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AN EXACTLY SOLUBLE MODEL FOR RESONANCE SCATTERING

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I. Introduction

In the course of development of the nuclear theory the description of nuclear reactions has been attempted mainly by two approaches. The first of them is characterized by a high degree of generality and exactness which is achieved without going into details of the internal dynamics of the many-nucleon system. The main aim of these general theories /1/ is to clarify the structure and the analytical properties of the collision matrix. The other, so-called microscopic, approach starts from a more or less detailed dynamical description of the interacting many-nucleon system and tries to derive the S-matrix explicitly in terms of wave functions and Hamiltonians. As a rule, these microscopic theories of nuclear reactions /2/ contain ab ovo some approximations, since at the present stage our ability to tackle the many nucleon problem is very limited. In the nuclear structure calculations, concerning the low-lying states of nuclei, the various microscopic approximations used in the shell model, in the pairing model, in the RPA etc. are very well studied and their validity and accuracy have often been tested.

As far as the microscopic nuclear reaction theories are concerned the situation is quite different. First, the number of actual calculations performed up to now is rather small, mainly because of the cumbersome numerical computations involved. Second, the compa-

rison of the calculated results with the experimental data does not allow definite conclusions to be drawn about the reliability of the approximations, because in addition to approximations characteristic of these theories, we always have to introduce further assumptions of more general type e.g. about the form of the interaction potentials, about the restriction of the configuration space etc. In the final results of a detailed calculation destined to reproduce some experimental data, the effect of these different approximations are completely mixed up and it is very difficult to decide which of them is responsible for the success or failure of the attempt. Therefore, it seems to us that the effect of the specific approximations used in microscopic reaction theories can be studied most conveniently on a model-system, for which an exact solution can be evaluated and can then be compared with the results given by the different approximations /3/.

In this paper our aim is to construct such a model and to work out an exact solution for it. The results of the numerical calculations and the comparison of this exact solution with the solutions obtained in different approximations will be reported in a subsequent paper. Now, the question arises: what kind of model system should be chosen. Of course, it must consist of an incoming particle and a target. If we want to go beyond the case of simple potential scattering we have to choose a target with an excitable internal structure. The simplest target that meets this requirement is a particle which has at least two bound states in a potential. This system is appropriate for the investigation of resonance effects in elastic and inelastic scattering and pick-up reactions. With a slight modification stripping reactions and composite particle elastic scattering can be studied as well. The two particles are taken to be equal mass fermions but their spins will not enter explicitly since all the interactions will be taken spin-independent and thus the fermion character of the particles will show up only in the symmetry properties of the spatial wave functions. In fact, the calculation can be carried out as if the particles were spinless and distinguishable, and the two

possible spin-states (triplet and singulet) and the Pauli-principle (if necessary) can be taken into account afterwards in a simple way. Solving the Schrödinger equation of the above described system we are essentially confronted with a three-body problem: mathematically, because the Hamiltonian contains three interaction terms (the two single-particle potentials and the two-particle interaction) and physically, because the single-particle potential, acting on both particles can be considered as a third particle with infinite mass. As it is well known the three-body problem can be solved by Faddeev's method leading to a system of coupled integral equations. Faddeev's /4/ method solves all the principal difficulties associated with the three-body problem, its practical application, however, is rather difficult, since it leads to coupled, two-variable integral equations. It was realized many years ago that the use of non-local, separable potentials simplifies the problem very much /5/. Since the main guide in the choice of the model was the mathematical simplicity, all the interactions entering the problem were taken to be of spin-independent, non-local and separable acting only in relative s-states. The form and the parameters of the potentials were chosen to yield two single-particle bound states and one bound state in the two-particle system.

In Section II we describe our model in detail, in Section III we derive the integral equations to be solved numerically and in Section IV the calcultions of the transition matrix elemetnts are discussed.

II. The Model

The investigated model consists of two particles: the first is bound in a potential well and the second is scattered on the first one. This system should be treated essentially as a three-body system since the potential which acts on both particles can be considered as a third particle with infinite mass. The Hamiltonian is the following:

$$H(1,2) = t(1) + t(2) + U(1) + U(2) + V(1,2), \qquad (1)$$

where t is the kinetic energy operator, U is the single particle potential (which is assumed to be the same for both particles) and $V(1,2)$ is the interaction between the two particles. Since the aim of the present work is not a comparison between certain theoretical calculations and experimental data but rather a study of the reliability and accuracy of some approximations used in nuclear reaction theories, the choice of the potentials U and V was governed mainly by mathematical simplicity. Therefore both U and V were taken to be non-local, and separable. It is the separability which simplifies the calculation. Also, for the sake of simplicity in both cases we restricted ourselves to the s-state interactions only. As we want to study not only direct, but also inelastic and resonance scattering, the potential U must have at least two bound states. The general form of the nonlocal separable potential which acts only in relative s-states is given in momentum representation by

$$\langle \vec{k}_1 \vec{k}_2 \mid V \mid \vec{k}_1' \vec{k}_2' \rangle = \delta(\vec{p} - \vec{p}') \sum_i \rho_i v_i(k) v_i(k'), \qquad (2)$$

where the total and relative momenta are denoted by $\vec{p}$ and $\vec{k}$, respectively. It has been proved /6/ that the number of bound states of such a potential is limited by the number of terms having negative strength parameter (ρ_i). In our model the two-particle system has only one bound state, therefore it is enough to retain one attractive term:

$$\langle \vec{k}_1 \vec{k}_2 \mid V(1,2) \mid \vec{k}_1' \vec{k}_2' \rangle = -\lambda^2 \delta(\vec{p} - \vec{p}') v(k) v(k'). \qquad (3)$$

The form factor $v(k)$ is defined as

$$v(k) = \frac{1}{\beta^2 + k^2} \qquad (4)$$

which was introduced at first by Yamagouchi /5/. The wave function of the bound state can be obtained very easily and is given by

$$\phi(k) = \frac{\sqrt{\alpha\beta(\alpha+\beta)}}{\pi} \cdot \frac{v(k)}{k^2 + \alpha^2} \tag{5}$$

or in coordinate representation:

$$\phi(r) = \sqrt{\frac{\alpha\beta(\alpha+\beta)}{2\pi}} \frac{1}{\beta-\alpha} \left(\frac{e^{-\alpha r} - e^{-\beta r}}{r} \right). \tag{6}$$

The binding energy $\epsilon = -\alpha^2 \frac{\hbar^2}{m}$ is determined by the range (β) and strength (λ^2) parameters:

$$\alpha = -\beta + \sqrt{\frac{m \pi^2 \lambda^2}{\hbar^2 \beta}}. \tag{7}$$

Turning to the single particle potential U first of all we note that in the expression (2) the factor $\delta(\vec{p}-\vec{p}')$ expressing the translational invariance should be omitted, thus

$$\langle \vec{k}_1 | U(1) | \vec{k}_1' \rangle = \sum_i \rho_i v_i(k_1) v_i(k_1'). \tag{8}$$

Later on, we shall need the matrix elements of the single particle potential in the space of two particle states, these can be expressed as

$$\langle \vec{k}_1 \vec{k}_2 | U(1) | \vec{k}_1' \vec{k}_2' \rangle = \delta(\vec{k}_2 - \vec{k}_2') \sum_i \rho_i v_i(k_1) v_i(k_1') \tag{9}$$

$$\langle \vec{k}_1 \vec{k}_2 | U(2) | \vec{k}_1' \vec{k}_2' \rangle = \delta(\vec{k}_1 - \vec{k}_1') \sum_i \rho_i v_i(k_2) v_i(k_2').$$

Here the δ-functions clearly express the one-body character of U. Since in our model we want to have two single particle bound states, the potential must contain, at least, two terms:

$$<\vec{k}_1\,\vec{k}_2 \mid U(1) \mid \vec{k}'_1\vec{k}'_2> = -\delta(\vec{k}_2 - \vec{k}'_2)\sum_{i=1}^{2} \lambda_i^2\, v_1(k_1)\, v_1(k'_1) \quad . \qquad (10)$$

For bound states the Schrödinger equation reads

$$k^2 \phi_\nu(\vec{k}_1) - \sum_{i=1}^{2} \Lambda_i\, v_1(k_1) \int \Lambda_1\, v_1(k'_1)\, \phi_\nu(\vec{k}'_1)\, d\vec{k}'_1 = -a_\nu^2\, \phi_\nu(\vec{k}_1), \qquad (11)$$

where $E_\nu = -\frac{2m}{\hbar^2} a_\nu^2$ is the ν-th energy eigenvalue ($\nu = 0,1$) and Λ_i is defined as $\sqrt{\frac{2m}{\hbar^2}}\,\lambda_i$. The wave function can be obtained directly as

$$\phi_\nu(\vec{k}_1) = \frac{1}{k_1^2 + a_\nu^2} \sum_{i=1}^{2} \Lambda_i\, v_1(k_1)\, N_1(a_\nu) \quad , \qquad (12)$$

$$N_1(a_\nu) = \int \Lambda_1\, v_1(k_1)\, \phi_\nu(\vec{k}_1)\, d\vec{k}_1 \quad . \qquad (13)$$

Thus the coefficients $N_1(a_\alpha)$ are the solutions of the following homogenuous equations:

$$\sum_{i=1}^{2} M_{j1}(a_\nu) N_1(a_\nu) = 0, \qquad (14)$$

where the symmetric matrix $M_{j1}(a_\nu)$ is defined as

$$M_{j1}(a_\nu) = \delta_{j1} - 4\pi \Lambda_1 \Lambda_j \int \frac{v_j(k_1)\, v_1(k_1)}{k_1^2 + a_\nu^2}\, k_1^2\, dk_1 \quad . \qquad (15)$$

The equation (14) has nontrivial solutions only if

$$\det\,(M_{j1}(a_\nu)) = 0. \qquad (16)$$

This is an algebraic equation for the determination of the energy eigenvalues. The model is completely specified by the explicit definition of the form factors which are given by:

$$v_1(k) = \frac{1}{k^2 + \beta_1^2} \qquad (17)$$

$$v_2(k) = \frac{1}{k^2+\beta_2^2} - \frac{\Gamma}{k^2+\gamma_2^2} \quad (\Gamma > 0). \tag{18}$$

In this case the matrix $M_{j1}(\alpha_\nu)$ can be expressed explicitely in terms of the potential parameters as follows:

$$M_{11}(\alpha_\nu) = 1 - 2\pi^2\Lambda_1^2 \frac{1}{2\beta_1(\beta_1+\alpha_\nu)^2} \tag{19a}$$

$$M_{22}(\alpha_\nu) = 1 - 2\pi^2\Lambda_2^2 \left\{ \frac{1}{2\beta_2(\beta_2+\alpha_\nu)^2} - \right.$$

$$\left. - \frac{2\Gamma}{(\gamma_2+\beta_2)(\gamma_2+\alpha_\nu)(\beta_2+\alpha_\nu)} + \frac{\Gamma^2}{2\gamma_2(\gamma_2+\alpha_\nu)^2} \right\} \tag{19b}$$

$$M_{12}(\alpha_\nu) = -2\pi^2\Lambda_1\Lambda_2 \left\{ \frac{1}{(\beta_1+\beta_2)(\beta_1+\alpha_\nu)(\beta_2+\alpha_\nu)} - \right.$$

(19c)

$$\left. - \frac{\Gamma}{(\beta_1+\gamma_2)(\beta_1+\alpha_\nu)(\gamma_2+\alpha_\nu)} \right\} .$$

This choice of the form factors can be motivated by the following consideration. The bound state are orthogonal to each other and both of them have zero orbital angular momentum, consequently they must have different number of nodes. A separable nonlocal potential characterized by the form factors (17) and (18) leads to a bound $1s$ state if $\Lambda_2=0$ and to a $2s$ state if $\Lambda_1=0$.

Therefore it is expected, that in the case when both Λ_1 and Λ_2 are different from zero, we get two bound states. It is worth while to mention that in the special case when both $v_1(k)$ and $v_2(k)$ are of Yamaguchi type, that is $\Gamma = 0$, we get two solutions with negative energy eigenvalues but these can not be interpreted as proper bound states, since one of the wave functions has wrong asymptotic behaviour, namely it decays more slowly in the asymptotic region than $e^{-\alpha_i r}$. This difficulty, however, does not arise if $\Gamma \neq 0$ and in this case it is possible to prove that with a proper choice of the parameters Λ_1, Λ_2, Γ, β_1, β_2, γ_2, the eigenvalue equation (16) has two positive roots α_1 and α_2 so that

$$\max(\alpha_1, \alpha_2) < \min(\beta_1, \beta_2, \gamma_2)$$

This condition guarantees the correct assymptotic behaviour of the bound state wave functions. Besides the roots α_1 and α_2 there exist two negative roots and a pair of complex roots which can give rise to a "potential resonance".

The main features of the model are summarised in the Fig.1, which shows the energy spectrum of the system and the thresholds of the various processes.

III. Integral Equations for the Wave Function

The Schrödinger equation in momentum space is the following:

$$(k_1^2 + k_2^2 + \tilde{U}(1) + \tilde{U}(2) + \tilde{V}(1,2) - e)\,\Psi(\vec{k}_1 \vec{k}_2) = 0, \qquad (20)$$

where $e = \frac{2m}{\hbar^2} E$ and E is the total energy, and the tilda means multiplication by $\frac{2m}{\hbar^2}$. In order to solve this equation the Faddeev /4/ method should be used in which the wave function Ψ is written as a sum of three terms and a system of coupled integral equations is obtained for these terms:

$$\Psi(\vec{k}_1 \vec{k}_2) = \Psi^1(\vec{k}_1, \vec{k}_2) + \Psi^2(\vec{k}_1, \vec{k}_2) + \Psi^{12}(\vec{k}_1 \vec{k}_2) \qquad (21a)$$

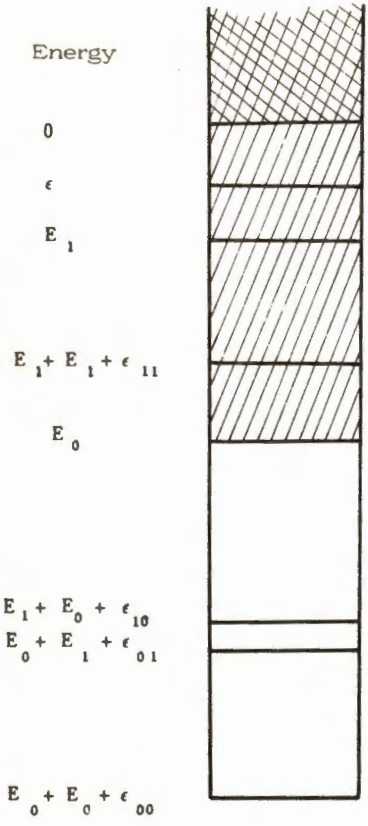

Fig. 1. The possible states of the Model System.

E_0 and E_1 are the bound state energies in the single particle potential, ϵ is the binding energy of the two particle system, ϵ_{00}, ϵ_{01}, ϵ_{11} and ϵ are the energy corrections to the "unperturbed" energies of U produced by the interaction V(12).

$$\begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^{12} \end{pmatrix} = \begin{pmatrix} \Phi^1_{o\vec{k}_a} \\ 0 \\ 0 \end{pmatrix} + G_0 \begin{pmatrix} 0 & T_1 & T_1 \\ T_2 & 0 & T_2 \\ T_{12} & T_{12} & 0 \end{pmatrix} \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^{12} \end{pmatrix} \qquad (21b)$$

Here G_0 is the Green-operator of the non-interacting system ; in the momentum representation:

$$\langle \vec{k}_1 \vec{k}_2 | G_0 | \vec{k}_1' \vec{k}_2' \rangle = \frac{\delta(\vec{k}_1 - \vec{k}_1')\delta(\vec{k}_2 - \vec{k}_2')}{e - k_1^2 - k_2^2 + i\eta}. \qquad (22)$$

The $\Phi^1_{o\vec{k}_a}$ is the wave function of the initial state, in which the first particle is in the bound ground state (indicated by the superscript 1 and subscript 0) and the second particle is described by an incident plane wave with momentum $\vec{k}_a$. The operators T_1, T_2 and T_{12} are the T-operators of two-body problems and they satisfy the equations:

$$\begin{aligned} T_1 &= \tilde{U}(1) + \tilde{U}(1) G_0 T_1 \\ T_2 &= \tilde{U}(2) + \tilde{U}(2) G_0 T_2 \\ T_{12} &= \tilde{V}(1,2) + \tilde{V}(1,2) G_0 T_{12} \end{aligned} \qquad (23)$$

The first task is to solve the equations (23). Using formulae (3) and (10), after some straight-forward calculations we obtain the matrix elements of the T-operators in momentum representation:

$$<\vec{k}_1\,\vec{k}_2\,|\,T_1\,|\,\vec{k}_1',\vec{k}_2'>=-\delta(\vec{k}_2-\vec{k}_2')\sum_{i,j}\Lambda_i\Lambda_j v_i(k_1)v_j(k_1')M^{-1}_{ij}(\sqrt{k_0^2+k_2^2}), \quad (24)$$

where the notations $\Lambda_i=\sqrt{\frac{2m}{\hbar^2}}\lambda_i$, and $e=-k_0^2$ were used and M^{-1} denotes the inverse of the matrix M defined in (16). A similar expression holds for the matrix element of T_2 only the indices 1 and 2 must be interchanged on the right hand side of (24). The matrix element of T_{12} is the following:

$$<\vec{k}_1\,\vec{k}_2\,|\,T_{12}\,|\,\vec{k}_1'\,\vec{k}_2'>=-\Lambda^2\delta(\vec{p}-\vec{p}')v(k)v(k')[M(\sqrt{p^2+2k_0^2})]^{-1}, \quad (25)$$

where again $\Lambda=\sqrt{\frac{2m}{\hbar^2}}\lambda$, $e=-k_0^2$ and the total and relative momenta were used. The quantity $M(x)$ is the following

$$M(x)=1-2\pi\Lambda^2\int_0^\infty\frac{v^2(k)k^2\,dk}{k^2+(\frac{x}{2})^2}= \quad (26)$$

$$=1-\frac{2\pi^2\Lambda^2}{\beta}\,\frac{1}{(x+2\beta)^2}\,.$$

Substituting (24) and (25) into (21b) we get the following expressions:

$$\Psi^1(\vec{k}_1\,\vec{k}_2)=\Phi^1_{0\vec{k}_a}(\vec{k}_1\,\vec{k}_2)+G_0\sum_i\Lambda_i v_i(k_1)F_i^1(\vec{k}_2) \quad (27a)$$

$$\Psi^2(\vec{k}_1\,\vec{k}_2)=G_0\sum_i\Lambda_i v_i(k_2)F_i^2(\vec{k}_1) \quad (27b)$$

$$\Psi^{12}(\vec{k}_1\,\vec{k}_2)=G_0\Lambda v(k)F^{12}(\vec{p})\,. \quad (27c)$$

On the right hand side of (27c) the total and relative momenta were introduced. The unknown quantities $F_i^1(\vec{x})$, $F^2_i(\vec{x})$, $F(\vec{x})$ satisfy the following equations:

$$M\left(\sqrt{x^2 + 2k_0^2}\right) F^{12}(\vec{x}) = \Lambda\, v\left(\frac{1}{2}\vec{x} - \vec{k}_a\right) \phi_0(\vec{x} - \vec{k}_a) + \Lambda \sum_i \int B_i(\vec{x}, \vec{y}) [F_i^1(\vec{y}) + F_i^2(\vec{y})]\, d\vec{y} \tag{28a}$$

$$\sum_j M_{ij}\left(\sqrt{x^2 + k_0^2}\right) F_j^1(\vec{x}) = \Lambda \int B_i(\vec{x}, \vec{y}) F^{12}(\vec{y})\, d\vec{y} + \sum_j \Lambda_j v_j(x) \int A_i(\vec{x}, \vec{y}) F_j^2(\vec{y})\, d\vec{y} \tag{28b}$$

$$\sum M_{ij}\left(\sqrt{x^2 + k_0^2}\right) F_j^2(\vec{x}) = \Lambda_i v_i(k_a) \phi_0(\vec{x}) + \Lambda \int B_i(\vec{x}, \vec{y}) F^{12}(\vec{y})\, d\vec{y} + \sum_j \Lambda_j v_j(x) \int A_i(\vec{x}, \vec{y}) F_j^1(\vec{y})\, d\vec{y}, \tag{28c}$$

where $\vec{k}_a$ is the momentum vector of the incident particle and the following energy equation holds: $E = E_0 + \frac{\hbar^2 k_a^2}{2m}$, $\phi_0(\vec{x})$ is the ground state wave function of the bound particle. Further, the notations:

$$B_i(\vec{x}, \vec{y}) = \Lambda_i \frac{v_i(\vec{y} - \vec{x})\, v\left(\vec{x} - \frac{1}{2}\vec{y}\right)}{k_0^2 + x^2 + (\vec{y} - \vec{x})^2}$$

$$A_i(\vec{x}, \vec{y}) = \Lambda_i \frac{v_i(y)}{k_0^2 + x^2 + y^2} \tag{29}$$

were used. Thus we got a system of coupled integral equations for the 5 unknown functions $F_i^1(\vec{x})$, $F_i^2(\vec{x})$, $F^{12}(\vec{x})$. Each of them depends on a vector variable and therefore all the integrals in (14) are three-fold. In order to get one-variable integral equations we expand the F-s as follows:

$$F(\vec{x}) = \sum_{\ell m} F_{\ell m}(x)\, Y_{\ell m}(\Omega_x) \tag{30}$$

The kernel $B_i(\vec{x},\vec{y})$ can also be expanded making use of the fact that it depends only on the relative angle between $\vec{x}$ and $\vec{y}$:

$$B_i(\vec{x},\vec{y}) = \sum_{\ell m} B_i^{\ell}(x,y)\, Y^*_{\ell m}(\Omega_x) Y_{\ell m}(\Omega_y) \ . \tag{31}$$

The $A_i(\vec{x},\vec{y})$ need not be expanded because it is independent of the angles of $\vec{x}$ and $\vec{y}$. To complete the expansion procedure we have to expand the first term on the right hand side of (28a):

$$v\left(\frac{1}{2}\vec{x}-\vec{k}_a\right)\phi_\nu(\vec{x}-\vec{k}_a) = \sum_{\ell m} \chi^{\ell}_{\nu}(x,k_a)\, Y_{\ell m}(\Omega_a) Y^*_{\ell m}(\Omega_x) \tag{32}$$

here the index ν can be either 0 or 1 according to the two bound states of the potential U. In the initial wave function, the particle 1. is in the ground state, therefore $\nu = 0$. If we assume - and that can be done without loss of generality - that the momentum of the incident particle is directed along the z- axis, then the expansion (32) takes the following form:

$$v\left(\frac{1}{2}\vec{x}-\vec{k}_a\right)\phi_\nu(\vec{x}-\vec{k}_a) = \sum_{\ell} \chi^{\ell}_{\nu}(x,k_a) Y_{\ell 0}(\Omega_x) \sqrt{\frac{2\ell+1}{4\pi}} \tag{32a}$$

Substituting the expressions (30), (31) and (32a) into equations (28), performing the integrations over the solid angle and using the orthonormality of the spherical harmonics, finally we get the following set of one-variable integral equations:

$$M(\sqrt{x^2+2k_0})F^{12}_{\ell m}(x) = \sqrt{\frac{2\ell+1}{4\pi}}\delta_{m0}\chi^{\ell}_0(x,k_a) + \\ + \Lambda\sum_j\int B^{\ell}_j(y,x)[F^1_{j\ell m}(y)+F^2_{j\ell m}(y)]y^2\,dy \qquad (33a)$$

$$\sum_i M_{ij}(\sqrt{x^2+k_a^2})F^1_{i\ell m}(x) = \Lambda\int B^{\ell}_i(x,y)F^{12}_{\ell m}(y)y^2\,dy + \\ + 4\pi\,\delta_{\ell 0}\delta_{m0}\sum_j\Lambda_j v_j(x)\int A_i(x,y)F^2_{j\ell m}(y)y^2\,dy \qquad (33b)$$

$$\sum_j M_{ij}(\sqrt{x^2+k_0^2})F^2_{j\ell m}(x) = \Lambda\int B^{\ell}_i(x,y)F^{12}_{\ell m}(y)\,dy + \\ + 4\pi\,\delta_{\ell 0}\delta_{m0}\sum_i\Lambda_j v_j(x)\int A_i(x,y)F^1_{j\ell m}(y)y^2\,dy + \\ + \sqrt{4\pi}\,\delta_{\ell 0}\delta_{m0}v_i(k_a)\phi_0(x). \qquad (33c)$$

The equations (33) are uncoupled in the quantum numbers ℓ and m. This is due to the separability of the potentials, that is to the fact, that in expansions (16) we could introduce ℓ and m in such a way, that in the total wave function they turn out to be the total orbital angular momentum quantum numbers. This can be seen immediately if one substitutes the expressions (27) and (30) into (21a). The total wave function can be written as a sum over terms with different angular momentum and each term consists of three parts: in the first part particle 1. has zero angular momentum and particle 2. caries the whole angular momentum ℓ, the second part has the same structure as the first, only the roles of the particles 1. and 2. are interchanged; in the third part the relative motion of the two particles has zero angular momentum, while the center-of-mass motion carries the total angular momentum ℓ.

There is an other feature of the solutions of eq. (33) that we can discover on the base of symmetry considerations. The whole

problem possesses an axial symmetry around the z-axis since the momentum vector of the incident particle is directed along the z-axis. This means that the wave function can not depend on the azimuthal angle. Thus in the expansions (30) only the term with $m = 0$ should be retained, that is the quantities $F_{\ell m}$ are proportional to δ_{m0}.

IV. Transition Matrix Elements

In the preceding section an exact solution was obtained for the wave function $\Psi_a(\vec{k}_1\ \vec{k}_2)$ of the scattering problem (there the subscript a indicates the initial state, described by $\Phi_{\vec{k}_a}$, see (21)). Our next task is to calculate the transition matrix elements

$$T_{ba} = \langle \Phi_b \mid V_b \mid \Psi_a \rangle \qquad (34)$$

which are connected with the observable cross-sections in a simple way. In (34) the Φ_b is the final state wave function and V_b, is the part of the total interaction which is "not contained" in Φ_b. More precisely this means, if Ψ_a satisfies the equation

$$H\Psi_a = E_a\Psi_a \qquad (35)$$

then Φ_b must satisfy

$$(H - V_b)\Phi_b = E_a\Phi_b \qquad (36)$$

To calculate the matrix element (34) we have to construct first of all the possible final states Φ_b. There are three basic types of final states:

a) Simple scattering: particle 1. remains in one of the bound states while the particle 2. is scattered into another free state, characterized by momentum $\vec{k}_b$. The wave function of such a final state is:

$$\Phi_b(\vec{k}_1\vec{k}_2) = \Phi^1_{\nu\vec{k}_b}(\vec{k}_1\vec{k}_2) = \phi_\nu(k_1)\delta(\vec{k}_2 - \vec{k}_b), \tag{37}$$

where the index ν refers to the bound state, and the superscript 1. indicates that in this wave function the particle 1. is bound. The vector $\vec{k}_b$ must satisfy the energy equation:

$$k_b^2 = \frac{2m}{\hbar^2}(E - E_\nu) > 0, \tag{38}$$

where E_ν and E are the energy of the ν-th bound state and the total energy, respectively. In this case:

$$V_b = U(2) + V(1,2). \tag{39}$$

b) Exchange scattering: particle 1. flies out with momentum $\vec{k}_b$ and particle 2. remains bound. The wave function:

$$\Phi_b(\vec{k}_1\vec{k}_2) = \Phi^2_{\nu\vec{k}_b}(\vec{k}_1\vec{k}_2) = \phi_\nu(k_2)\delta(\vec{k}_1 - \vec{k}_b). \tag{40}$$

The energy equation coincides with (38). The remaining interaction has the form:

$$V_b = U(1) + V(1,2) \tag{41}$$

c) "Pick-up" reaction: particles 1. and 2. form a bound state and together fly out with total momentum $\vec{k}_b$. The wave function:

$$\Phi_b(\vec{k}_1\vec{k}_2) = \Phi^{12}_{\vec{k}_b}(\vec{k}_1\vec{k}_2) = \phi(k)\ \delta(\vec{p} - \vec{k}_b), \tag{42}$$

where $\vec{p}$ and $\vec{k}$ are the total and relative momenta and $\phi(k)$ is the wave function of the bound state of potential $V(12)$.

The energy equation is now:

$$\frac{1}{2}k_b^2 = \frac{2m}{\hbar^2}(E - \epsilon) > 0. \tag{43}$$

In this case

$$V_b = U(1) + U(2). \tag{44}$$

A final state with both particles 1. and 2. in free states can not occure since we have restricted ourselves to negative E values only. Forming matrix elements (34) with the final state wave functions of these three types, we can evaluate the cross-section for any energetically possible process. Up to this point we treated the particles 1. and 2, as distinguishable, but it can be shown, that in a symmetrized treatment the cross- section can be expressed in terms of the same matrix elements. In order to calculate matrix elements (1) it is convenient to use the following identities:

$$\begin{aligned} U(1)\,\Psi_a &= (E - H_0)\,\Psi^1 \\ U(2)\,\Psi_a &= (E - H_0)\,\Psi^2 \\ V(1,2)\,\Psi_a &= (E - H_0)\,\Psi^{12}. \end{aligned} \tag{45}$$

These formulae can be obtained from equations (21) and (23) using some operator algebra. With the help of these equations the three types of matrix elements (34) can be easily evaluated. The results are the following:

Simple Scattering:

$$T^D(\nu \vec{k}_b, o\vec{k}_a) = \sum_\ell T_\ell^D(\nu k_b, o k_a) P_\ell(\cos\theta) \tag{46}$$

where the arguments of the transition matrix elements indicate the initial and final states., and θ is the scattering angle (the angle between vectors k_a and k_b). The meaning of T_ℓ^D is the following:

$$T_\ell^D(\nu k_b, o k_a) = -\frac{h^2}{2m}\sqrt{2\ell+1}\,\{\sqrt{4\pi}\,\delta_{\ell 0}\sum_i \Lambda_i v_i(k_b)\, I_{i\nu}^2 + \Lambda\, I_{\ell\nu}^{12}\} \tag{47}$$

with

$$I_{i\nu}^a = \int \phi_\nu(x)\, F_{i0}^a(x)\, x^2\, dx \qquad a = 1, 2 \qquad \text{and} \tag{48}$$

$$I^{12}_{\ell\nu} = \int \chi^{\ell}_{\nu}(x, k_b) F^{12}_{\ell}(x) x^2 \, dx \tag{49}$$

Exchange Scattering:

$$T^{E}(\nu \vec{k}_b, o\vec{k}_a) = \sum_{\ell} T^{E}_{\ell}(\nu k_b, o k_a) P_{\ell}(\cos\theta) \tag{50}$$

here the notations are the same as in the preceding case.

$$T^{E}_{\ell}(\nu k_b, o k_a) = -\frac{h^2}{2m}\sqrt{2\ell+1}\{\delta_{\ell 0}[(k_0^2 + k_a^2 + k_b^2)\phi_{\nu}(k_a)\phi_a(k_b) + \tag{51}$$

$$+ \sqrt{4\pi}\sum_{i}\Lambda_i v_i(k_b) I^{i}_{i\nu}] + \Lambda I^{12}_{\ell\nu}\}$$

Pick-up Reaction

$$T^{P}(\vec{k}_b, o\vec{k}_a) = \sum_{\ell} T^{P}_{\ell}(\vec{k}_b, o\vec{k}_a) P_{\ell}(\cos\theta) \tag{52}$$

$$T^{P}_{\ell}(k_b, o k_a) = -\frac{h^2}{2m}\{\omega_{\ell}(k_b, k_a) + \sqrt{\frac{2\ell+1}{4\pi}}\sum_{i}\Lambda_i(R^{1}_{i\ell} + R^{2}_{i\ell})\}, \tag{53}$$

where the quantities $\omega_{\ell}(k_b, k_a)$ and $R^{a}_{i\ell}$ are defined as follows:

$$\phi(\frac{1}{2}\vec{k}_b - \vec{k}_a)\phi_0(\vec{k}_b - \vec{k}_a)(k_0^2 + k_a^2 + (k_b - k_a)^2) = \tag{54}$$

$$= \sum_{\ell}\omega_{\ell}(k_a, k_b) P_{\ell}(\cos\theta)$$

and

$$R^{a}_{i\ell} = \int \eta^{i}_{\ell}(k_b, x) F^{a}_{i\ell}(x) x^2 \, dx \, . \tag{55}$$

The definition of the $\eta_\ell^1(x,y)$ is the following:

$$\phi\left(\frac{1}{2}\vec{x}-\vec{y}\right)a_1\left(\vec{x}-\vec{y}\right)=\sum_{\ell m}\eta_\ell^1(x,y)\,Y_{\ell m}(\Omega_x)Y^*_{\ell m}(\Omega_y) \qquad (56)$$

In eqs. (54) and (56) $\phi(x)$ denotes the wave function of the bound state of the potential $V(12)$.

It is well known that the transition matrix is diagonal in total angular momentum quantum number, but since we have used as initial and final states not angular momentum eigen states., our transition matrix elements (46a), (50) and (52) do not correspond to a definite angular momentum. But the diagonal matrix elements of the transition operator, corresponding to a definite ℓ can be extracted immediately just from equations (46), (50) and (52): they are the coeffieients of $P_\ell(\cos\theta)$ in these equations.

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