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INTERPRETATION OF UNCERTAINTY RELATIONS FOR THREE OR MORE OBSERVABLES

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1. INTRODUCTION

Different types and generalizations of the quantum uncertainty relations (UR) are known. For example, the uncertainty of observables may be described not by dispersions but in other ways, e.g., see [1-3]. For generalization to the case of non-Hermitian operators see [3,4]. Here I consider the extension of UR to the case of three and more observables, see Refs. 5, 6, 3, 7-9. Specifically, I shall discuss extensions of the so-called Schroedinger URs, see Refs. 10, 11, 3, 12, 13. The relation of Schroedinger UR to the well-known Heisenberg UR is discussed in Sec. 2.

My aim is to elucidate what new information the extension of UR to several observables provides.

Different kinds of inequalities are known which may be considered as extensions of UR to several observables, see Refs. 5, 6, 3, 7-9. In order to explain which of them are used here (they are called generalized uncertainty relations (GUR)) I present in Sec. 3 a derivation of GUR. It is similar to UR derivation starting with Cauchy inequality, given by Schroedinger [11]. I shall begin with derivation of the known generalized Cauchy inequality (GCI) (the term being used in [14]), and hence obtain GUR. The reason for this way of GUR derivation is that I need a separate expression for GCI in order to interprete GCI in a manner similar to the GUR interpretation. The relation of our GUR to other extensions of UR known in the literature is discussed at the end of Sec. 3.

I show in Sec. 4 what new infromation GUR and GCI provide. For summary see Sec. 5.

2. SCHROEDINGER AND HEISENBERG UNCERTAINTY RELATIONS

Robertson [10] and Schroedinger [11] obtained the inequality

$$\sigma_A^2 \sigma_B^2 \geq |(\psi, \Delta A \Delta B \psi)|^2 , \tag{1}$$

A and B are two observables, and ψ is a state vector;

$$\Delta A \equiv A - (\psi, A\psi), \quad \sigma_A^2 = (\psi, (\Delta A)^2 \psi). \tag{2}$$

Usually (1)is called the Schroedinger uncertainty relation, e.g., see [12,13]. The well-known Heisenberg uncertainty relation

$$\sigma_A^2 \sigma_B^2 \geq |(\psi, [A, B]\psi)|^2 \tag{3}$$

is a particular case of (1 , see below Subsec. 2.1. Therefore, (1) is natural subject for generalization to three or more observables. It is the inequality (1) which is implied here when using the term "uncertainty relation" (UR) without adjectives (Schroedinger or Heisenberg).

2.1. Note that (1) relates measurable quantities. This is evident for dispersion σ_A^2 and σ_B^2 which are mean values of Hermitian operators. As to $(\psi, \Delta A \Delta B \psi)$, it is not real when A and B do not commute and $\Delta A \Delta B$ is not Hermitian. But Schroedinger [11] pointed out a way of measuring $(\psi, \Delta A \Delta B \psi)$ in this case. Represent $\Delta A \Delta B$ as

$$\begin{aligned} \Delta A \Delta B &= \frac{1}{2}\{\Delta A \Delta B + \Delta B \Delta A\} + \frac{1}{2}[\Delta A \Delta B - \Delta B \Delta A] \\ &\equiv R + iJ\,, \end{aligned} \tag{4}$$

R and J are Hermitian operators, i.e., observables. Denoting

$$(\psi, R\psi) = r, \quad (\psi, J\psi) = j \tag{5}$$

we have $(\psi, \Delta A \Delta B \psi) = r + ij$, and Eq. (4) can be rewritten in the form

$$\sigma_A^2 \sigma_B^2 \geq r^2 + j^2. \tag{6}$$

If (6) holds, then of course

$$\sigma_A^2 \sigma_B^2 \geq j^2 = |(\psi, (-\frac{i}{2})[A, B]\psi)|^2. \tag{7}$$

So we obtain (3) from (1). In the general case (3) is less informative than (1): the region of possible values of $\sigma_A^2 \sigma_B^2$ which is allowed by (7) is greater than the region allowed by (6). In other words (6) is more restrictive than (7). For example, when A and B commute (3) turns into $\sigma_A^2 \sigma_B^2 \geq 0$ which is the trivial inequality giving no information on σ_A^2 and σ_B^2 (they are positive by definition, see Eq. (2)). Meanwhile (6) shows that in the case $\sigma_A^2 \sigma_B^2$ must be greater than a nonzero (generally) quantity r^2. However, (7) may be useful if one does not know how to obtain information which is lost when passing from (6) to (7). Heisenberg UR usually is considerably simpler than Schroedinger UR.

2.2. The simplest Heisenberg UR $\sigma_x^2 \sigma_p^2 \geq h^2/4$ has the well-known interpretation: the dispersions σ_x^2 and σ_p^2 in the same state ψ cannot be arbitrarily small: their product cannot be less than $h^2/4$, regardless ψ. The interpretation is not suitable in general for (1) and (3) because their right-hand sides depend upon ψ, A, B and $\sigma_A^2 \sigma_B^2$ has no definite lower bound. When rh sides assume zero values the inequalities (1) and (3) turn into the trivial inequality $\sigma_A^2 \sigma_B^2 \geq 0$. The following meaning of (1) may be more appropriate:

"The module of the ratio $(\psi, \Delta A \Delta B \psi)/\sigma_A \sigma_B$ of measurable quantities $(\psi, \Delta A \Delta B \psi)$ and $\sigma_A \sigma_B$ cannot exceed 1, this upper bound being independent of ψ, A, B".

Here σ_A denotes $|(\sigma_A^2)^{1/2}|$.

2.3. Schroedinger [11] obtained (1) starting with the known Cauchy-Bunyakowskii-Schwarz inequality (which will be called the Cauchy inequality in what follows):

$$(\alpha_1, \alpha_1)(\alpha_2, \alpha_2) \geq |(\alpha_1, \alpha_2)|^2, \tag{8}$$

where α_1 and α_2 are two vectors and (α_1, α_2) is their scalar product (this derivation will be reproduced incidentally in Sec. 3). Let us rewrite (8) in the form similar to $|(\psi, \Delta A \Delta B \psi)|^2/\sigma_A^2 \sigma_B^2 \leq 1$, namely

$$|\alpha_1, \alpha_2|^2/|\alpha_1|^2|\alpha_2|^2 \leq 1, \quad |\alpha_1|^2 \equiv (\alpha_1, \alpha_1). \tag{9}$$

In fact, the quantum-mechanical meaning of lhs. of (9) is well-known: is it the probability to find the state α_1 in the state α_2. Inequality (9) ensures that upper bound of this probability does not exceed 1 for any α_1 and α_2.

3. GENERALIZED CAUCHY INEQUALITY AND UNCERTAINTY RELATION

My aim now is to present the derivation of the known generalization of the Cauchy inequality (8) to the case of three and more vectors α_1, α_2, $\alpha_3, \ldots$ Hence, the generalized uncertainty relation will follow.

3.1. Let α_i, $i = 1, 2, \ldots, n$ denote several vectors describing possible physical states. Consider their superposition $\Phi = \sum_i \mu_i \alpha_i$, where μ_i are complex numbers. We have

$$(\Phi, \Phi) = \sum_{ij} (\alpha_i, \alpha_j) \mu_i^* \mu_j. \tag{10}$$

Since $(\Phi, \Phi) \geq 0$ the Hermitian form in rhs. of Eq. (10) is positive (nonnegative) for all μ_i. Therefore, the $n \times n$ matrix

$$\begin{pmatrix} (\alpha_1, \alpha_1) & (\alpha_1, \alpha_2) & (\alpha_1, \alpha_3) & \ldots \\ (\alpha_2, \alpha_1) & (\alpha_2, \alpha_2) & (\alpha_2, \alpha_3) & \ldots \\ (\alpha_3, \alpha_1) & (\alpha_3, \alpha_2) & (\alpha_3, \alpha_3) & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{11}$$

must be positive (nonnegative) definite. The necessary and sufficient conditions for its positivity are positivity of all principal minors of (11), e.g., see [15]. The simplest of these determinants are (α_i, α_i), $i = 1, 2, \ldots, n$ and they are evidently positive. Positivity of the principle minors of the second order gives the Cauchy inequalities

$$|\alpha_i|^2 |\alpha_j|^2 \geq |(\alpha_i, \alpha_j)|^2 \tag{12}$$

for each pair α_i, α_j out of α_1, α_2, $\alpha_3, \ldots$ The notation $|\alpha_i|^2 \equiv (\alpha_i, \alpha_i)$ and the property $(\alpha_i, \alpha_j) = (\alpha_j, \alpha_i)^*$ are used.

The positivity of the third order minors, in particular, of the determinant of the matrix explicitly written in (11) gives the inequality

$$
\begin{aligned}
&|\alpha_1|^2|\alpha_2|^2|\alpha_3|^2 + 2\mathrm{Re}(\alpha_1, \alpha_2)(\alpha_1, \alpha_3)(\alpha_3, \alpha_1) \\
&-|(\alpha_1, \alpha_2)|^2|\alpha_3|^2 - |(\alpha_2, \alpha_3)|^2|\alpha_1|^2 - |(\alpha_3, \alpha_1)|^2|\alpha_2|^2 \geq 0\,. \qquad (13)
\end{aligned}
$$

The equality holds if α_1, α_2, α_3 are linear dependent.

3.2. Uncertainty relations may now be obtained by putting

$$\alpha_i = (A_i - \bar{a}_i)\psi \equiv \Delta A_i \psi, \quad \bar{a}_i \equiv (\psi, A_i\psi); \qquad (14)$$

A_i are n observables A_1, A_2, $A_3, \ldots$, ψ is a vector. Let us denote

$$\sigma_i^2 \equiv (\alpha_i, \alpha_i) = (\psi, (\Delta A_i)^2\psi), \qquad (15)$$

$$\langle i, j\rangle \equiv (\alpha_i, \alpha_j) = (\psi, \Delta A_i \Delta A_j \psi). \qquad (16)$$

One obtains Scroedinger UR (1) for each pair A_i, A_j out of A_1, A_2, $A_3, \ldots$

$$\sigma_i^2\sigma_j^2 \geq |\langle i, j\rangle|^2. \qquad (17)$$

The generalized uncertainty relation (GUR) follows from (13)

$$
\begin{aligned}
&\sigma_1^2\sigma_2^2\sigma_3^2 + 2\mathrm{Re}\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle - \\
&-|\langle 1,2\rangle|^2\sigma_3^2 - |\langle 2,3\rangle|^2\sigma_1^2 - |\langle 3,1\rangle|^2\sigma_2^2 \geq 0. \qquad (18)
\end{aligned}
$$

3.3. One can derive (17) and (18) for the case when a physical state is described not by a vector ψ but by a density matrix W, e.g., see [3] and references therein. The same expressions (17) and (18) result, but with changed notation for σ_i^2 and $\langle i, j\rangle$:

$$\sigma_i^2 = \mathrm{Sp}W(\Delta A_i)^2, \quad \langle i, j\rangle = \mathrm{Sp}W\Delta A_i \Delta A_j. \qquad (19)$$

3.3. It is inequality (18) that is used (and called) here as generalization of UR for several observables. Other inequalities were deduced [5–9] using positivity of the matrix with the elements $\langle i, j\rangle$, see (11) and (16). These inequalities do not coincide with (18). Robertson himself [5] referred to them as "assuredly weaker" than (18). For example, in the case of three operators A_1, A_2, A_3 Robertson's inequality for $\sigma_1^2\sigma_2^2\sigma_3^2$ (see inequality (3.3) together with (3.5) in [5] or inequality (10) in [9]) turns into the trivial one $\sigma_1^2\sigma_2^2\sigma_3^2 \geq 0$. I need not comment further on this subject because positivity of principle minors is the necessary and *sufficient* condition, and suffice it to interprete only (18). Robertson [5] treated (18) as unmanageable, but I shall be able to interprete it in the next section.

4. INTERPRETATION OF GENERALIZED UNCERTAINTY RELATION

In the case of three observables A_1, A_2, A_3 we have three usual uncertainty relations (17) which give the following restrictions on the measurable quantities "Module of the ratios $\langle i,j\rangle/\sigma_i\sigma_j$ cannot exceed 1", see Sec. 2. I am going to demonstrate the validity of the following statement: GUR, see (18), provides additional restrictions on these three ratios $\langle i,j\rangle/\sigma_i\sigma_j$.

4.1. The lhs. of (18) depends upon three real quantities σ_1, σ_2, σ_3 and three complex ones $\langle i,j\rangle$ (six real). It is remarkable that (18) can be represented as an inequality containing three complex (six real) ratios $\langle i,j\rangle/\sigma_i\sigma_j$ considered above (indeed, divide lhs. of (18) by $\sigma_1^2\sigma_2^2\sigma_3^2$). Let us denote

$$\langle i,j\rangle/\sigma_i\sigma_j = \rho_{ij}\exp\varphi_{ij}. \tag{20}$$

Note that usual URs (17) restrict only ρ_{ij} ($\rho_{ij} \leq 1$) imposing no restriction on the phases φ_{ij}. In terms of ρ_{ij} and φ_{ij} inequality (18) takes the form

$$1 + 2\rho_{12}\rho_{23}\rho_{31}\cos\Sigma - \rho_{12} - \rho_{23} - \rho_{31} \geq 0, \quad \Sigma \equiv \varphi_{12} + \varphi_{23} + \varphi_{31}. \tag{21}$$

We can see that really lhs. of (21) depends on four real variables: ρ_{ij} and Σ.

If A_1, A_2, A_3 commute, then $\langle i,j\rangle$ are real, positive or negative. In the case, φ_{ij} assume only two values ($\varphi_{ij} = 0$ or π) and $\cos\Sigma$ is equal to ± 1.

One may verify that (21) is satisfied if all ρ_{ij} do not exceed 1/2. This is the example of allowed values of ρ_{ij}. Let us demonstrate that not all ρ_{ij} values (from intervals $(0,1)$) satisfy (21).

Consider at first instead of (21) its weakened consequence

$$1 + 2\rho_{12}\rho_{23}\rho_{31} - \rho_{12}^2 - \rho_{23}^2 - \rho_{31}^2 \geq 0 \tag{22}$$

(the inequality $\cos\Sigma \leq 1$ is used). One may verify that lhs. of (22) is not positive in the following region:

$$\frac{\sqrt{3}}{2} < \rho_{12} \leq 1, \quad \frac{\sqrt{3}}{2} < \rho_{31} \leq 1, \quad 0 \leq \rho_{23} < \frac{1}{2} \tag{23}$$

as well as in the analogous regions obtained from (23) by substitutions $\rho_{12} \rightleftarrows \rho_{23}$, $\rho_{31} \rightleftarrows \rho_{23}$. So (23) is the example of forbidden ρ_{ij} values. As (22) gives less information than (21), we expect that when $\cos\Sigma < 1$ the forbidden region is even larger as compared to (23).

Note that lhs. of (21) is the function (of four variables) the explicit form of which does not depend on a particular choice of A_1, A_2, A_3, ψ. So does the bound of allowed values of ρ_{ij} and Σ which is determined by equality (21) (cf. Subsec. 2.2).

4.2. Let us mention a particular GUR application. Let A_1, A_2, A_3 be some projections of spin operators of three particles 1, 2, 3 which originate in a reaction of the type $a + b \to 1 + 2 + 3$. In this case $\langle i, j\rangle$ are called correlations of polarixations. It was shown above that the measured values of the ratios $|\langle i, j\rangle|/\sigma_i\sigma_j$ cannot get into, e.g., the region (23).

4.2. In order to interprete generalized Cauchy inequality (GCI) (13) let us rewrite it in terms of the following variables ρ_{ij} and φ_{ij}

$$\rho_{ij} \exp \varphi_{ij} = (\alpha_i, \alpha_j)/|\alpha_i||\alpha_j|. \tag{24}$$

I use in Eq. (24) the same letters as in Eq. (20), but now ρ_{ij}^2 are probabilities which are ≤ 1 due to (12). In terms of ρ_{ij} and φ_{ij} inequality (13) assumes the form (21). The restrictions which this inequality imposes on the probabilities ρ_{ij} have already been discussed in Subsec. 4.1.

Let us mention a particular case of GCI which is specific of the probabilistic interpretation. Let α_2 and α_3 be orthogonal vectors, then $\rho_{23} = 0$ and the inequality under discussion assumes the simple form $\rho_{12}^2 + \rho_{31}^2 \leq 1$. This is the restriction on the possible values of the probabilities ρ_{12}^2 and ρ_{31}^2 to find α_2 or α_3 in the state α_1: they both cannot be close to 1.

5. SUMMARY

In the case of three observables A_1, A_2, A_3 we have three conventional uncertainty relations (URs) for each pair (A_1, A_2), (A_2, A_3), (A_3, A_1) out of A_1, A_2, A_3, see (17). In addition we have the generalized uncertainty relation (GUR), see (18), including dispersion of all observables. It is demonstrated that GUR gives new information. Namely, GUR provides restrictions on possible values of the quantities $\rho_{ij} = |(\psi, \Delta A_i \Delta A_j \psi)|/\sigma_i\sigma_j$. The restrictions complement the constraints $\rho_{ij} \leq 1$ which give conventional URs for each pair out of A_1, A_2, A_3.

The known Cauchy inequality $|\alpha_1|^2|\alpha_2|^2 \geq |(\alpha_1, \alpha_2)|^2$ may be given quantum-mechanical interpretation: it ensures that the ratio $|(\alpha_1, \alpha_2)|^2/|\alpha_1|^2|\alpha_2|^2$ can be interpreted as probability to find the state described by the vector α_1 in the state α_2. Generalizations (13) of the Cauchy inequality for three and more vectors α_1, α_2, $\alpha_3, \ldots$ are known. It is shown that they provide (in complete analogy to the above GUR) regions of forbidden values for the corresponding probability amplitudes $(\alpha_i, \alpha_j)/|\alpha_i||\alpha_j|$.

The above restrictions are universal in the sense that they do not depend upon a particular choice of A_1, A_2, A_3, ψ. They are consequences of most basic quantum postulates (such as "physical state is to be described by a

Hilbert space vector"), but do not include dynamical assumptions, e.g., the Schroedinger equation.

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