix elements[x]:

$$\langle\vec{k}|\tilde{V}|\vec{k}'\rangle = \sum_{i,j=1}^{N}\langle\vec{k}|V|\chi_i\rangle[d^{-1}]_{ij}\langle\chi_j|V|\vec{k}'\rangle\,, \qquad (3)$$

where

$$d_{ij} = \langle\chi_i|V|\chi_j\rangle\,.$$

There parameters α_i can be fixed minimizing the functional (4) with respect to these parameters:

$$\chi^2(\alpha_i) = \frac{\int|\langle\vec{k}|V|\vec{k}'\rangle - \langle\vec{k}|\tilde{V}|\vec{k}'\rangle|^2\rho(\vec{k},\vec{k}')\,d\vec{k}}{\int|\langle\vec{k}|V|\vec{k}'\rangle|^2\rho(\vec{k},\vec{k}')\,d\vec{k}}\,, \qquad (4)$$

x/ Discussion on the possibility of the separable expansion for a two-body potential can be found also in works[4,5].

Substituting (3) into (1) we obtain for the approximate t-matrix:

$$\langle\vec{k}|\tilde{t}(z)|\vec{k}'\rangle = \sum_{i,j=1}^{N}\eta_i(\vec{k})[\vec{A}(z)]_{ij}\eta_j(\vec{k}')\,, \qquad (5)$$

where the complex matrix $A_{ij}(z)$ is of the form:

$$A_{ij}(z) = d_{ij} - \int\eta_i(\vec{k})G_0(k^2;z)\eta_j(\vec{k})\,d\vec{k} \equiv d_{ij} - I_{ij}(z)\,. \qquad (6)$$

Thus, eqs. (5) and (6) define in quadratures the solution to the three-dimensional Lippmann-Schwinger equation with an arbitrary short-range potential as the series in the known functions $\eta_i(\vec{k})$.

The short range character of a potential is required for the practical use of the expansion (3). The point is that the number of terms in this expansion cannot be smaller than that of bound states in the potential $V(r)$. Therefore if we would try to apply this expansion, e.g., to the Coulomb attractive potential we should involve, generally speaking, infinite number of terms.

Now, we introduce N various vectors $\vec{\alpha}_i$ on the scattering plane, i.e.,on the plane formed by vectors $\vec{k}$ and $\vec{k}'$, and as linearly independent functions $\chi_i(\vec{r})$ we take the plane waves, i.e.,

$$\chi_i(\vec{r}) = (2\pi)^{-3/2}e^{i\vec{\alpha}_i\vec{r}}\,. \qquad (7)$$

In this case the expansion (3) represents the special interpolation formula for the Fourier transform of the potential (1). Indeed, from (3) it follows that if at least one of the vectors $\vec{k}$, $\vec{k}'$ coincides with one of the vectors $\vec{\alpha}_i$, then the approximate potential (3) coincides with the exact one. Hence, the expansion (3) with the functions $X_i(\vec{z})$ (eq.(7)) is the three-dimensional generalization of the known Bateman expansion /2/.

Consider now the concrete case of the Gaussian potential $V(z)=V_0 e^{-ar^2}$, and we investigate properties of the approximate t-matrix given by (5).

For the Fourier transform of the potential we have

$$\langle \vec{k}|V|\vec{k}'\rangle = \frac{V_0}{8(\pi a)^{3/2}} e^{-(\vec{k}-\vec{k}')^2/4a},$$
$$\eta_i(\vec{k}) = \frac{V_0}{8(\pi a)^{3/2}} e^{-(\vec{k}-\vec{\alpha}_i)^2/4a}. \qquad (8)$$

In this case the matrix $A_{ij}(z)$ has the form:

$$A_{ij}(z) = \frac{V_0}{8(\pi a)^{3/2}} e^{-(\alpha_i^2+\alpha_j^2)/4a}\left[e^{\vec{\alpha}_i\cdot\vec{\alpha}_j/4a} - \frac{V_0}{\sqrt{\pi a}\,|\vec{\alpha}_i+\vec{\alpha}_j|}\int_0^\infty e^{-\frac{k^2}{2a}} sh\left(\frac{k}{2a}|\vec{\alpha}_i+\vec{\alpha}_j|\right) G_0(k,z)\,k\,dk\right] \qquad (9)$$

The factor $e^{-(\alpha_i^2+\alpha_j^2)/4a}$ in (9) and the analogous factor in the product $\eta_i(\vec{k})\eta_j(\vec{k}')$ in (5) cancel out.

Making use of the expression (5) we find the forward scattering on-mass-shell amplitude for the potential (8). To this end we choose auxiliary vectors $\vec{\alpha}_i$ along the scattering direction.

Remembering of the interpolation character of (3) for the potential, such a choice of the vectors $\vec{\alpha}_i$ seems to be natural. Lengths of the vectors $\vec{\alpha}_i$ can be chosen as usual by the condition (4). However, in our calculations $|\vec{\alpha}_i|$ were taken in such a way that the approximate potential coincides with the exact one at points of the interval $0 \le \frac{\alpha_i^2}{2\mu} \le 400$ MeV. And varying the position of inner points α_i proved to influence not strongly the magnitude of the forward scattering amplitude $f^N(\vec{k},\vec{k}')$ $(N=6)$. The parameters V_0 and a of the potential (8) were taken to be:

$$V_0 = -67.97 \text{ MeV}; \quad a = 0.424 \text{ fm}^{-2}.$$

Such a potential desoribes the triplet S-wave scattering at low energies.

The real and imaginary parts of the forward on-mass-shell scattering amplitude (in the approximation $N=6$) are drawn in Figs. 1 and 2. For comparison the Born and eikonal amplitudes as energy functions are presented. It is interesting that beginning from energy ≤ 100 MeV $Ref(E,0)$ becomes rather close to the Born amplitude. The eikonal amplitude for the potential under consideration is of the form:

$$f(k,k) = \frac{c}{a}\left\{1+\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{(2n+1)(2n+1)!}\left(\frac{2c}{k}\right)^{2n} + i(-1)^{n+1}\frac{2^{2(n-1)}}{n(2n)!}\left(\frac{c}{k}\right)^{2n-1}\right]\right\},$$

where $c = -\sqrt{\frac{\pi}{a}}V_0 m$.

As it was to be expected, at sufficiently high energies it becomes slightly different from the approximate amplitude (5). If the amplitude (5) is treated as "exact" one then it should be concluded that no range of application does exist for the eikonal approximation.

Note should be taken that unlike, e.g., eikonal amplitude the approximate amplitude (5) has a pole at the negative energy not very different from the deuteron binding energy.

In Figs 3 and 4 $Re f(E,0)$ and $Im f(E,0)$ are pictured for two cases: 1) N=4 and 2) N=5 with $\alpha_S = \sqrt{E}$. As is seen from the Figures, the interpolation parameter of the potential depending on energy governs essentially the behaviour of the amplitude at high energies[3]. Figs. 5 and 6 illustrate the convergence of $Re f(E,\theta)$ and $Im f(E,\theta)$ at $\theta = \frac{\pi}{4}$ (the convergence of the forward scattering amplitude in N is roughly the same). It is seen that through out the whole considered energy region the expansion terms (eq.7) with $N \geq 7$ are practically not essential, and $Re f(E,\theta)$ as the forward scattering amplitude is close to the Born value already at 100 MeV. It should be noted that our choice of the parameters $\vec{\alpha}_i$ is not the best one, therefore one can expect that for optimal choice of these parameters the number of essential terms in (5) becomes smaller. Choice of parameters $\vec{\alpha}_i$ is shown in Fig.7.

In Figs. 8 and 9 the sensitivity of the approximate potential (3) on the mass shell to small variations of angles and lengths of the vectors $\vec{\alpha}_i$ with respect to some choice which appears to be the best, is shown. It is seen that at low energies ($E \leq 50$ MeV) these variations change slighly the potential value.

Let us note that the variations of angles of the vectors $\vec{\alpha}_i$ at fixed lengths shift simultaneously the whole curve with respect to the exact one $V(E,\theta)$. At the same time the variation of vector lengths at fixed angles results in the energy dependent change of the potential.

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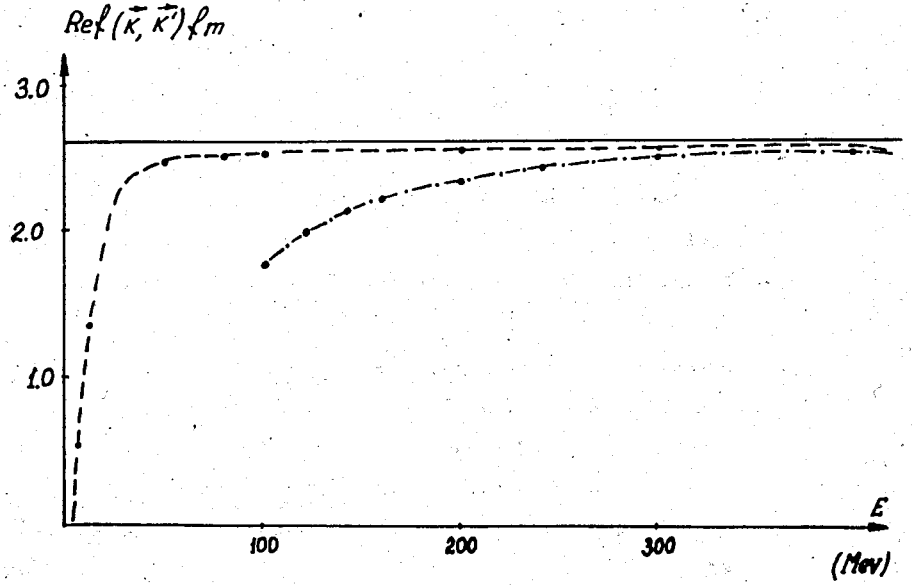

Fig.1. The real part of the forward scattering amplitude. Solid line represents the Born approxiмation, dot-dashed line -the eikonal approximation, dashed line - the Bateman method (N=6) (eq.5)

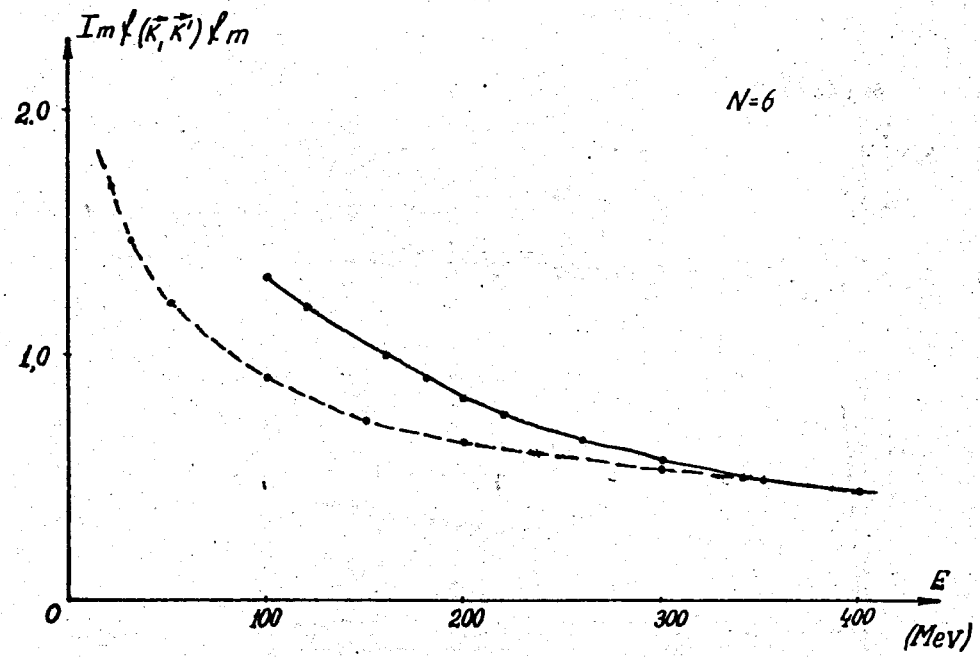

Fig.2. The imaginary part of the forward scattering amplitude. Solid line represents the eikonal approximation, dashed line - the Bateman method (N=6).

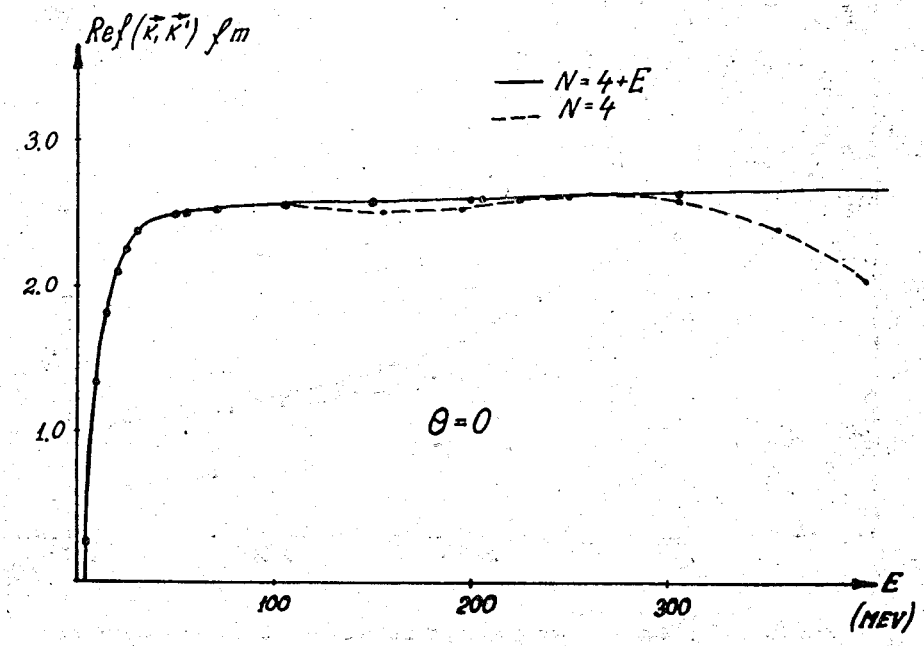

Fig.3. The real part of the forward scattering amplitude (both curves Bateman approximation).

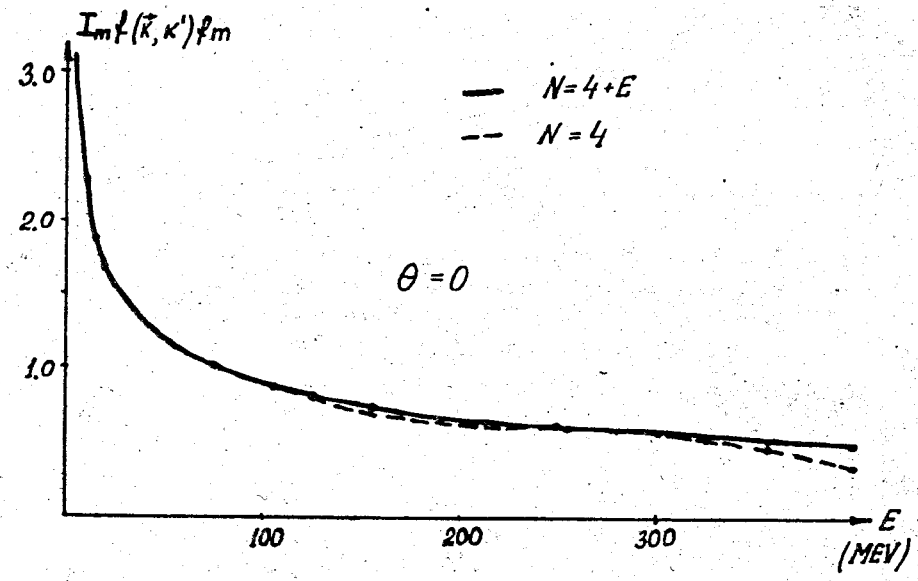

Fig.4. The imaginary part of the forward scattering amplitude (the same approximations, as in Fig.3).

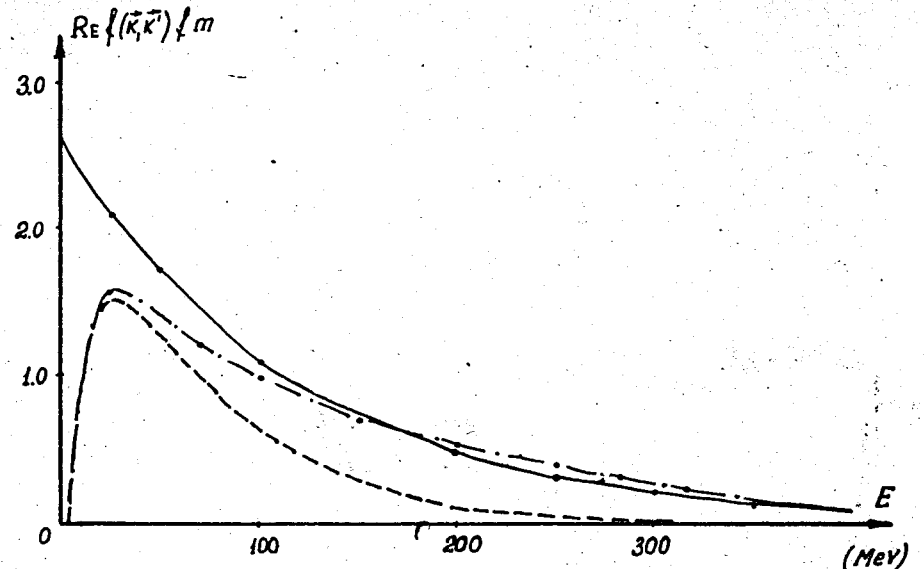

Fig.5. Dependence of the real part of scattering amplitude on the number of separable terms in the factorized potential ($\Theta=\pi/4$). Solid line–Born approximation; dot-dashed line–N=7,10,13; dashed line – N=4.

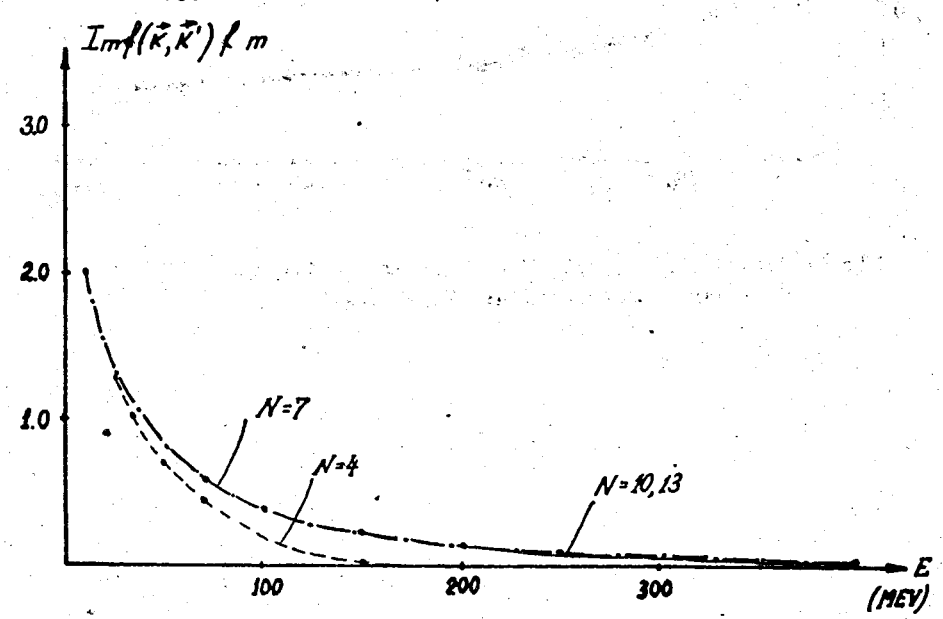

Fig.6. Dependence of the imaginary part of the scattering amplitude on the number of separable terms in the factorized potential ($\Theta=\pi/4$).

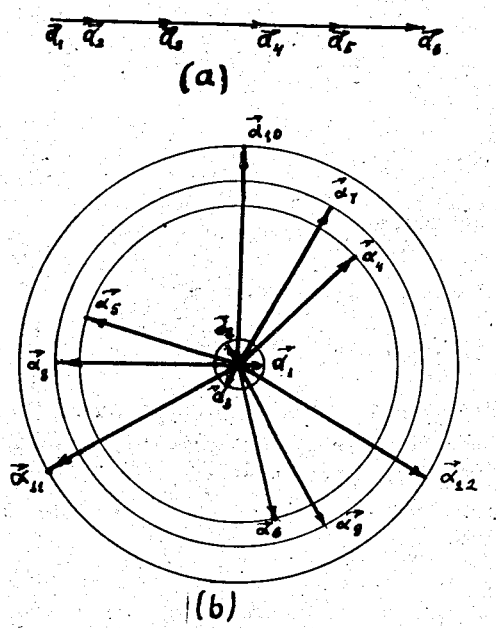

Fig.7. The choice of the parameters $\vec{\alpha}_i$ (in dimensionless quantities, the parameter of passing to dimensionless quantities is α =0.3408 fm.).

a) the case: Θ =0:

$|\vec{\alpha}_i| = 0;\ 0.2;\ 0.5;\ 0.9;\ 1.2;\ 1.5$

b) the case: $\Theta = \pi/4$

$\alpha_1=\alpha_2=\alpha_3=0.1;\ \alpha_4=\alpha_5=\alpha_6=0.75;$

$\alpha_7=\alpha_8=\alpha_9=0.9;\ \alpha_{10}=\alpha_{11}=\alpha_{12}=1.1;\ \alpha_{13}=0.$

$\Theta_{1,2}\equiv(\hat{\vec{\alpha}_1,\vec{\alpha}_2})=\frac{2}{3}\pi;\ \Theta_{1,3}=\frac{4}{3}\pi;\ \Theta_{1,4}=\frac{7}{30}\pi;$

$\Theta_{1,5}=0.9\pi;\ \Theta_{1,6}=\frac{47}{30}\pi;\ \Theta_{1,7}=\frac{\pi}{3};\ \Theta_{1,8}=\pi;\ \Theta_{1,9}=\frac{5}{3}\pi;$

$\Theta_{1,10}=\frac{\pi}{2};\ \Theta_{1,11}=\frac{7}{6}\pi;\ \Theta_{1,12}=\frac{11}{6}\pi.$

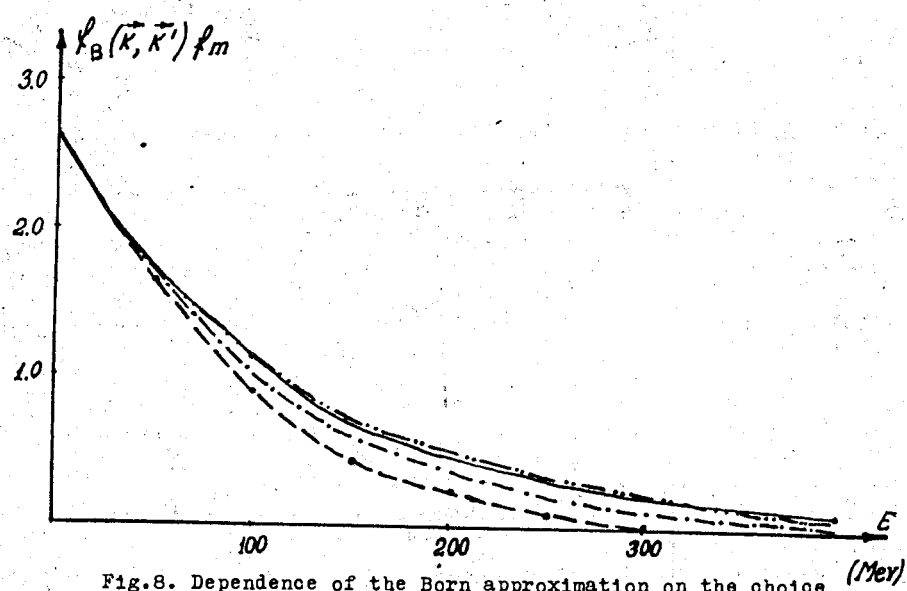

Fig.8. Dependence of the Born approximation on the choice of parameters of factorization. The second circle (see Fig.7) was rotated. Solid line – $V(E,\theta)$.

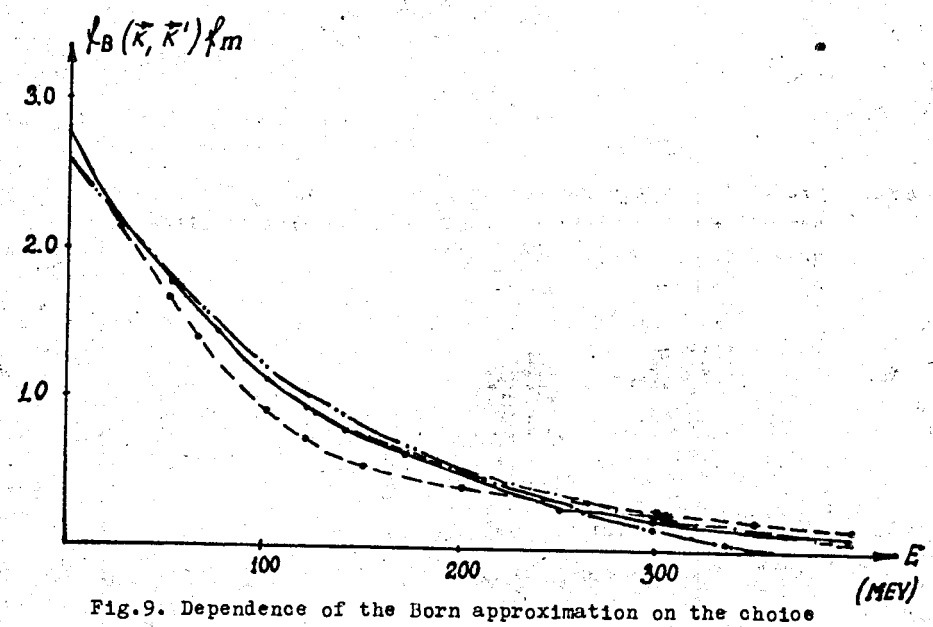

Fig.9. Dependence of the Born approximation on the choice of parameters of potential factorization. The radius of the second circle is under change. Solid line – $V(E,\theta)$.