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SOME APPROACHES FOR SOLVING THE THREE-BODY PROBLEM AT POSITIVE ENERGY

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1. Introduction

Series of works have been successfully performed at present, investigating the properties of the three-nucleon bound states by means of different approaches. Most of them are quite well worked up and can be applied to the solution of the three nucleon problems involving rather complicated realistic interactions between the nucleons.

However, when the total energy for the three-nucleon system is positive, each of the proposed approaches contains its specific difficulties. For instance, using the variation approaches or those based on an expansion in terms of K-harmonics, one is faced with the problem of the asymptotic behaviour of the wave function in the configuration space[1]. Furthermore, in the integral equation formulation of the three-nucleon problem[2] the consideration of positive energies leads to the appearing of moving singularities of pure kinematic origin, which rather complicates the solution of the corresponding integral equation. In spite of these serious difficulties a certain progress in solving the Fadeev-equation at positive energies has recently been achieved[3].

Two possibilities of overcoming the above-mentioned difficulties will be considered below. From a methodical point of view it is worth comparing some different methods of solving the integral equation using a model equation which has a rather simple structure and contains at the same time all the singularities of the realistic case. The equation of Skornyakov-Ter-Martirosyan for the s-wave quartet nd-scattering is just of this type. This equation has the following form:

$$\frac{\sqrt{\frac{3}{4}k^2-E}-\alpha}{k^2-k_0^2}\,a(k,k_0)=-\frac{G_0(k,k_0)}{2kk_0}-\frac{2}{\pi}\int_0^\infty k'^2dk'\,\frac{G_0(k,k')}{2kk'}\cdot\frac{a(k',k_0)}{k'^2-k_0^2-i\varepsilon}, \qquad (1)$$

where $a(k_0, \kappa_0)$ is the s-component of the quartet nd-soattering amplitude and

$$G_0(k,k') = \int_{-1}^{1} \frac{dx}{Q+x-i\varepsilon} \quad , \quad Q = \frac{k^2+k'^2-E}{kk'} \quad ,$$

where $E = -\alpha^2 + \frac{3}{4}\kappa_0^2$ stands for the total energy of the system and the quantities α^2 and κ_0^2, z are given by

$$\alpha^2 = \frac{m\varepsilon_d}{\hbar^2} \quad , \quad \kappa_0^2 = \frac{4}{3}\cdot\frac{mE_n}{\hbar^2} \quad , \quad z = E + i\varepsilon \, .$$

Let us now consider the dimensionless variables defined by

$$k = \sqrt{\tfrac{4}{3}E}\cdot x, \quad k' = \sqrt{\tfrac{4}{3}E}\cdot y, \quad k_0 = \sqrt{\tfrac{4}{3}E}\cdot x_0, \quad \alpha = \sqrt{\tfrac{4}{3}E}\cdot\gamma \quad ,$$

and instead of the function $a(k_0, k_0)$ introduce a new function

$$B(x) \equiv B(x, x_0) = \frac{\sqrt{E}\, a(x, x_0)}{\left(2\gamma/\sqrt{3} + \sqrt{x^2-1}\right)\left[1 + i x_0 a(x_0, x_0)\right]}$$

which satisfies the following equation:

$$B(x) = -\frac{G_0(x,x_0)}{2x x_0} - \frac{8}{3\pi} P\int_0^\infty y^2 dy \, \frac{G_0(x,y)}{2xy} \, \frac{\gamma - i\frac{\sqrt{3}}{2}\sqrt{1-y^2}}{y^2 - x_0^2} B(y) \quad , \quad (2)$$

where

$$G_0(x,y) = G^{(1)}(x,y) + i\pi G^{(2)}(x,y) \quad ,$$

$$G^{(1)}(x,y) = \ln\left|\frac{x^2+y^2+xy-3/4}{x^2+y^2-xy-3/4}\right| \, ; \quad G^{(2)}(x,y) = \theta\left(\frac{x^2+y^2-3/4}{xy}\right) , \quad (3)$$

$$\theta(t) = \begin{cases} 1 & |t|<1 \\ 0 & |t|>1 \, . \end{cases}$$

As it is seen from (3) the kernel of the integral equation (2) has moving singularities of the logarithmic type defined by the following equations:

$$\left.\begin{aligned} y_1 &= -\frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-x^2} \\ y_3 &= \frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-x^2} \end{aligned}\right\} \quad 0 \le x \le \frac{\sqrt{3}}{2}$$

$$\left.\begin{aligned} y_4 &= \frac{x}{2} - \frac{\sqrt{3}}{2}\sqrt{1-x^2} \\ y_3 &= \frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-x^2} \end{aligned}\right\} \quad \frac{\sqrt{3}}{2} \le x \le 1$$

The position of the singularities of the kernel in the (x,y)-plane is shown in Fig.1.

2. The Expansion of the Iterated Kernel

As it was shown in Ref.[2] the logarithmic singularities of the kernel in the Faddeev equation can be weakened or abolished performing a suitable number of iterations. Therefore, instead of equation (2) we shall consider an iterated one:

$$B(x) = -\frac{G_0(x,x_0)}{2x x_0} + \frac{8}{3\pi}R(x,x_0) + \left(\frac{8}{3\pi}\right)^2 P\int_0^\infty z^2 dz \, R(x,z)\, \frac{\gamma - i\frac{\sqrt{3}}{2}\sqrt{1-z^2}}{z^2 - x_0^2} B(z) \, , \quad (4)$$

where

$$R(x,z) = \frac{1}{4xz} P\int_0^\infty dy \, G_0(x,y) \, \frac{\gamma - i\frac{\sqrt{3}}{2}\sqrt{1-y^2}}{y^2 - x_0^2} \, G_0(y,z) \, .$$

The kernel $R(x,z)$ emerging after the iterations does not contain moving singularities and therefore it is natural to anticipate that its expansion in the corresponding set of orthogonal functions will converge faster than the analogous expansion of the original kernel $G_0(x,y)$.

If one considers in more details the structure of the iterated kernel $R(x,z)$, we have :

$$R(x,z) = R_1(x,z) + i R_2(x,z) \quad ,$$

and for the quantities R_1 and R_2 the following decompositions take place:

$$R_1(x,z) = R_1^{(1)}(x,z) + R_1^{(2)}(x,z) + R_1^{(3)}(x,z) + R_1^{(3)}(z,x) ,$$

$$R_2(x,z) = R_2^{(1)}(x,z) + R_2^{(2)}(x,z) + R_2^{(3)}(x,z) + R_2^{(3)}(z,x) ,$$

where

$$R_1^{(1)}(x,z) = \frac{\gamma}{4xz}\int_0^1 \frac{dy}{y^2-x_0^2} G^{(1)}(x,y) G^{(1)}(y,z) + P\int_1^\infty \frac{dy}{4xz}\frac{[x+\frac{\sqrt{3}}{2}\sqrt{y^2-1}]}{(y^2-x_0^2)} G^{(1)}(x,y) G^{(1)}(y,z),$$

$$R_1^{(2)}(x,z) = -\frac{\pi^2\gamma}{4xz}\int_0^1 \frac{dy}{y^2-x_0^2} G^{(2)}(x,y) G^{(2)}(y,z) ,$$

$$R_1^{(3)}(x,z) = \frac{\sqrt{3}}{2}\cdot\frac{\pi}{4xz}\int_0^1 dy \frac{\sqrt{1-y^2}}{y^2-x_0^2} G^{(1)}(x,y) G^{(2)}(y,z) ,$$

$$R_2^{(1)}(x,z) = -\frac{\sqrt{3}}{2}\cdot\frac{1}{4xz}\int_0^1 dy \frac{\sqrt{1-y^2}}{y^2-x_0^2} G^{(1)}(x,y) G^{(1)}(y,z) ,$$

$$R_2^{(2)}(x,z) = \frac{\sqrt{3}}{2}\cdot\frac{\pi^2}{4xz}\int_0^1 dy \frac{\sqrt{1-y^2}}{y^2-x_0^2} G^{(2)}(x,y) G^{(2)}(y,z) ,$$

$$R_2^{(3)}(x,z) = \frac{\gamma\pi}{4xz}\int_0^1 dy \frac{1}{y^2-x_0^2} G^{(1)}(x,y) G^{(2)}(y,z) .$$

Taking into account the behaviour of the component $R_j^{(i)}(x,z)$ of the iterated kernel in the region of the points 0, 1 and ∞ of the variables x,y we introduce three systems of orthonormal functions for the expansion of the integral kernel.

In the region [0, 1] the components $R_j^{(i)}(x,z)$ will be expanded in terms of the functions $P_n(x)$ and $\varphi_\nu(x)$, while in the region [1,∞] - in terms of the functions $P_n(1/x)$. Here the $P_n(x)$ are the usual Legendre polynominals and the $\varphi_\nu(x)$ are defined by

$$\varphi_\nu(x) = x\sqrt{1-x}\sqrt{\frac{(\nu+3)(2\nu+4)}{\nu+1}}\, P_\nu^{(1,2)}(2x-1) ,$$

where the $P_\nu^{(1,2)}$ are the Jacobi polynomials. The functions $\varphi_\nu(x)$ are introduced for those components $R_j^{(i)}(x,z)$ which contain the functions $G^{(2)}(x,y)$ as integrand. Using these three orthonormal functions for the expansion in eq.(4), one obtains for its solution:

$$B(x) = \begin{cases} -\frac{G_0(x,x_\nu)}{2xx_0} + \frac{8}{3\pi}R(x,x_\nu) + \left(\frac{8}{3\pi}\right)^2 \sum_m C_m^{(1)} \frac{1}{x} P_m(x^{-1}) & \text{at } x>1 \\ -\frac{G_\nu(x,x_\nu)}{2xx_0} + \frac{8}{3\pi}R(x,x_0) + \left(\frac{8}{3\pi}\right)^2 \left[\sum_m C_m^{(2)} \frac{1}{x} P_m(x) + \sum_\nu c_\nu \frac{1}{x}\varphi_\nu(x)\right] & \text{at } x<1. \end{cases} \quad (6)$$

Here the unknown coefficients $c_m^{(i)}$ and c_ν are to be found as a result of solving the corresponding system of algebraic equations.

3. The Interpolation Method

The second approach for the solution of eq.(2) is a well known standard method for solving integral equations. This method involves the transformation of the integral equation to a set of algebraic equations for the unknown function B(x) calculated at some knot points. However, because of the presence of the singularities in the kernel of eq.(2), this solution procedure of an integral equation cannot be applied. Providing that the solution of eq.(2) is a smooth function in each of the intervals [0,1] and [1,∞] it can be approximated within the intervals by means of a Lagrange polynomial. A similar approach was also used in Ref.[3e]. Considering the above-mentioned the integral in the right-hand side of eq.(2) can be decomposed as a sum of two integrals

$$P\int_0^\infty dy\, K(x,y) B(y) = \int_0^1 dy\, K(x,y) B(y) + P\int_1^\infty dy\, K(x,y) B(y), \quad (7)$$

where

$$K(x,y)=y^2\cdot\frac{G_0(x,y)}{2xy}\cdot\frac{\gamma-i\frac{\sqrt{3}}{2}\sqrt{1-y^2}}{y^2-x_0^2}. \qquad (8)$$

As it follows from expressions (3) and (8) the first sum of the right-hand side in (7) contains the moving logarithmic singularities, while the second one contains that part, which corresponds to the pole term. Now we can single out the singular factor $G_0(x,y)$ in the first integral in (7) and put down this integral as a sum of integrals over intervals $h = 1/N$:

$$\int_0^1 dy\, G_0(x,y)\Phi(y)=\sum_{j=1}^{N}\int_{(j-1)h}^{jh} dy\, G_0(x,y)\Phi(y), \qquad (9)$$

where

$$\Phi(y)=\frac{y^2}{2xy}\cdot\frac{\gamma-i\frac{\sqrt{3}}{2}\sqrt{1-y^2}}{y^2-x_0^2}B(y). \qquad (10)$$

Since it was supposed that the function $\Phi(y)$ is a smooth one in the interval $[0,1]$, it can be approximated in each of the subintervals by a Lagrange interpolation polynomial with a high accuracy:

$$\Phi_j(y)=\sum_k L_k^{(j)}(y)\Phi(y_k), \qquad (11)$$

here $\Phi(y_k)$ are the values of the function $\Phi(y)$ at the interpolation knots. On inserting (11) into (9) one can see that the corresponding integrals may be obtained analytically.

The second integral in eq.(7) can be treated in the same manner by decomposing the interval $[1,\infty]$ into two parts.

In so doing the integral for the first interval can be taken by the method described above and to evaluate the integral for the second interval we use the Gauss quadrature approximation. A simpler method usually used for solving the principal value of an integral can be applied:

$$P\int_1^\infty dy\, K(x,y)B(y)=\int_1^\infty\frac{dy}{y^2-x_0^2}\left[F(x,y)B(y)-F(x,x_0)B(x_0)\right]+ \qquad (12)$$

$$+F(x,x_0)B(x_0)P\int_1^\infty\frac{dy}{y^2-x_0^2},$$

with

$$F(x,y)=y^2\frac{G_0(x,y)}{2xy}\left(\gamma+\frac{\sqrt{3}}{2}\sqrt{y^2-1}\right).$$

In the first integral at the right-hand side of (12) the integrand is a smooth function at $y > 1$ and therefore it can be calculated using again the Gauss quadrature approximation. It should be noted that the function $\Phi(y)$ in (10) at $y = 1$ has a derivative which becomes infinitive and therefore, in general, it is impossible to use the approximation (11). However, in the sense of integrals the approximation (11) is sufficiently good. We have analogous situation for the function $F(x,y)$ in the expression (12).

4. Results and Discussions

Using the two approaches described above, the solutions of the integral equation (2) have been obtained for some incident neutron energies from 4–20 MeV. The expansion coefficients, defining according to (6) the solution of equation (2), obtained by the first proposed method, are given in Table 1. As it is seen all coefficients decrease rather rapidly

with increasing the number of expansion functions. Moreover, considering that the orthonormal polynomials are bounded, the coefficients in the Table indicate a good convergence of this method. The solution of eq.(2) for the real and imaginary part for two different incident neutron energies 6.7 MeV and 13.4 MeV, respectively, is given in Fig.2.

Using the second approach for the solution of eq.(2) the number of subintervals in the interval $[0,1]$ in the expression (9) has been varied from N = 6 to N = 30. Moreover, the desired solution according to (11) has been approximated for each subinterval by a Lagrange interpolation polynomial of the 4th degree. Representation (12) has been used for the interval from 1 up to ∞, whose regular part has been again evaluated by means of the Gauss quadrature approximation with 18 knots. It was found out from the calculations that the differences between the solutions for the cases with N = 20 and N = 30 do not exceed 1%.

The relative differences between the solutions of eq.(2) predicted by the two approaches proposed are shown in Fig.3. As it can be seen the agreement between the two solutions is rather satisfactory and in all cases the difference does not exceed a few percents.

In Fig.4 we show the dependence of $\kappa \operatorname{ctg}\delta_1$ on the incident neutron energy, where δ_1 is the real part of the s-wave phase. The relative differences of the phases predicted by the two approaches amount to one percent. As it can be seen in the figure, the curves obtained by calculations with the δ -function potential reproduce the shape of the experimental dependence of the quartet phase on the energy.

5. Conclusions

In the present paper we have shown that the two completely different approaches to the solution of the singular equation (2) result in practical coincidences. It is worth noting that the second approach which is based on a polynomial approximation of the resultant function is simpler to use than the one, based on an expansion of the iterated kernel of the equation (2) in terms of orthonormal functions. The use of the iteration for the solution of the equation (2) should be considered as a methodical task in order to check the simpler second approach.

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Table

The real and imaginary part of the coefficients, defined by expression (6) for equation (2) and calculated by the first described expansion method are given for two different energies of the incident neutrons.

N = 11

	E = 6,69 MeV		E= 13,36 MeV	
m	Re C_m	Im C_m	Re C_m	Im C_m
I	0.84I03	-0.38700	4.0I040	0.03696
2	0.I2392	-0.0026I	0.3I427	0.284I3
3	0.0I796	+0.0I098	-0.00055	0.05450
4	-0.00II3	0.0I256	-0.00I29	0.0336I
5	-0.0006	-0.0II00	0.0086I	0.04I33
6	0.00I78	-0.02047	0.0II24	0.07027
7	0.00073	-0.0I079	0.00469	0.03526
8	0.00002	-0.00I8I	0.000I9	0.00390
9	-0.00009	0.00I99	-0.00I54	-0.00559
I0	0.0007I	0.00I67	0.00287	-0.00388
II	0.00I45	-0.00024	0.00744	-0.0028I
m	Re $C_m^{(1)}$	Im $C_m^{(1)}$	Re $C_m^{(1)}$	Im $C_m^{(1)}$
I	-I.07984	-0.6262I	-0.52708	-3.25708
3	0.23443	-0.798I7	I.8242I	-2.78396
5	0.08427	0.05823	0.47234	0.40002
7	0.05I57	0.0II27	0.28566	0.I5426
9	0.020I9	-0.00625	0.I79I6	-0.09864
II	0.02006	-0.000I7	0.I3068	0.00383
m	Re $C_m^{(2)}$	Im $C_m^{(2)}$	Re $C_m^{(2)}$	Im $C_m^{(2)}$
I	-0.8II69	-0.46209	0.57243	-2.I34I4
3	0.04888	-0.5I804	-0.I3947	-I.96408
5	-0.07325	-0.05446	0.30938	-0.I4I5I
7	+0.09528	-0.0326I	0.I5722	-0.II5I6
9	-0.08445	-0.0I350	0.III04	-0.04I4I
II	0.07735	-0.0I220	0.07377	-0.04765

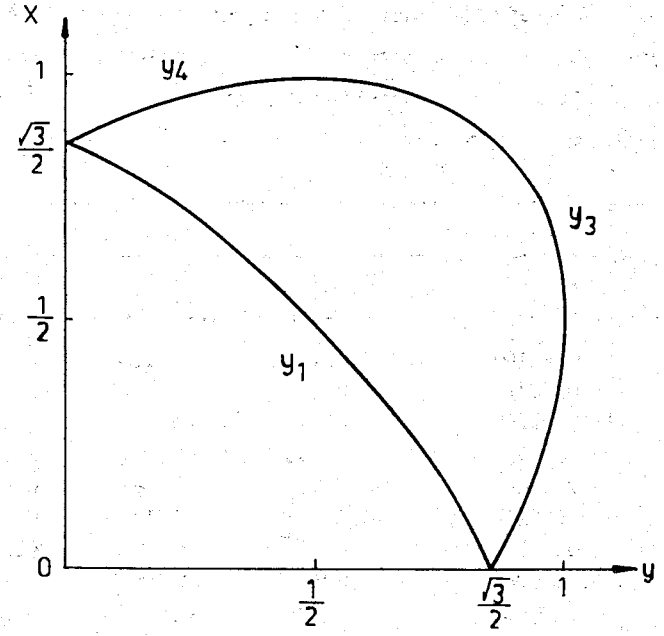

Fig.1: The position of the moving logarithmic singularities of the integral kernel of eq.(2).

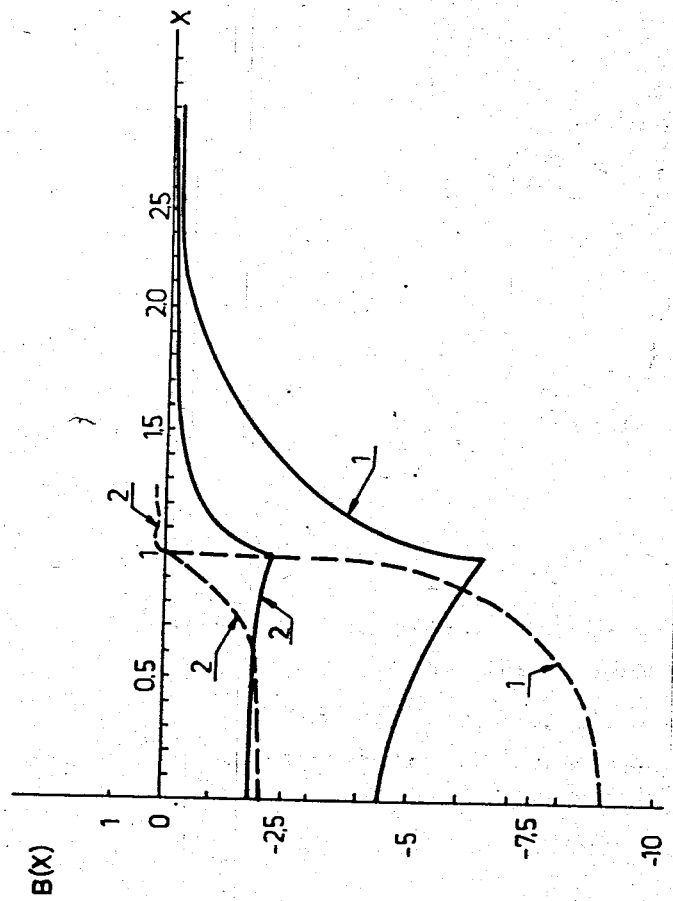

Fig.2: The solid and dashed lines are respectively the real and imaginary part of the solution of eq.(2).
1. - E_n = 13.4 MeV
2. - E_n = 6.7 MeV

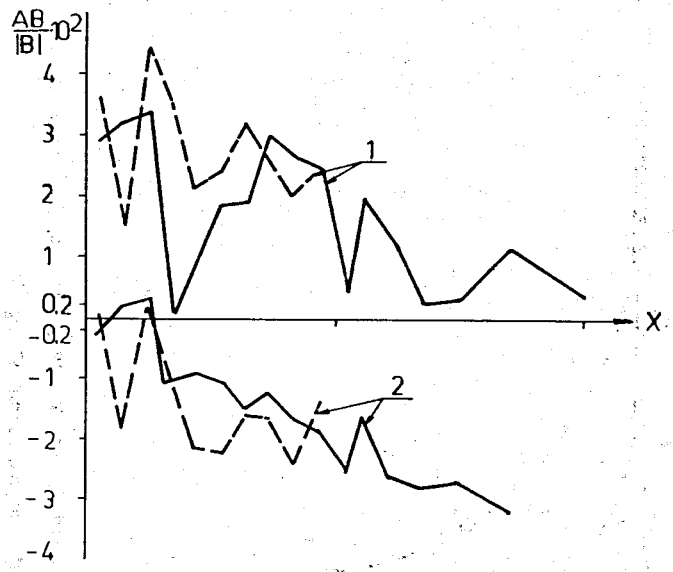

Fig.3: The relative deviations obtained by the proposed two methods for the solution of eq.(2)

———— real part of the deviation

---------- imaginary part of the deviation

1. - E_n = 13.4 MeV

2. - E_n = 6.7 MeV

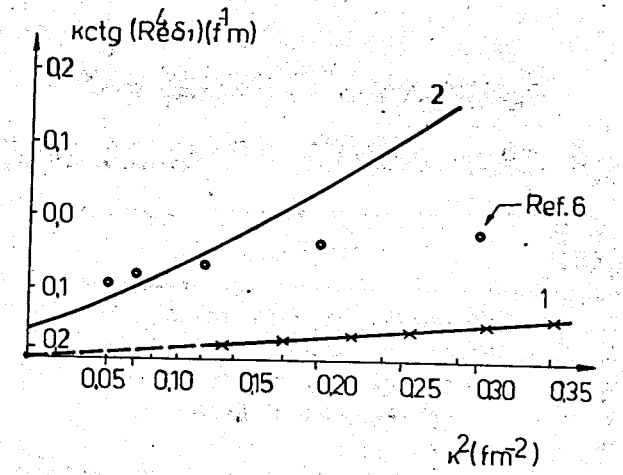

Fig.4: Curve 1 is that obtained by the authors while the second one is the result of Ref. 5. The results of a phase analysis are from Ref. 6 .