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COORDINATES AND OBSERVABLES IN THE NUCLEAR THREE-BODY PROBLEM

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I. Introduction

Many interesting problems arise in the quantum mechanics in general, and in nuclear physics in particular, when we start to look at the properties of a three-body system. In the classical mechanics, or more precisely in the celestial mechanics fruitful theories were developped on this subject. But even here only systems with Newton interactions were examined in detail. The case of arbitrary forces was practically not considered, or at least nothing was told about it in the standard text books.

In the quantum mechanics much less attention was devoted to the detailed study of the three-particle problem. There is quite a number of papers in which the energy levels of tritium and of He^3 are calculated. The problem of nucleon-deuteron scattering is considered in many papers too. But most of these works are rather utilitary, without giving a general formulation of the problem.

A consistent investigation of the question is given by Badalyan and Simonov /1,2/. The authors begin with the construction of the basis functions for a system of particles without interaction, and evaluate a perturbation theory in which the functions of the interacting particles are expanded over this basis. The basis is formed by so-called K-polynomials, which are harmonic functions corresponding to the Laplace operator on the six dimensional sphere. Further, a complete set of solutions is considered. Five commuting operators are given, and it is shown, how in principle one can construct the polynomials being eigenfunctions of these operators. The calculations are done, however, in a rather complicated way, so that the authors obtain only the lowest polynomials, which can be characterized completely by 4 quantum numbers.

Another possibility of constructing a basis is demonstrated by Zickendraht[3], but the method he used is also too complicated, and doesn't give a sufficiently general result.

In the paper of Lévy-Leblond and Lévy-Nahas[4] the connection between the basis and the representations of SU(3) is discovered. The authors use a proper parametrization, and obtain the Laplace operator expressed in terms of angular variables. Yet, they do not discuss a general solution either.

If one intends to construct harmonic functions for the three-particle system, analogous to the spherical functions forming the basis in case of two particles, it is natural to use angular variables on the six dimensional sphere or on the three dimensional complex sphere, and build up the wanted functions in terms of these coordinates. The purpose of the present work is to carry out this program. It is worth while to note that the full group of motion on the 6-sphere is too large for our aim. The problem is just to find the suitable subgroup.

Introducing angular variables, we have to separate similarity transformations, and take into consideration those transformations only, under which the sum of squares of coordinates of the three particles is invariant, i.e. the radius of the six dimensional sphere remains constant.

Consider now a triangle, the vertices of which are determined by three particles. If we exclude the similarity transformations, two possible types of transformations are left: rotations in the ordinary three dimensional space which are described by the group O(3), and deformations of the triangle respectively. It can be easily seen, that the deformations lead to SU(2).

Now, it is obvious, that different forms of a deformed, non-rotating triangle can be considered as the projections onto its plane of all the possible positions of a rotating rigid triangle.

Studying both types of transformations together, one can say, that all the transformations of a triangle besides the similarity transformations are described by the projections onto the three-dimensional space of a rigid triangle which rotate in thefour-dimensional space. That means, that an arbitrary motion of three particles is equivalent to the rotation of a rigid triangle and the similarity transformations[*]. This way we come to the local $O(4)$ symmetry, which will be also considered.

Both the representation of the group of motions on the six-dimensional sphere and its reduction to $SU(3)$ or $O(4)$ involve the representation of the permutation group $P_2(3)$. That's why this description is extraordinary convenient for the system of three equivalent particles. In the present paper we will restrict ourselves to this simple case. In the general case of arbitrary masses some new features will appear only when we expand the amplitudes or the wave functions of the interacting particles over the basis functions/5/. As it is well-known, the boundary of the definition of the functions depends on the masses.

Concerning the construction of the basis, a question arise, whether it is necessary to build up the basis with help of K-polynomials of the harmonic functions of $O(6)$. Obviously, if the interaction between the particles is weak and their motion differs only slightly from the free one, so this choice of basis functions will be natural. If, on the contrary, the particles are strongly bounded and form an almost rigid triangle, a basis not obeying the Laplace equation on the six dimensional sphere turns out to be more

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[*] Let's turn our attention to a formal analogy with the Kepler problem. The planetary motion along the elliptical trajectory can be described as the projection of the motion along the great circle on the four dimensional sphere onto its equatorial section. Ellipses with equal major axes are corresponding to different great circles. After carrying out the transformation of time, we can show, that the Kepler motion will be described by the free motion of a point on the four-dimensional sphere. (That is the famous Fock symmetry).

convenient. As an example of such a basis, so-called B-polynomials will be constructed at the end of the present work.

From a group-theoretical point of view the most interesting questions in the three-body problem are connected with the features of the fifth quantum number Ω (see[6]). The introduction of this quantity becomes necessary, because it is not sufficient to use only the quantum numbers coming from the reduction $O(6)=O(3)\times O(2)$, i.e. quantum numbers characterizing the rotations and the permutations. Recently it was shown by Surkov[7], that for a system consisting of more then three particles one has to introduce some additional quantum numbers, namely 3 quantum numbers in the case of 4 particles, and 4 in the case of 5 particles. In the case of a system including 6 particles or more, 5 new quantum numbers are necessary. It is rather a remarkable fact, that for more than 6 particles the number of additional quantum numbers remains constant.

To the investigation of the features of these new quantum numbers Ω a separate work will be devoted.

Finally, it is worth while to study the energy spectrum of the three-particle system in the case when the triangle formed by the three particles is getting rigid. The transition from the spectrum of non-interacting particles to the spectrum of the top may be investigated with help of the basis given in the present paper. This problem will be discussed separately.

II. Choice of Coordinates and Group Properties

The usual way of choosing the coordinates is the following. Let $x_i (i = 1,2,3)$ be the radius-vectors of the three particles, and fix

$$x_1 + x_2 + x_3 = 0 . \tag{2.1}$$

The Jacobi coordinates for equal masses will be defined as

$$\xi = -\sqrt{\frac{3}{2}}\,(x_1 + x_2) , \tag{2.2}$$
$$\eta = \sqrt{\frac{1}{2}}\,(x_1 - x_2) , \tag{2.3}$$

$$\xi^2 + \eta^2 = 2x_1^2 + 2x_1x_2 + 2x_2^2 = x_1^2 + x_2^2 + x_3^2 = \rho^2 . \tag{2.4}$$

We might define similar coordinates in the momentum space as well. In that case condition (2.1) means that we are in the center-of-mass frame, and ρ^2 is a quantity proportional to the energy.

The quadratic form (2.4) can be understood as an invariant of the O(6) group. In fact, we are interested in the direct product O(3) x O(2), as we have to introduce the total angular momentum observables L and M (group O(3)), and quantum numbers of the three-particle permutation group O(2).

To characterize our three-particle system we need 5 quantum numbers. Thus the O(6) group is too large for our purposes and it is convenient to deal with SU(3) symmetry, in case of which we dispose exactly of the necessary 5 quantum numbers.

Let us introduce the complex vector

$$z = \xi + i\eta , \tag{2.5}$$

$$z^* = \xi - i\eta . \tag{2.6}$$

The permutation of two particles leads in terms of these coordinates to rotations in the complex z -plane.

$$P_{12}\begin{pmatrix} z \\ z^* \end{pmatrix} = \begin{pmatrix} z^* \\ z \end{pmatrix} \qquad P_{13}\begin{pmatrix} z \\ z^* \end{pmatrix} = \begin{pmatrix} e^{i\frac{\pi}{3}} z^* \\ e^{-i\frac{\pi}{3}} z \end{pmatrix} \qquad P_{23}\begin{pmatrix} z \\ z^* \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\pi}{3}} z^* \\ e^{i\frac{\pi}{3}} z \end{pmatrix} . \tag{2.7}$$

The condition

$$\xi^2 + \eta^2 = |z|^2 = \rho^2 \tag{2.8}$$

gives the invariant of the group $SU(3) \subset O(6)$. In the following we will take $\rho = 1$.

The generators of SU(3) we define as usual

$$A_{ik} = i z_i \frac{\partial}{\partial z_k} - i z_k^* \frac{\partial}{\partial z_i^*} . \tag{2.9}$$

The chain $SU(3) \supset SU(2) \supset U(1)$ familiar from the theory of unitary symmetry of hadrons is of no use to us, because it doesn't contain O(3), i.e. going this way we can't introduce the angular momentum quantum numbers. Instead of that, we consider two subgroups $O(6) \supset O(4) \sim SU(2) \times O(3)$ and $O(6) \supset SU(3)$. In other words, we have to separate from (2.8) the antisymmetric tensor-generator of the rotation group O(3).

$$L_{ik} = \frac{1}{2}\left(A_{ik} - A_{ki}\right) = \tag{2.10}$$
$$= \frac{1}{2}\left(iz_i\frac{\partial}{\partial z_k} - iz_k\frac{\partial}{\partial z_i} + iz_i^*\frac{\partial}{\partial z_k^*} - iz_k^*\frac{\partial}{\partial z_i^*}\right),$$

The remaining symmetric part

$$B_{ik} = \frac{1}{2}\left(A_{ik} + A_{ki} - 2\,\mathrm{Sp}\,A\,\delta_{ik}\right) = \tag{2.11}$$
$$= \frac{1}{2}\left(iz_i\frac{\partial}{\partial z_k} + iz_k\frac{\partial}{\partial z_i} - iz_i^*\frac{\partial}{\partial z_k^*} - iz_k^*\frac{\partial}{\partial z_i^*} - 2iz_\ell\frac{\partial}{\partial z_\ell}\delta_{ik} + 2iz_\ell^*\frac{\partial}{\partial z_\ell^*}\delta_{ik}\right)$$

is the generator of the group of deformations of the triangle which turns out to be locally isomorphic with the rotation group. Finally, we introduce a scalar operator

$$N = \frac{1}{2i}\,\mathrm{Sp}\,A = \frac{1}{2}\sum_i\left(z_i\frac{\partial}{\partial z_i} - z_i^*\frac{\partial}{\partial z_i^*}\right), \tag{2.12}$$

For characterizing our system, we choose the following quantum numbers:

K(K+ 4) -eigenvalue of the Laplace operator (quadratic Casimir operator for SU(3)),

L(L +1) -eigenvalue of the square of the angular momentum operator $L^2 = 4 \sum_{i>k} L_{ik}^2$, (2.13)

M -eigenvalue of $L_3 = 2L_{12}$,

ν -eigenvalue of N .

Although the generator (2.12) is not a Casimir operator of SU(3), the representation might be characterized by means of its eigenvalue, because, as it can be seen, the eigenvalue of the Casimir operator of third order can be written as a combination of K and ν . (If the harmonic function belongs to the representation (p,q) of SU(3), then it is the eigenfunction of Δ and N with eigenvalues K(K+4) and ν respectively, where $K = p + q$ and $\nu = p-q$).

The fifth quantum number is not included in any of the considered subgroups, we have to take it from O(6). We define it as the eigenvalue of

$$\Omega = \sum_{ijk} L_{ij} B_{jk} L_{ki} = \mathrm{Sp}\, LBL \quad . \qquad (2.14)$$

This cubic generator was first introduced by Racah/6/. Its physical meaning we will discuss later.

III. Parametrization of a Complex Sphere

Dealing with a 3 particle system, we have to introduce coordinates which refer explicitely to the moving axes. One of the possible parametrizations of the vectors Z and Z^* is the following:

$$z = e^{-i\frac{\lambda}{2}}\left(\cos\frac{\alpha}{2}\,\ell_1 + i\sin\frac{\alpha}{2}\,\ell_2\right) \,, \tag{3.1}$$

$$z^* = e^{i\frac{\lambda}{2}}\left(\cos\frac{\alpha}{2}\,\ell_1 - i\sin\frac{\alpha}{2}\,\ell_2\right) \,, \tag{3.2}$$

$$|z|^2 = 1 \,, \tag{3.3}$$

$$\ell_1^2 = \ell_2^2 = 1 \,, \quad \ell_1\ell_2 = 0 \,. \tag{3.4}$$

The three orthogonal unit vectors ℓ_1, ℓ_2 and $\ell = \ell_1 \times \ell_2$ form the moving system of coordinates. Their orientation to the fixed coordinate system can be described with help of the Euler anges φ_1, θ, φ_2.

$$\ell_1 = \left\{\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2\cos\theta;\ -\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2\cos\theta;\ \sin\varphi_1\sin\theta\right\} \,, \tag{3.5}$$

$$\ell_2 = \left\{\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2\cos\theta;\ -\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2\cos\theta;\ -\cos\varphi_1\sin\theta\right\} \,, \tag{3.6}$$

$$\ell = \left\{\sin\varphi_2\sin\theta;\ \cos\varphi_2\sin\theta;\ \cos\theta\right\} \,. \tag{3.7}$$

We note here, that this parametrization of the basic vectors is used by Lévy-Leblond and Lévy-Nahas /4/. However, this parametrization is not too convenient for real calculations.

In the following it will be simpler to introduce a new angle $a = \alpha - \frac{\pi}{2}$ and work with the vectors

$$z = e^{-i\frac{\lambda}{2}}\left(\cos\frac{a}{2}\,l_+ - \sin\frac{a}{2}\,l_-\right) , \tag{3.8}$$

$$z^* = e^{i\frac{\lambda}{2}}\left(\cos\frac{a}{2}\,l_- - \sin\frac{a}{2}\,l_+\right) , \tag{3.9}$$

where

$$\begin{aligned} l_+ &= \frac{1}{\sqrt{2}}(l_1 + il_2) , \\ l_- &= \frac{1}{\sqrt{2}}(l_1 - il_2) . \end{aligned} \tag{3.10}$$

The vectors l_+ , l_- have the obvious properties

$$\begin{aligned} &l_+^2 = l_-^2 = 0 \\ &l_0 = l_+ \times l_- = -il \\ &l_+ l_- = 1 \qquad l_+^+ = l_- . \end{aligned} \tag{3.11}$$

Let's turn our attention to the fact, that the components of l_+ and l_- may be expressed through the Wigner D-functions, defined as

$$D^l_{mn}(\varphi_1\,\theta\,\varphi_2) = e^{-i(m\varphi_1 + n\varphi_2)}P^l_{mn}(\cos\theta) \tag{3.12}$$

in the following way:

$$l_- = \left\{ -D^1_{1-1}(\varphi_1\theta\varphi_2);\ -iD^1_{10}(\varphi_1\theta\varphi_2);\ D^1_{11}(\varphi_1\theta\varphi_2) \right\}$$

$$l_0 = \left\{ iD^1_{0-1}(\varphi_1\theta\varphi_2);\ -D^1_{00}(\varphi_1\theta\varphi_2);\ -iD^1_{01}(\varphi_1\theta\varphi_2) \right\} \tag{3.13}$$

$$l_+ = \left\{ D^1_{-1-1}(\varphi_1\theta\varphi_2);\ -iD^1_{-10}(\varphi_1\theta\varphi_2);\ -D^1_{-11}(\varphi_1\theta\varphi_2) \right\} .$$

These equations demostrate the possibility of construction of Wigner functions from the unit vectors corresponding to the moving coordinate system, in a way similar to the construction of spherical harmonics from the unit vectors of the fixed coordinate system. However, we see, that the traditional parametrization of the vectors l_i which we have introduced is not too fortunate; it would be much more aesthetical to go over to a parametrization in which $D^1_{mn} = l_m k_n$, where l_m and k_n are unit vectors of the moving and fixed coordinate systems respectively. Yet in the present work we will not change the parametrization.

The vectors z and z^* can be written as

$$z_M = \sum_{M'=\pm\frac{1}{2}} a(M',M) D^{\frac{1}{2}}_{\frac{1}{2},M'}(\lambda, a, 0) D^1_{-2M',M}(\varphi_1\theta\varphi_2) , \tag{3.14}$$

$$z^*_M = -\sum_{M'=\pm\frac{1}{2}} a(M',M) D^{\frac{1}{2}}_{-\frac{1}{2},M'}(\lambda, a, 0) D^1_{2M',M}(\varphi_1\theta\varphi_2) , \tag{3.15}$$

where

$$a\left(\tfrac{1}{2}, M\right) = i\exp\left(-(M+1)\right)$$
$$a\left(-\tfrac{1}{2}, M\right) = i\exp M \qquad (3.16)$$
$$M = -1, 0, 1 \ .$$

Now we have to add a few words considering the meaning of the coordinates. By definition, the D-functions include an exponential factor. In (3.14) there figure a factor $\exp 2iM'\varphi_1$, which provides the necessary dependence between the two D-functions. Indeed, this factor is due to the angle φ_1 in the second D-function; but it could belong to the first D-function too, if we put $D^{k}_{k,M'}(\lambda_1 a_1 - 2\varphi_1)$. Later on we will use a new variable Ω_3 (in the co-moving system), and the Euler angles in the first D-function will be written as λ, a, $-2\Omega_3$. Thus, the angle φ_1 (or Ω_3) plays a double role as an Euler angle in two spaces simultaneously.

IV. The Laplace Operator

We have now to write down the operators, the eigenvalues of which we are looking for. First let us construct the Laplace operator. We could do that by a straightforward calculation of $\Delta = |A_{ik}|^2$, but we choose a simpler way. We calculate

$$dz = -\frac{i}{2} z\, d\lambda + \frac{i}{2} e^{-i\lambda}(1 \times z^*)\, da - (d\omega \times z) \ . \quad (4.1)$$

This rather simple expression is obtained by introducing the infinitesimal rotation $d\omega$. Its projections onto the fixed coordinates $k_1 = (1,0,0.)$, $k_2 = (0,1,0)$, $k_3 = (0,0,1)$ given in terms of the Euler angles are well-known;

$$\left.\begin{aligned} d\omega_1 &= \sin\varphi_2 \sin\theta \, d\varphi_1 + \cos\varphi_2 \, d\theta \\ d\omega_2 &= \cos\varphi_2 \sin\theta \, d\varphi_1 - \sin\varphi_2 \, d\theta \\ d\omega_3 &= \cos\theta \, d\varphi_1 + d\varphi_2 \end{aligned}\right\} \qquad (4.2)$$

We shall use the corresponding projections onto the rotating coordinates as well, they are by definition

$$d\Omega_i = l_i \, d\omega . \qquad (4.3)$$

(Note, that they act on an arbitrary vector A in the following way:

$$\frac{\partial}{\partial\Omega_i} A = - l_i \times A \qquad (4.4)).$$

It is useful to write down the explicit expressions for $\frac{\partial}{\partial\Omega_i}$:

$$\left.\begin{aligned} \frac{\partial}{\partial\Omega_1} &= -\sin\varphi_1 \operatorname{ctg}\theta \frac{\partial}{\partial\varphi_1} + \sin\varphi_1 \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi_2} + \cos\varphi_1 \frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\Omega_2} &= \cos\varphi_1 \operatorname{ctg}\theta \frac{\partial}{\partial\varphi_1} - \cos\varphi_1 \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi_2} + \sin\varphi_1 \frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\Omega_3} &= \frac{\partial}{\partial\varphi_1} \end{aligned}\right\} \qquad (4.5)$$

From the last equation it is getting clear, that in fact $\Omega_3 = \varphi_1$.

Let's calculate the square of $|dz|$. From (4.1) we obtain

$$ds^2 = dz\,dz^* = g_{ik}\,x^i x^k =$$

$$= \frac{1}{4}da^2 + \frac{1}{4}d\lambda^2 + \frac{1}{2}(1+\sin a)\,d\Omega_1^2 + \tag{4.6}$$

$$+ \frac{1}{2}(1-\sin a)\,d\Omega_2^2 + d\Omega_3^2 - \cos a\, d\Omega_3\, d\lambda \quad .$$

This expression determines the components of the metric tensor g_{ik}, and it becomes easy to calculate the Laplace operator

$$\Delta = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i} g^{ik}\sqrt{g}\frac{\partial}{\partial x^k} =$$

$$= 4\Big\{\frac{\partial^2}{\partial a^2} + 2\,\mathrm{ctg}\,2a\frac{\partial}{\partial a} + \frac{1}{\sin^2 a}\Big(\frac{\partial^2}{\partial \lambda^2} + \cos a\frac{\partial^2}{\partial\lambda\,\partial\Omega_3} + \frac{1}{4}\frac{\partial^2}{\partial\Omega_3^2}\Big) +$$

$$+ \frac{1}{2\cos^2 a}\Big(\frac{\partial^2}{\partial\Omega_1^2} + \frac{\partial^2}{\partial\Omega_2^2}\Big) - \frac{\sin a}{2\cos^2 a}\Big(\frac{\partial^2}{\partial\Omega_1^2} - \frac{\partial^2}{\partial\Omega_2^2}\Big)\Big\} \quad . \tag{4.7}$$

In the following it will be more convenient to consider $\Delta' = \frac{1}{4}\Delta$. If ϕ is the eigenfunction of Δ', corresponding to a certain representation, it has to fulfill

$$\Delta'\phi = -\frac{1}{4}K(K+4)\phi = -\frac{K}{2}\Big(\frac{K}{2}+2\Big)\phi \tag{4.8}$$

and

$$N\phi = \nu\phi \,, \qquad \text{where} \qquad N = i\frac{\partial}{\partial\lambda} \quad . \tag{4.9}$$

Expressing (4.7) in terms of the Euler angles, we get the Laplace operator in the form obtained in /4/:

$$\Delta' = \Delta_a - \operatorname{tg} a \frac{\partial}{\partial a} + \frac{1}{2\cos a}\left(\Delta_\theta - \frac{\partial^2}{\partial \varphi_1^2}\right) -$$

$$- \frac{\sin a}{2\cos^2 a}\left[\cos 2\varphi_1\left(-\Delta_\theta + \frac{\partial^2}{\partial \varphi_1^2} + 2\frac{\partial^2}{\partial \theta^2}\right) + \right. \tag{4.10}$$

$$\left. + \sin 2\varphi_1\left(\frac{1+\cos^2\theta}{\sin^2\theta}\frac{\partial}{\partial \varphi_1} - 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial \varphi_2} - 2\operatorname{ctg}\theta\frac{\partial^2}{\partial \varphi_1 \partial \theta} + 2\frac{1}{\sin\theta}\frac{\partial^2}{\partial \varphi_2 \partial \theta}\right)\right],$$

$$\Delta_a = \frac{\partial^2}{\partial a^2} + \operatorname{ctg} a \frac{\partial}{\partial a} + \frac{1}{\sin^2 a}\left(\frac{\partial^2}{\partial \lambda^2} + \cos\frac{\partial^2}{\partial \lambda \partial \Omega_3} + \frac{1}{4}\frac{\partial^2}{\partial \Omega_3^2}\right), \tag{4.11}$$

$$\Delta_\theta = \frac{\partial^2}{\partial \theta^2} + \operatorname{ctg}\theta\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial \varphi_1^2} - 2\cos\frac{\partial^2}{\partial \varphi_1 \partial \varphi_2} + \frac{\partial^2}{\partial \varphi_2^2}\right). \tag{4.12}$$

Formally, substituting

$$\frac{\partial}{\partial \chi_1} = \sqrt{2}\sin\left(\varphi_2 + \frac{\pi}{4}\right)\left[-\operatorname{ctg}\theta\frac{\partial}{\partial \varphi_2} + \frac{1}{\sin\theta}\frac{\partial}{\partial \varphi_1}\right] + \sqrt{2}\cos\left(\varphi_2 + \frac{\pi}{4}\right)\frac{\partial}{\partial \theta}$$

$$\tag{4.13}$$

$$\frac{\partial}{\partial \chi_2} = -\sqrt{2}\cos\left(\varphi_2 + \frac{\pi}{4}\right)\left[-\operatorname{ctg}\theta\frac{\partial}{\partial \varphi_2} + \frac{1}{\sin\theta}\frac{\partial}{\partial \varphi_1}\right] + \sqrt{2}\sin\left(\varphi_2 + \frac{\pi}{4}\right)\frac{\partial}{\partial \theta}$$

we can express the Laplace operator in a symmetric (but rather formal) way:

$$\Delta' = \frac{\partial^2}{\partial a^2} + \mathrm{ctg}\,a\frac{\partial}{\partial a} + \frac{1}{\sin^2 a}\left(\frac{\partial^2}{\partial\lambda^2} + \cos a\frac{\partial^2}{\partial\lambda\partial\Omega_3} + \frac{1}{4}\frac{\partial^2}{\partial\Omega_3^2}\right) - \tag{4.14}$$

$$- \mathrm{tg}\,a\frac{\partial}{\partial a} + \frac{1}{\cos^2 a}\left(\frac{\partial^2}{\chi_1^2} - 2\sin a\frac{\partial^2}{\partial\chi_1\partial\chi_2} + \frac{\partial^2}{\partial\chi_2}\right) .$$

However, we don't need such a long form of the Laplace operator, and use practically only the shortened form (4.7), exploiting the advantage of the moving coordinate system.

V. Calculation of the Generators L_{ik} and B_{ik}

To obtain the generators directly from dz , we have to invert a 5x5 matrix in the case of three particles. That requires rather a long calculation, which is getting hopeless for a larger number of particles. Instead of performing the straightforward calculation, we get the wanted expressions in the following way. Let us first consider L_{ik} , or rather its special case L_{12}. We introduce a parameter σ_{ik} which define the displacement along the particular trajectory which corresponds to the action of the operator L_{ik} . Thus, formally we can write

$$L_{12} = \frac{1}{2}\left(iz_1 \frac{\partial}{\partial z_2} - iz_2 \frac{\partial}{\partial z_1} + iz_1^* \frac{\partial}{\partial z_2^*} - iz_2^* \frac{\partial}{\partial z_1^*} \right) \equiv \frac{\partial}{\partial \sigma_{12}} , \qquad (5.1)$$

Acting with L_{12} on the vectors z and z^*

$$L_{12}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -iz_2 \\ iz_1 \\ 0 \end{pmatrix} \qquad L_{12}\begin{pmatrix} z_1^* \\ z_2^* \\ z_3^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -iz_2^* \\ iz_1^* \\ 0 \end{pmatrix} \qquad (5.2)$$

we see, that σ_{12} has to be imaginary. In the following, we will make use of the equations

$$z L_{12} z = 0 , \qquad z^* L_{12} z^* = 0 , \qquad (5.3)$$

$$z^* L_{12} z = \frac{i}{2}\left(z \times z^* \right)_3 , \qquad (5.4)$$

$$\ell L_{12} z = -\frac{i}{2}\left(\ell \times z \right)_3 . \qquad (5.5)$$

Using the expression (4.1) for dz, we can write

$$L_{12} z = \frac{\partial z}{\partial \sigma_{12}} = -\frac{i}{2} z \frac{d\lambda}{d\sigma_{12}} + \frac{i}{2} e^{-i\lambda} (\ell \times z^*) \frac{da}{d\sigma_{12}} - \left(\frac{d\omega}{d\sigma_{12}} \times z \right) \qquad (5.6)$$

and

$$L_{12} z^* = \frac{\partial z^*}{\partial \sigma_{12}} = \frac{i}{2} z^* \frac{d\lambda}{d\sigma_{12}} - \frac{i}{2} e^{i\lambda} (\ell \times z) \frac{da}{d\sigma_{12}} - \left(\frac{d\omega}{d\sigma_{12}} \times z^* \right) . \qquad (5.7)$$

Substituting

$$\left.\begin{aligned} (l \times z^*) &= ie^{i\frac{\lambda}{2}}\left(\cos\frac{a}{2}\, l_- + \sin\frac{a}{2}\, l_+\right) \\ (l \times z) &= -ie^{-i\frac{\lambda}{2}}\left(\cos\frac{a}{2}\, l_+ + \sin\frac{a}{2}\, l_-\right) \end{aligned}\right\} , \tag{5.8}$$

$$\left.\begin{aligned} z^2 &= -e^{-i\lambda}\sin a \\ z^{*2} &= -e^{i\lambda}\sin a \end{aligned}\right\} \tag{5.9}$$

we obtain from (5.3)

$$\frac{da}{d\sigma_{12}} = \frac{d\lambda}{d\sigma_{12}} = 0 \tag{5.10}$$

Similarly, (5.4) gives

$$\frac{d\Omega_3}{d\sigma_{12}} = -\frac{i}{2}\, l^{(3)} \tag{5.11}$$

and finally (5.5) leads to

$$\frac{d\Omega_1}{d\sigma_{12}} = -\frac{i}{2}\, l_1^{(3)} , \tag{5.12}$$

$$\frac{d\Omega_2}{d\sigma_{12}} = -\frac{i}{2}\, l_2^{(3)} , \tag{5.13}$$

where $l_i^{(k)}$ stands for the k-th component of vector l_i. Thus we obtain

$$L_{12} = -\frac{i}{2}\left[\ell_1^{(3)}\frac{\partial}{\partial\Omega_1} + \ell_2^{(3)}\frac{\partial}{\partial\Omega_2} + \ell^{(3)}\frac{\partial}{\partial\Omega_3}\right] = -\frac{i}{2}\frac{\partial}{\partial\omega_3} \quad . \tag{5.14}$$

Similarly

$$L_{23} = -\frac{i}{2}\left[\ell_1^{(1)}\frac{\partial}{\partial\Omega_1} + \ell_2^{(1)}\frac{\partial}{\partial\Omega_2} + \ell^{(1)}\frac{\partial}{\partial\Omega_3}\right] = -\frac{i}{2}\frac{\partial}{\partial\omega_1} \quad , \tag{5.15}$$

$$L_{31} = -\frac{i}{2}\left[\ell_1^{(2)}\frac{\partial}{\partial\Omega_1} + \ell_2^{(2)}\frac{\partial}{\partial\Omega_2} + \ell^{(2)}\frac{\partial}{\partial\Omega_3}\right] = -\frac{i}{2}\frac{\partial}{\partial\omega_2} \quad . \tag{5.16}$$

Introducing the notation

$$L_1 = 2L_{23} \qquad L_2 = 2L_{31} \qquad L_3 = 2L_{12} \tag{5.17}$$

we can write the general expression for the angular momentum operator

$$L_k = -i\left[\ell_1^{(k)}\frac{\partial}{\partial\Omega_1} + \ell_2^{(k)}\frac{\partial}{\partial\Omega_2} + \ell^{(k)}\frac{\partial}{\partial\Omega_3}\right] \quad . \tag{5.18}$$

It fulfils the commutation relations

$$\begin{aligned} [L_1, L_2] &= -iL_3 \\ [L_2, L_3] &= -iL_1 \\ [L_3, L_1] &= -iL_2 \quad . \end{aligned} \tag{5.19}$$

The square of the angular momentum operator is

$$L^2 = \left(\frac{\partial^2}{\partial\Omega_1^2} + \frac{\partial^2}{\partial\Omega_2^2} + \frac{\partial^2}{\partial\Omega_3^2}\right) = \Delta_\theta \quad . \tag{5.20}$$

Now let's turn our attention to the operator B_{ik}. We oonsider

$$B_{12} = \frac{1}{2}\left(iz_1\frac{\partial}{\partial z_2} + iz_2\frac{\partial}{\partial z_1} - iz_1^*\frac{\partial}{\partial z_2^*} - iz_2^*\frac{\partial}{\partial z_1^*}\right) \equiv \frac{\partial}{\partial \beta_{12}} . \quad (5.21)$$

From the action of B_{12} on z and z^*

$$B_{12}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} iz_2 \\ iz_1 \\ 0 \end{pmatrix} \qquad B_{12}\begin{pmatrix} z_1^* \\ z_2^* \\ z_3^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -iz_2^* \\ -iz_1^* \\ 0 \end{pmatrix} \quad (5.22)$$

it is obvious, that β_{12} is real. We make use of the conditions

$$z B_{12} z = iz_1 z_2 , \qquad z^* B_{12} z^* = -iz_1^* z_2^* , \quad (5.23)$$

$$z^* B_{12} z = \frac{i}{2}\left(z_1^* z_2 + z_1 z_2^*\right) , \quad (5.24)$$

$$\ell B_{12} z = \frac{i}{2}\left(\ell^{(1)} z_2 + \ell^{(2)} z_1\right) \quad (5.25)$$

and of (4.1) and (5.8). Let us introduce the notation

$$b_{ik}^{(\ell m)} = \frac{1}{2}\left(\ell_i^{(\ell)}\ell_k^{(m)} + \ell_i^{(m)}\ell_k^{(\ell)}\right) . \quad (5.26)$$

Then, following a procedure similar to that in the case of L_{ik} , we obtain from (5.23)

$$\frac{da}{d\beta_{12}} = 2\, b_{12}^{(12)} \,, \tag{5.27}$$

$$\frac{d\lambda}{d\beta_{12}} = \frac{1}{\sin a}\left[b_{11}^{(12)}(1-\sin a) - b_{22}^{(12)}(1+\sin a)\right] . \tag{5.28}$$

Equations (5.24) and (5.25) lead to

$$\frac{d\Omega_1}{d\beta_{12}} = - \frac{1-\sin a}{\cos a}\, b_{13}^{(12)} \,, \tag{5.29}$$

$$\frac{d\Omega_2}{d\beta_{12}} = - \frac{1+\sin a}{\cos a}\, b_{23}^{(12)} \,, \tag{5.30}$$

$$\frac{d\Omega_3}{d\beta_{12}} = \frac{1}{2}\,\mathrm{ctg}\, a \left(b_{11}^{(12)} - b_{22}^{(12)} \right) . \tag{5.31}$$

Thus the expression for B_{12} will be

$$\begin{aligned} B_{12} = {} & 2\, b_{12}^{(12)} \frac{\partial}{\partial a} + b_{11}^{(12)}\left[\frac{1-\sin a}{\sin a}\frac{\partial}{\partial \lambda} + \frac{1}{2}\,\mathrm{ctg}\, a \frac{\partial}{\partial \Omega_3}\right] - \\ & - b_{22}^{(12)}\left[\frac{1+\sin a}{\sin a}\frac{\partial}{\partial \lambda} + \frac{1}{2}\,\mathrm{ctg}\, a \frac{\partial}{\partial \Omega_3}\right] - \\ & - \frac{1-\sin a}{\cos a}\, b_{13}^{(12)} \frac{\partial}{\partial \Omega_1} - \frac{1+\sin a}{\cos a}\, b_{23}^{(12)} \frac{\partial}{\partial \Omega_2} \end{aligned} \tag{5.32}$$

and the generator B_{ik} of the group of deformations of the triangle obtains the form

$$
\begin{aligned}
B_{ik} = {} & 2 b_{12}^{(ik)} \frac{\partial}{\partial a} + b_{11}^{(ik)} \left[\frac{1-\sin a}{\sin a} \frac{\partial}{\partial \lambda} + \frac{1}{2} \operatorname{ctg} a \frac{\partial}{\partial \Omega_3} \right] - \\
& - b_{22}^{(ik)} \left[\frac{1+\sin a}{\sin a} \frac{\partial}{\partial \lambda} + \frac{1}{2} \operatorname{ctg} \frac{\partial}{\partial \Omega_3} \right] - \frac{1-\sin a}{\cos a} b_{13}^{(ik)} \frac{\partial}{\partial \Omega_1} - \\
& - \frac{1+\sin a}{\cos a} b_{23}^{(ik)} \frac{\partial}{\partial \Omega_2} + 2 \frac{\partial}{\partial \lambda} \delta_{ik} \quad .
\end{aligned}
\tag{5.33}
$$

Acting in the space of polynomials of z, there exist the following operatorial identity:

$$
i \frac{1-\sin a}{\cos a} \frac{\partial}{\partial \Omega_1} = \frac{\partial}{\partial \Omega_2} \quad . \tag{5.34}
$$

(Naturally, functionally that is not true). Thus, in the space of polynomials of z B_{ik} might be written as

$$
\begin{aligned}
B_{ik} = {} & 2 b_{12}^{(ik)} \frac{\partial}{\partial a} + b_{11}^{(ik)} \left[\frac{1-\sin a}{\sin a} \frac{\partial}{\partial \lambda} + \frac{1}{2} \operatorname{ctg} a \frac{\partial}{\partial \Omega_3} \right] - \\
& - b_{22}^{(ik)} \left[\frac{1+\sin a}{\sin a} \frac{\partial}{\partial \lambda} + \frac{1}{2} \operatorname{ctg} a \frac{\partial}{\partial \Omega_3} \right] + \\
& + i b_{13}^{(ik)} \frac{\partial}{\partial \Omega_2} - i b_{23}^{(ik)} \frac{\partial}{\partial \Omega_1} + 2 \frac{\partial}{\partial \lambda} \delta_{ik} \quad .
\end{aligned}
\tag{5.35}
$$

This remarkable feature of B_{ik} is due to the fact, that we are dealing with a system of just three (and not more) particles. Indeed, the three particles form a triangle, so that the deformations take place on a plane, while $\frac{\partial}{\partial \Omega_1}$ and $\frac{\partial}{\partial \Omega_2}$ act in the three-dimensional space.

For the sake of completeness let's write down the commutation relations

$$
\begin{aligned}
&[B_{12}, B_{23}] = -\tfrac{i}{2} L_{31} \qquad [B_{31}, B_{12}] = -\tfrac{i}{2} L_{23} \qquad [B_{23}, B_{31}] = -\tfrac{i}{2} L_{12} \\
&[B_{12}, B_{11}] = -i L_{12} \qquad [B_{12}, B_{22}] = i L_{12} \\
&[B_{13}, B_{11}] = -i L_{13} \qquad [B_{13}, B_{33}] = i L_{13} \\
&[B_{23}, B_{22}] = -i L_{23} \qquad [B_{23}, B_{33}] = i L_{23} \\
&[B_{12}, B_{33}] = [B_{13}, B_{22}] = [B_{23}, B_{11}] = 0 \\
&[B_{11}, B_{22}] = [B_{11}, B_{33}] = [B_{22}, B_{33}] = 0
\end{aligned}
\tag{5.36}
$$

$$
\begin{aligned}
&[B_{11}, L_{12}] = i B_{12} \qquad [B_{11}, L_{13}] = i B_{13} \qquad [B_{11}, L_{23}] = 0 \\
&[B_{22}, L_{12}] = -i B_{12} \qquad [B_{22}, L_{13}] = 0 \qquad [B_{22}, L_{23}] = i B_{23} \\
&[B_{33}, L_{12}] = 0 \qquad [B_{33}, L_{13}] = -i B_{13} \qquad [B_{33}, L_{23}] = -i B_{23} \\
&[B_{12}, L_{12}] = -\tfrac{i}{2}(B_{11} - B_{22}) \\
&[B_{12}, L_{13}] = \tfrac{i}{2} B_{23} \qquad [B_{12}, L_{23}] = \tfrac{i}{2} B_{13} \\
&[B_{13}, L_{12}] = \tfrac{i}{2} B_{23} \qquad [B_{13}, L_{23}] = -\tfrac{i}{2} B_{12} \\
&[B_{13}, L_{13}] = -\tfrac{i}{2}(B_{11} - B_{33}) \\
&[B_{23}, L_{12}] = -\tfrac{i}{2} B_{13} \qquad [B_{23}, L_{13}] = -\tfrac{i}{2} B_{12} \\
&[B_{23}, L_{23}] = -\tfrac{i}{2}(B_{22} - B_{33})
\end{aligned}
\tag{5.37}
$$

From these relations it can be seen, that the three operators L_{12}, $-\frac{1}{2}(B_{11}-B_{22})$ and B_{12} form the SU(2) subgroup. This fact allows to check the meaning of the coordinates.

The square of B_{ik} is as follows

$$\sum_{i,k} B_{ik}^2 = 2\Big\{\frac{\partial^2}{\partial a^2} + \operatorname{ctg} a\frac{\partial}{\partial a} + \frac{1}{\sin^2 a}\Big(\frac{\partial^2}{\partial\lambda^2} + \cos a\frac{\partial^2}{\partial\lambda\partial\Omega_3} + \frac{1}{4}\frac{\partial^2}{\partial\Omega_3^2}\Big) - \tag{5.38}$$

$$- \operatorname{tg} a\frac{\partial}{\partial a} - \frac{1}{4}\frac{\partial^2}{\partial\Omega_3^2} + 3\frac{\partial^2}{\partial\lambda^2} +$$

$$+ \frac{1+\sin a}{4\cos^2 a}\Big(\frac{\partial^2}{\partial\Omega_1^2} + \frac{\partial^2}{\partial\Omega_2^2}\Big) - \frac{\sin a}{2\cos^2 a}\Big(\frac{\partial^2}{\partial\Omega_1^2} - \frac{\partial^2}{\partial\Omega_2^2}\Big)\Big\} .$$

VI. The Cubic Operator Ω

Let us finally calculate the operator

$$\Omega = \sum_{i,j,k} L_{ij} B_{jk} L_{ki} =$$

$$= -\frac{1}{4}\Big\{\frac{\partial}{\partial\lambda}\Big[\frac{1}{\sin a}\Big(\frac{\partial^2}{\partial\Omega_+^2} + \frac{\partial^2}{\partial\Omega_-^2}\Big) + 2\frac{\partial^2}{\partial\Omega_3^2}\Big] +$$

$$+ \Big(\frac{1}{2}\operatorname{ctg} a + \operatorname{tg} a\Big)\frac{\partial}{\partial\Omega_3}\Big(\frac{\partial^2}{\partial\Omega_+^2} + \frac{\partial^2}{\partial\Omega_-^2}\Big) - \tag{6.1}$$

$$- i\Big(\frac{1}{2}\operatorname{tg} a + \operatorname{ctg} a\Big)\Big(\frac{\partial^2}{\partial\Omega_+^2} - \frac{\partial^2}{\partial\Omega_-^2}\Big) - i\frac{\partial}{\partial a}\Big(\frac{\partial^2}{\partial\Omega_+^2} - \frac{\partial^2}{\partial\Omega_-^2}\Big) +$$

$$+ \Big(2\frac{\partial}{\partial\lambda} - \frac{1}{\cos a}\frac{\partial}{\partial\Omega_3}\Big)\Big(\frac{\partial^2}{\partial\Omega_+\partial\Omega_-} + \frac{\partial^2}{\partial\Omega_-\partial\Omega_+}\Big) - \frac{1}{2}\frac{1}{\cos a}\frac{\partial}{\partial\Omega_3}\Big\} ,$$

where

$$\left.\begin{aligned} \frac{\partial}{\partial\Omega_\pm} &= \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial\Omega_1} \mp i\frac{\partial}{\partial\Omega_2}\Big) \\ \frac{\partial}{\partial\Omega_-} &= \frac{i}{\sqrt{2}} e^{i\varphi_1}\Big[-i\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi_2} + \operatorname{ctg}\theta\frac{\partial}{\partial\varphi_1}\Big] \\ \frac{\partial}{\partial\Omega_+} &= -\frac{i}{\sqrt{2}} e^{-i\varphi_1}\Big[i\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi_2} + \operatorname{ctg}\theta\frac{\partial}{\partial\varphi_1}\Big] \end{aligned}\right\} \tag{6.2}$$

To obtain the operator in a form, in which the connection with the SU(2) operator is expressed explicitly, it is better to go over to another notation in B_{ik}. Introducing

$$\left.\begin{aligned} b^{(ik)}_{++} &= l^{(i)}_{+} l^{(k)}_{+} \qquad b^{(ik)}_{--} = l^{(i)}_{-} l^{(k)}_{-} \\ b^{(ik)}_{+-} &= \frac{1}{2}\left(l^{(i)}_{+} l^{(k)}_{-} + l^{(k)}_{+} l^{(i)}_{-}\right) \\ b^{(ik)}_{+0} &= \frac{1}{2}\left(l^{(i)}_{+} l^{(k)}_{0} + l^{(k)}_{+} l^{(i)}_{0}\right) \\ b^{(ik)}_{-0} &= \frac{1}{2}\left(l^{(i)}_{-} l^{(k)}_{0} + l^{(k)}_{-} l^{(i)}_{0}\right) \end{aligned}\right\} \tag{6.3}$$

and making use of (5.33), we get

$$\begin{aligned} B_{ik} = & -i\sqrt{2}\left[b^{(ik)}_{++} H_{+} - b^{(ik)}_{--} H_{-}\right] + 2\left(\delta_{ik} - b^{(ik)}_{+-}\right)\frac{\partial}{\partial\lambda} - \\ & -\frac{1}{\sqrt{2}} b^{(ik)}_{+0}\left[\frac{\partial}{\partial\Omega_2} - i\frac{\partial}{\partial\Omega_1}\right] - \frac{1}{\sqrt{2}} b^{(ik)}_{-0}\left[\frac{\partial}{\partial\Omega_2} + i\frac{\partial}{\partial\Omega_1}\right] . \end{aligned} \tag{6.4}$$

Operators H_{+} and H_{-} are the usual raising and lowering operators in SU(2) taken at the value of the second Euler angle $-2\Omega_3 = 0$.

$$H_{+} = \frac{i}{\sqrt{2}}\left[-i\frac{\partial}{\partial a} + \frac{1}{\sin a}\frac{\partial}{\partial\lambda} + \frac{1}{2}\,\mathrm{ctg}\, a\,\frac{\partial}{\partial\Omega_3}\right] . \tag{6.5}$$

$$H_- = -\frac{i}{\sqrt{2}}\left[i\frac{\partial}{\partial a} + \frac{1}{\sin a}\frac{\partial}{\partial\lambda} + \frac{1}{2}\operatorname{ctg} a\frac{\partial}{\partial\Omega_3}\right]. \qquad (6.6)$$

Applying these expressions, Ω oan be written in the form

$$\Omega = \frac{i}{2\sqrt{2}}\left(\frac{\partial^2}{\partial\Omega_-^2}H_+ - \frac{\partial^2}{\partial\Omega_+^2}H_-\right) - \frac{1}{4}\frac{\partial^2}{\partial\Omega_3^2}\frac{\partial}{\partial\lambda} . \qquad (6.7)$$

VII. Solution of the Eigenvalue Problem

We shall look now for the harmonic function ϕ , which fulfills equations (4.8) and (4.9), i.e. which is the eigenfunction of the Laplace operator and the operators N with the eigenvalues K(K + 4) and ν respectively. The general

form of it is the following:

$$\phi_M^L = \sum_B b(B)\sum_{M'=-B}^{B} a(M',M)D^B_{\nu,M'}(\lambda,a,-2\Omega_3=0)D^L_{-2M',M}(\varphi_1\theta\varphi_2). \qquad (7.1)$$

The first D-function is the eigenfunction of $B^2 = \frac{1}{4}\sum_{i,k} B_{ik}^2$ and N (quantum numbers B and ν). The second D-function is the eigenfunction of L^2 and L_3 (quantum numbers L and M), and it can be considered as a wave function of a rotating rigid top with the projection of angular momentum on the moving axis equal to $-2M'$. This projection is not conserved in our case, and we have to sum over different values of M'. That is just the point where we need an additional operator to orthogonalize the obtained linear space.

As we told already, we have chosen the operator Ω for this role. In that way we get the basis which is determined by five operators B^2, ν , L^2, M and Ω . The last step is to perform the unitary transformation to the eigenvectors of the Laplace operator. Since the Laplace operator doesn't commute only with B^2, we have to introduce the transformation coefficient $b(B)$ depending on the quantum number B only.

The function ϕ_M^L fulfills

$$L^2 \phi_M^L = -L(L+1)\phi_M^L , \tag{7.2}$$

$$L_3 \phi_M^L = -M\phi_M^L . \tag{7.3}$$

Concerning the operator B^2, every term in the sum (7.1) is its eigenfunction with the eigenvalue $B(B + 1)$. However, Φ^L_M is not an eigenvalue of B^2 because B^2 does not commute with the Laplace operator of the three-body system.

The coefficients $a(M,M')$ can be determined with help of the operator Ω, which connects the rotation and the permutation of the particles. Obviously, this operator doesn't lead out of the $D^B_{\nu,M'}(\lambda,a,0)D^L_{-2M',M}(\varphi_1\theta\varphi_2)$ space, since it raises M' in one and lowers -2M' in the other D-function with the same value. However, the determination of the mentioned coefficients demands a tedious calculation and it will be carried out in another paper.

Let us finally consider a few special cases of the solution. As it is discussed by Dragt[8], in the low-dimensional representations of SU(3) (L=0,1) the Ω is not needed. Indeed, in the case of L = 0 the Laplace operator obtains the form

$$\Delta = 4\frac{\partial^2}{\partial(2a)^2} + 4\,\mathrm{ctg}\,2a\frac{\partial}{\partial(2a)} + \frac{1}{\sin^2 a}\frac{\partial^2}{\partial\lambda^2} . \qquad (7.4)$$

Obviously, the eigenfunction will be the following:

$$\Phi_0 = D^{\frac{K}{4}}_{\nu,-\nu}\left(\frac{\lambda}{2}, 2a, -\frac{\lambda}{2}\right) \qquad (7.5)$$

which obeys the equation

$$\Delta\Phi_0 = -\frac{K}{4}\left(\frac{K}{4}+1\right)\Phi_0 . \qquad (7.6)$$

This solution demonstrates clearly the SU(2) - nature of a non-rotating triangle.

In the case of L=I the solution is

$$z^{M} = \sum_{M_1=\pm\frac{1}{2}} a(M_1 M)\, D_{\frac{1}{2}}^{\frac{1}{2} M_1}(\gamma' \alpha' 0)\, D_{1}^{-\frac{1}{2} M_1 M}(\varphi' \theta \varphi^{5})$$

fulfilling the Laplace equation with the value K = 1.

Simultaneously z_M obeys the equations

$$L^2 z_M = -2 z_M \quad , \tag{7.7}$$

$$L_3 z_M = -M z_M \qquad M = -1, 0, 1 \quad , \tag{7.8}$$

and
$$B^2 z_M = -\frac{3}{4} z_M \quad . \tag{7.9}$$

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