СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E4-2002-121

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THE KRÜGER PROBLEM

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2002

1 Introduction

Spin-flippers and spin rotators are very common devices in neutron physics now. One of such devices is radiofrequency spin-flipper. Its work is described by the well known Rabi formula

$$w_{fl}=\frac{\omega_1^2}{\omega_1^2+(\omega-\omega_0)^2}\sin^2\left(\sqrt{\omega_1^2+(\omega-\omega_0)^2}\frac{D}{v_0}\right), \tag{1}$$

derivation of which we shall remind below. Formula (1) gives probability of spin flipping w_{fl} for neutron with velocity v_0 after passage through the coil of length D. Inside the coil there is a constant magnetic field, $\boldsymbol{B}_0=(0,0,B_0)$, and a periodically varying in time radio frequency (RF) magnetic field, $\boldsymbol{B}_1=B_1(\cos 2\omega t,\sin 2\omega t,0)^1$, as shown in fig. 1. The terms $\omega_{0,1}$ in (1) denote $|\mu B_{0,1}|/\hbar$ respectively, where μ is the neutron magnetic moment.

However interaction of neutrons with RF field leads not only to spin rotation, but also to change of the neutron energy. If energy of the previous neutron before interaction with the field is $E_0=mv_0^2/2$, where m is the neutron mass, then after interaction with RF field it becomes $E=E_0\pm\hbar 2\omega$, which is equivalent to emission or absorption of RF quantum or inelastic scattering on the field. This change of energy is not accounted by (1).

We want to show how to find spin-flip probability taking into account inelastic interaction with RF field. For simplicity we suppose that all the fields are confined in some area of space and have no other dependence on space coordinates.

This problem was formulated before by Krüger [1], he presented solution as a combination of 16 waves with 16 unknown coefficients. He could solve the system of 16 equations only approximately. Complete solution of this system was given 14 years later [2]. We show how to avoid this formidable task and to obtain a compact physically transparent solution, from which all the interesting values can be calculated immediately.

So, let us consider one dimensional elastic and inelastic scattering of the neutron on magnetic field confined within the slab of space of thickness D: $0<x<D$ as shown in fig. 1. The field consists of a constant part directed along z-axis, $\boldsymbol{B}_0=(0,0,B_0)$, and RF part, $\boldsymbol{B}_1=B_1(\cos 2\omega t,\sin 2\omega t,0)$, rotating in $x-y$ plane with frequency 2ω. The incident neutron is described by a plane wave $\psi=\exp(ik_0x-i\mathcal{E}_0t)\xi_0$, where $k_0=mv_0/\hbar$, $\mathcal{E}_0=E_0/\hbar$,

[1] For convenience we use frequency 2ω instead of ω. This eliminates factor 1/2 in many formulas. However one should remember that the quantum of this RF field is $\hbar 2\omega$, not $\hbar\omega$.

$E_0 = mv_0^2/2$, and ξ_0 is an arbitrary spinor related to an arbitrary polarization. We need to find transmission, $\hat{\tau}$, and reflection, $\hat{\rho}$ matrix amplitudes, which contain information about neutron polarization and neutron energy after the interaction.

2 Scattering of high energy neutrons. Rabi formula

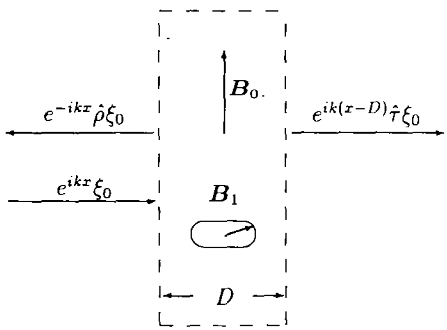

Fig. 1. Reflection and transmission by a slab of thickness D with constant $\boldsymbol{B}_0 = (0, 0, B_0)$ and RF $\boldsymbol{B}_1 = B_1(\cos 2\omega t, \sin 2\omega t, 0)$ magnetic fields, for a neutron with an arbitrary polarization described by the spinor ξ_0.

If the neutron energy E_0 is sufficiently high, i.e. $E_0 \gg \hbar 2\omega$, one may neglect the potential energy $-\boldsymbol{\mu B}_0$ and the change of velocity: $v - v_0 = \sqrt{v_0^2 - 2m\boldsymbol{\mu B}_0} - v_0 = 0$, when the neutron enters the area D. In that case we can bind the reference frame to the moving particle, and look only for time evolution of the neutron spinor $\xi(t)$, which is determined by the short Schrödinger equation

$$i\hbar d\xi(t)/dt = -\mu[\boldsymbol{\sigma B}_0 + \boldsymbol{\sigma B}_1(t)]\xi(t), \tag{2}$$

($\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$ are the Pauli matrices), and by initial condition $\xi(0) = \xi_0$.

For general solution of this equation we use the representation of $\boldsymbol{\sigma B}_1$:

$$\boldsymbol{\sigma B}_1(t) = \sigma_x \cos 2\omega t + \sigma_y \sin 2\omega t = \exp(-i\omega\sigma_z t)\sigma_x \exp(i\omega\sigma_z t), \tag{3}$$

and substitute it into (2). As the result we get

$$id\xi(t)/dt = [\sigma_z\omega_0 + \exp(-i\omega\sigma_z t)\sigma_x\omega_1 \exp(i\omega\sigma_z t)]\xi(t), \tag{4}$$

where $\omega_i = -\mu_n B_i/\hbar$ for $i = 0, 1$.

Solution to (4) can be represented in the form $\xi(t) = \exp(-i\omega\sigma_z t)\phi(t)$. Substitution of it into (4) gives

$$id\phi(t)/dt = [\sigma_z(\omega_0 - \omega) + \sigma_x\omega_1]\phi(t). \tag{5}$$

The expression in square brackets doesn't depend on time, thus $\phi(t)$ should be represented as $\phi(t) = \exp(i\boldsymbol{\Omega\sigma} t)\phi(0)$, where vector $\boldsymbol{\Omega}$ is $(\omega_1, 0, \delta\omega)$, and

$\delta\omega = \omega_0 - \omega$. The initial condition $\xi(0) = \xi_0$ is equivalent to $\phi(0) = \xi_0$, so the final solution to (4) is

$$\xi(t) = \exp(-i\omega\sigma_z t)\exp(i\boldsymbol{\Omega}\boldsymbol{\sigma}t)\xi(0) = \exp(-i\omega\sigma_z t)\left[\cos\Omega t + \frac{\boldsymbol{\Omega}\boldsymbol{\sigma}}{\Omega}\sin\Omega t\right]\xi_0, \tag{6}$$

where $\Omega = \sqrt{(\omega_0 - \omega)^2 + \omega_1^2}$.

Let us find spin-flip probability, when initially the spin is oriented parallel to z-axis, i.e. $\sigma_z\xi_0 = \xi_0$. The factor $\boldsymbol{\sigma}\boldsymbol{\Omega} = \delta\omega\sigma_z + \omega_1\sigma_x$ contains non-diagonal σ_x matrix, which flips the polarization. Thus flipping probability is

$$w_{fl} = \left|\frac{\Omega_x}{\Omega}\sin(\Omega t_1)\right|^2 = \frac{\omega_1^2}{\omega_1^2 + (\omega - \omega_0)^2}\sin^2\left(\sqrt{\omega_1^2 + (\omega - \omega_0)^2}t_1\right). \tag{7}$$

Time $t_1 = D/v_0$ is the flight time through fields in the area D. The equation (7), which coincides with (1), is the well known Rabi-formula for RF spin rotation. The polarization becomes completely flipped ($w_{fl} = 1$) at the resonance $\omega = \omega_0$, when $t_1\omega_1 = \pi/2$.

Now we can see the drawback of Rabi approach. After the interaction, the neuron wave function is $\psi(t) = \exp(ik_0x - i\mathcal{E}_0t)\xi(t_1)$, i.e. it has the same initial momentum k_0 and energy $\mathcal{E}_0$. However intuitively we feel that interaction with the RF field should change the energy due to absorption or emission of RF quanta 2ω.

To see the full picture of the interaction we need to take into account the change of the neutron energy and velocity. We can do that only by solving the complete Schrödinger equation. This is the Krüger problem.

3 The Krüger problem

The complete Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \mu[\boldsymbol{\sigma}\boldsymbol{B}_0(x) + \boldsymbol{\sigma}\boldsymbol{B}_1(t,x)]\right\}\psi(t,x), \tag{8}$$

where

$$\boldsymbol{B}_0(x) + \boldsymbol{B}_1(t,x) = \Theta(0 \le x \le D)[\boldsymbol{B}_0 + \boldsymbol{B}_1(t)],$$

and we introduced Θ-function, which is equal to 1, when inequality in its argument is satisfied, or 0 in the opposite case. In principle we can consider the case, when the permanent magnetic field exists in the whole space, but it will only result in unimportant mathematical complications, which will obscure the final results. We prefer to avoid such complications, and assume that outside the area $0 \le x \le D$ the space is empty.

We need to solve equation (8), which means to find reflected and transmitted waves, their energies and polarizations, when the incident wave is

$$\Theta(x < 0)\exp(ik_0x - i\mathcal{E}_0t)\xi_0. \tag{9}$$

To solve equation (8) we first suppose that $D = \infty$, i.e. the field exists in semiinfinite space $x \ge 0$, as is shown in fig. 2. In that case we have an interface at $x = 0$ between free space and space filled with fields, and we need to find the waves reflected and refracted at it.

3.1 Reflection and refraction at the interface

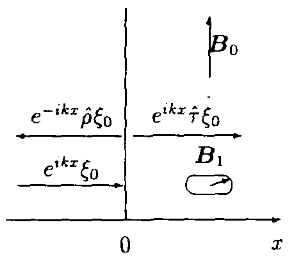

Fig. 2. Reflection and transmission at the interface at $x = 0$. To the left of it we have free space, and to the right there is the radiofrequency field $\boldsymbol{B}_1 = B_1(\cos\omega t, \sin\omega t, 0)$, and the permanent magnetic field $\boldsymbol{B}_0 = (0, 0, B_0)$. The $\hat{\rho}$ and $\hat{\tau}$ are reflection and refraction matrices, ξ_0 is a spinor of the incident particle.

Now we need to solve the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \mu[\boldsymbol{\sigma}\boldsymbol{B}_0 + \boldsymbol{\sigma}\boldsymbol{B}_1(t)]\Theta(x > 0)\right\}\psi(t,x), \tag{10}$$

with incident wave (9). This is a nonstationary equation with time dependent interaction potential, however with some simple transformation we can reduce equation (10) to a stationary one.

Indeed, the term $\boldsymbol{\sigma}\boldsymbol{B}_1(t)$ is represented in the form (3), and the solution of the equation (10) in the form

$$\psi(t,x) = e^{-i\omega\sigma_z t}\phi(t,x). \tag{11}$$

Substitution of (11) into (10) gives

$$\left\{i\frac{\partial}{\partial t} + \frac{\Delta}{2} + \sigma_z\omega - (\boldsymbol{\sigma}\boldsymbol{\Omega}_0)\Theta(x > 0)\right\}\phi(t,x) = 0, \tag{12}$$

where vector $\boldsymbol{\Omega}_0 = (\omega_1, 0, \omega_0)$, $\omega_0 = \mu B_0$, $\omega_1 = \mu B_1$, and we chose the unities $\hbar = m = 1$.

In equation (12) potential does not depend on time, so we may look for a stationary solution:

$$\phi(t,x) = e^{-iEt}\{\Theta(x<0)[\exp(i\hat{k}x)+\exp(-i\hat{k}x)\hat{\rho}]+\Theta(x>0)\exp(i\hat{k}'x)\hat{\tau}\}\xi_0, \tag{13}$$

where

$$\hat{k}=\sqrt{k^2+2\sigma_z\omega}, \quad \hat{k}'=\sqrt{k^2-2\boldsymbol{\Omega}\boldsymbol{\sigma}}, \quad \boldsymbol{\Omega}=(\omega_1,0,\omega_0-\omega), \quad k^2=2E, \tag{14}$$

and matrix amplitudes $\hat{\rho}$, $\hat{\tau}$ are defined by matching the function (13) and its derivative at the interface $x=0$. This matching gives two equations

$$1+\hat{\rho}=\hat{\tau}, \qquad \hat{k}(1-\hat{\rho})=\hat{k}'\hat{\tau}, \tag{15}$$

where 1 is the unit 2x2 matrix. Equations (15) have the solution

$$\hat{\tau}=(\hat{k}+\hat{k}')^{-1}2\hat{k} \quad \hat{\rho}=\hat{\tau}-1=(\hat{k}+\hat{k}')^{-1}(\hat{k}-\hat{k}'). \tag{16}$$

After substitution of (13) into (11) we get the total wave-function

$$\psi(t,x,E)=$$
$$e^{-iEt-i\omega\sigma_z t}\{\Theta(x<0)[\exp(i\hat{k}x)+\exp(-i\hat{k}x)\hat{\rho}]+\Theta(x>0)\exp(i\hat{k}'x)\hat{\tau}\}\xi_0. \tag{17}$$

Of course a linear combination of (17) with different E also satisfies the equation (10), and later we shall find that we need not a single wave (17), but a combination of two such waves with different E.

The first time factor $\exp(-iEt-i\omega\sigma_z t)$ in (17) looks very familiar and innocent, but σ_z here is very important and describes all the effects. The exponent in this time factor becomes $-i(E\pm\omega)$, if spinor after it is proportional to $\xi=\xi_{u,d}$, or, in other words, if the neutron is polarized along, ξ_u, or opposite, ξ_d, to z-axis, where $\xi_{u,d}$ are eigen spinors of the Pauli matrix σ_z with eigen values ±1: $\sigma_z\xi_{u,d}=\pm\xi_{u,d}$.

Let us check the role of this time factor in the incident wave

$$\exp(-iEt-i\omega\sigma_z t)\exp(i\hat{k}x)\xi_0$$

in expression (17). Since every spinor ξ_0 can be represented as a superposition

$$\xi_0=\alpha_u\xi_u+\alpha_d\xi_d, \qquad |\alpha_u|^2+|\alpha_d|^2=1, \tag{18}$$

the incident wave becomes

$$\psi_0(t,x,E)=\alpha_u e^{-i(E+\omega)t}e^{ik_+x}\xi_u+\alpha_d e^{-i(E-\omega)t}e^{ik_-x}\xi_d, \quad k_\pm=\sqrt{k^2\pm2\omega}, \tag{19}$$

which means that the two components of the incident wave have different energies and different velocities $v_\pm = k_\pm$ depending on frequency of the RF field, with which it have not yet interacted. This looks very strange and unacceptable.

If we want to consider a neutron with a definite energy E_0, then we can define parameter E as $E = E_0 - \omega$. Then the first part of the function (19) will look $\alpha_u e^{-iE_0t}e^{ik_0x}\xi_u$, where $k_0 = \sqrt{2E_0}$, as we wanted, but the second component will have different energy $E_0-2\omega$ and velocity $\sqrt{k_0^2-4\omega}$, which is also unacceptable. It shows that the solution of the form (17) is not appropriate for the incident wave with a general spinor ξ_0 and fixed primary energy E_0. If ξ_0 is a superposition (18), and both components in free space have the same energy, we cannot use a single stationary solution of (12) but must take the superposition of solutions (17):

$$\psi(t,x) = \alpha_u\psi_u(t,x,E_u) + \alpha_d\psi_d(t,x,E_d) =$$

$$\alpha_u e^{-iE_ut-i\omega\sigma_zt}\{\Theta(x<0)[e^{i\hat{k}_ux} + e^{-i\hat{k}_ux}\hat{\rho}_u] + \Theta(x>0)\exp(i\hat{k}'_ux)\hat{\tau}_u\}\xi_u +$$
$$\alpha_d e^{-iE_dt-i\omega\sigma_zt}\{\Theta(x<0)[e^{i\hat{k}_dx} + e^{-i\hat{k}_dx}\hat{\rho}_d] + \Theta(x>0)\exp(i\hat{k}'_dx)\hat{\tau}_d\}\xi_d, \quad (20)$$

with two different E, which we denoted $E_{u,d}$ for spinors $\xi_{u,d}$.

To have the incident wave

$$\psi_0(t,x) = \exp(-iE_0t + ik_0x)\xi_0 = \exp(-iE_0t + ik_0x)[\alpha_u\xi_u + \alpha_d\xi_d] \quad (21)$$

we must choose $E_u = E_0 - \omega$ and $E_d = E_0 + \omega$. Then (20) will look

$$\psi(t,x) = \alpha_u\psi_u(t,x,E_0-\omega) + \alpha_d\psi_d(t,x,E_0+\omega) =$$

$$\alpha_u e^{-iE_0t-i\omega(\sigma_z-1)t}\{\Theta(x<0)[e^{ik_0x} + e^{-i\hat{k}_ux}\hat{\rho}_u] + \Theta(x>0)\exp(i\hat{k}'_ux)\hat{\tau}_u\}\xi_u +$$
$$\alpha_d e^{-iE_0t-i\omega[\sigma_z+1]t}\{\Theta(x<0)[e^{ik_0x} + e^{-i\hat{k}_dx}\hat{\rho}_d] + \Theta(x>0)\exp(i\hat{k}'_dx)\hat{\tau}_d\}\xi_d, \quad (22)$$

where

$$\hat{k}_{u,d} = \sqrt{k_0^2 \mp 2\omega + 2\sigma_z\omega}, \quad \hat{k}'_{u,d} = \sqrt{k_0^2 \mp 2\omega - 2\boldsymbol{\sigma\Omega}}, \quad k_0^2 = 2E_0, \quad (23)$$

and the vector $\boldsymbol{\Omega}$ is defined in (14).

Let us look at the incident wave (21) again, representing it in the form

$$\psi_0(t,x) = \alpha_u e^{-iE_0t-i\omega(\sigma_z-1)t}e^{i\hat{k}_ux}\xi_u + \alpha_d e^{-iE_0t-i\omega(\sigma_z+1)t}e^{i\hat{k}_dx}\xi_d. \quad (24)$$

If we take into account that σ_z has eigen value ±1, when it acts on $\xi_{u,d}$, we reduce (24) to (21). However, let us imagine, that because of interaction with RF field the spinors $\xi_{u,d}$ are flipped upside down. Then $\xi_{u,d}$ in (24) are replaced with $\xi_{d,u}$, and instead of (24) we get

$$\psi_{fl}(t,x) = \alpha_u e^{-iE_0t-i\omega(\sigma_z-1)t}e^{ik_ux}\xi_d + \alpha_d e^{-iE_0t-i\omega(\sigma_z+1)t}e^{ik_dx}\xi_u. \qquad (25)$$

Now again we can replace all σ_z, entering this expression, with their eigen values, and as a result we obtain

$$\psi_{fl}(t,x) = \alpha_u e^{-i(E_0-2\omega)t}e^{ik_1x}\xi_d + \alpha_d e^{-i(E_0+2\omega)t}e^{ik_2x}\xi_u, \qquad (26)$$

where $k_{1,2} = \sqrt{k_0^2 \mp 4\omega}$. The function (25) contains two waves. The first one, obtained by flipping from the up state to down state, has the lower energy because of emission of RF quantum. The second wave after flipping from down to up state is accompanied by absorption of the quantum 2ω and increase of energy and velocity. .

It is interesting to mention that the up-down flipping proceeds with emission and down-up with absorption of RF quantum even in the absence of the permanent field $\boldsymbol{B}_0$, when the up and down states are degenerate. Such inequivalence of two degenerate states is related to handedness of rotation of the RF field and has nothing to do with parity violation.

The result of this subsection is represented by the solution (22), which is valid for incident neutron of a given energy E_0 and an arbitrary polarization. Now we can analyze polarization and energy of reflected neutrons. It is not difficult, however, if the area $0 \le x < infty$ is filled only with fields, this reflection is small, and we can neglect it. Reflection becomes important, when the space $0 \le x < \infty$ contains besides the fields also reflecting matter, as was considered in [3]. We shall consider this case later. As for now, we neglect reflection amplitude ρ, and approximate $\tau \approx 1$. In this approximation we can consider the finite thickness of the space with the fields, because reflection at the second interface can be also neglected. Thus we arrive at the original Krüger problem with the fields at $0 \le x \le D$, and start to analyze the wave function for the neutron transmitted through the fields i.e. at $x > D$.

3.2 The waves transmitted through the fields

Let us go back to the original Krüger problem, in which the fields are present in space of finite width D as shown in fig. 1. We suppose that at both interfaces $\rho_{u,d} = 0$ and $\tau_{u,d} = 1$. Then the transmitted wave at $x > D$ according to (22) becomes

$$\psi^{tr}(t,x) =$$

$$\alpha_u e^{-iE_0t-i\omega(\sigma_z-1)t}e^{i\hat{k}_u[x-D]}e^{i\hat{k}'_uD}\xi_u+\alpha_d e^{-iE_0t-i\omega(\sigma_z+1)t}e^{i\hat{k}_d[x-D]}e^{i\hat{k}'_dD}\xi_d, \quad (27)$$

where factor $\exp(i\hat{k}'_{u,d}D)$ describes propagation inside RF field, and factor $\exp(i\hat{k}_{u,d}[x-D])$ describes propagation in free space after it.

The factor $\exp(-iE_0t-i\omega[\sigma_z\mp 1]t)\exp(i\hat{k}_{u,d}[x-D])$ contains only σ_z matrix, so it cannot flip the $\xi_{u,d}$ spinors. Only the propagation factor

$$T_{u,d}(\boldsymbol{\sigma\Omega})=\exp(i\hat{k}'_{u,d}D) \quad (28)$$

can do this because it contains σ_x matrix. To find its explicit dependence on σ_x and then flipping and nonflipping probabilities, we use the following relation that is valid for arbitrary function $f(x)$:

$$f(\boldsymbol{q\sigma})=f_+(q)+\frac{\boldsymbol{q\sigma}}{q}f_-(q), \quad f_\pm=\frac{1}{2}[f(q)\pm f(-q)]. \quad (29)$$

One can easily check it acting by operator $f(\boldsymbol{q\sigma})$ upon eigen spinors $\xi_{\pm q}$ of the matrix $\boldsymbol{q\sigma}$.

Application of (29) to functions (28) gives

$$T_{u,d}(\boldsymbol{\sigma\Omega})=\frac{1}{2}\left[e^{ik'_{u,d}(\Omega)D}+e^{ik'_{u,d}(-\Omega)D}\right]+\frac{\boldsymbol{\sigma\Omega}}{2\Omega}\left[e^{ik'_{u,d}(\Omega)D}-e^{ik'_{u,d}(-\Omega)D}\right], \quad (30)$$

where $\Omega=\sqrt{\omega_1^2+(\omega-\omega_0)^2}$,

$$\hat{k}'_{u,d}=\sqrt{\tilde{k}_{1,2}^2-2\boldsymbol{\sigma\Omega}}, \quad k'_{u,d}(\pm\Omega)=\sqrt{\tilde{k}_{1,2}^2\mp 2\Omega}, \quad \tilde{k}_{1,2}^2=k_0^2\mp 2\omega. \quad (31)$$

Now we can easily find transmitted waves with nonflipped and flipped polarization, when the incident wave has polarization of the general form (18). For nonflipped polarization the wave function is

$$\psi_{nfl}^{tr}=\exp(ik_0[x-D]-iE_0t)\xi_{nfl}, \quad \xi_{nfl}=\alpha'_u\xi_u+\alpha'_d\xi_d, \quad \alpha'_{u,d}=\beta_{u,d}\alpha_{u,d}, \quad (32)$$

where

$$\beta_{u,d}=\frac{1}{2}[\exp(ik'_{u,d}(\Omega)D)+\exp(ik'_{u,d}(-\Omega)D)]\pm$$
$$\frac{\omega_0-\omega}{2\Omega}[\exp(ik'_{u,d}(\Omega)D)-\exp(ik'_{u,d}(-\Omega)D)], \quad (33)$$

and the sign $\pm$ is related to u,d components respectively. The spinor ξ_{nfl} of the wave (32) differs from ξ_0, but energy E_0 and velocity k_0 are the same as in the incident wave.

For flipped polarization we have the wave function

$$\psi_{fl}^{tr}=\gamma_u\alpha_u e^{-iE_0t-i\omega(\sigma_z-1)t}e^{ik_u(x-D)}\xi_d+\gamma_d\alpha_d e^{-iE_0t-i\omega(\sigma_z+1)t}e^{ik_d(x-D)}\xi_u, \quad (34)$$

where

$$\gamma_{u,d} = \frac{\omega_1}{2\Omega}\left[\exp(ik'_{u,d}(\Omega)D) - \exp(ik'_{u,d}(-\Omega)D)\right], \tag{35}$$

and $k'_{u,d}(\pm\Omega)$ are shown in (31).

The wave (34) is of the form (25), and everything, which was said there, is valid for (34). Flipped wave function consists of two plane waves with different energies and precisely defined polarizations.

Probability of spin flip

$$w^{fl}_{u,d} = |\gamma_{u,d}|^2 = \frac{\omega_1^2}{\sqrt{\omega_1^2 + (\omega - \omega_0)^2}}\frac{1}{4}\left|\exp(ik'_{u,d}(\Omega)D) - \exp(ik'_{u,d}(-\Omega)D)\right|^2$$

can be reduced to (7) only for $k_0^2 \gg 2\omega + \Omega$, which can be expected.

3.3 Application of the Krüger problem

The wave function (34) is a coherent superposition of a single neutron in two different energy states. These two states correspond also to two different neutron velocities, so with time these two states become separated in space. This separation can be cancelled with the help of second area with the fields, identical to the first one shown in fig. 1. After second transmission velocities of two components become interchanged, slower component becomes faster and overtakes the second component, which was first fast, and after second transmission became slower. At the position, where two components meet, one can observe interference - time oscillation of polarization. This longitudinal Mach-Zehnder interferometer is described in [4].

Here we want to describe another effect. If the space $x > D$ to the right of the area shown in fig. 1 is not empty but filled with the permanent magnetic field $\boldsymbol{B} = (0, 0, -B)$, with such B that $\mu B = 2\omega$, then the velocities of both components in (34) will be the same $v = v_0$. The wave function $\psi(x,t)$ of neutrons with the same velocities but different energies of two components represents the beam, which polarization on any axis other than z is oscillating in time. We can observe this time oscillation of polarization at every point downstream the neutron beam. Indeed, such a wave function is representable in the form

$$\psi(x,t) = e^{ik_0(x-D)}\left[e^{-i(E_0-2\omega)t}c_u\xi_u + e^{-i(E_0+2\omega)t}c_d\xi_d\right]. \tag{36}$$

Let us find the polarization on axis x. It is

$$< \psi^*|\sigma_x|\psi >= |c_u^* c_d|\cos(2\omega t - \phi), \tag{37}$$

where ϕ is the phase of the factor $c_u c_d^*$. If we put on the beam way an analyzer of polarization along x-axis, we shall obtain the beam with oscillating intensity. The beam with oscillating intensity can be represented by an intensity wave, which is coherent and can be used [5] for neutron holography.

It is also important to note, that the neutrons with precessing spin (precession frequency ω) both components of which are mowing with the same speed v can be represented by a wave (spin wave) $\exp(ikr - i\omega t)$ $(k = \omega/v)$, which is not the de Broglie wave, because its frequency is not $mv^2/2\hbar$. This plane wave is coherent and its phase is determined by RF field in the area $0 \leq x \leq D$ of fig. 1. Thus this plane wave can be used for neutron holography, as pointed out in [4]. This holography can be used without reference beam or with an artificial computerized reference wave. The computerized reference wave has phase determined by the phase of the same RF field, and it can be superposed with scattered spin wave at position sensitive detector, every sell of which is supplied with time-of flight analyser.

4 The Krüger problem at total reflection

The next problem, which we shall consider here, is the total reflection from an interface separating free space and the space filled with magnetic fields and matter, as shown in fig. 3.

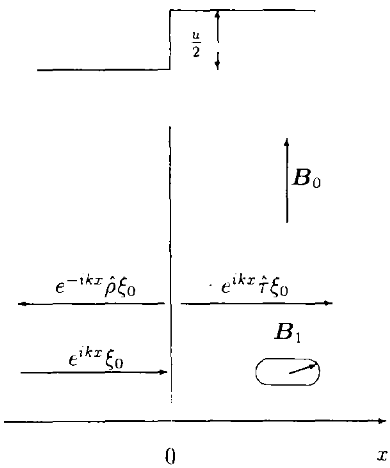

Fig. 3. Total reflection from the interface at $x = 0$ separating free space and the space $x > 0$ filled with constant $\boldsymbol{B}_0 = (0, 0, B_0)$ and RF $\boldsymbol{B}_1 = B_1(\cos\omega t, \sin\omega t, 0)$ fields and with the matter described by the optical potential shown in the upper part of the figure. The $\hat{\rho}$ and $\hat{\tau}$ are reflection and refraction matrices, ξ_0 is a spinor of the incident particle.

This problem is related to physics of ultracold neutrons, and it was solved for separate spin components in [3]. Here we present more compact solution using the results of section (3.1). The presence of matter is manifested by an optical potential of neutron-matter interaction, which can be represented as $\hbar^2 u/2m$, where $u = 4\pi N_0 b$, N_0 is atomic density of the matter, and b is coherent scattering length of neutrons on the matter nuclei. With this interaction the Schrödinger equation (10) becomes

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \left[\mu\boldsymbol{\sigma}\boldsymbol{B}_0 + \mu\boldsymbol{\sigma}\boldsymbol{B}_1(t) + \frac{\hbar^2}{2m}u\right]\Theta(x > 0)]\right\}\psi(t,x), \tag{38}$$

and after substitution of $\psi(t,x) = e^{-i\omega\sigma_z t}\phi(t,x)$ with account of (3) it is transformed to

$$\left\{i\frac{\partial}{\partial t} + \frac{\Delta}{2} + \sigma_z\omega - \left[\boldsymbol{\sigma}\boldsymbol{\Omega}_0 + \frac{u}{2}\right]\Theta(x > 0)\right\}\phi(t,x) = 0. \tag{39}$$

Solution of this equation with incident wave (21) is represented by (22), where $\hat{k}'_{u,d}$ inside matter are now $\hat{k}'_{u,d} = \sqrt{k_0^2 \mp 2\omega - 2\boldsymbol{\sigma}\boldsymbol{\Omega} - u}$. If $k_0^2 < u$ reflection of some components is total.

We can expect three reflected waves. The first one has the same energy as the incident wave but slightly changed polarization. Two other waves are completely polarized, and their energies are changed by emission or absorption of RF field quantum. Reflection of all the waves is described by reflection matrices $\hat{\rho}_{u,d}$ given by (16)

$$\hat{\rho}_{u,d} = (\hat{k}_{u,d}(\omega\sigma_z) + \hat{k}'_{u,d}(\boldsymbol{\sigma}\boldsymbol{\Omega}))^{-1}(\hat{k}_{u,d}(\omega\sigma_z) - \hat{k}'_{u,d}(\boldsymbol{\sigma}\boldsymbol{\Omega})), \tag{40}$$

where

$$\hat{k}_{u,d}(\omega\sigma_z) = \sqrt{\tilde{k}_{1,2}^2 + 2\omega\sigma_z}, \quad \hat{k}'_{u,d} = \sqrt{\tilde{k}_{1,2}^2 - u - 2\boldsymbol{\sigma}\boldsymbol{\Omega}}, \quad \tilde{k}_{1,2}^2 = k_0^2 \mp 2\omega.$$

We see that $k_{u,d}$ contains only σ_z, which cannot flipp the spin, while $k'_{u,d}$ contains σ_x, which is responsible for spin flipping.

To find flipped and nonflipped part of reflection we must represent $\hat{\rho}$ in the form $A + \boldsymbol{B}\boldsymbol{\sigma}$, and find A and $\boldsymbol{B}$. Such a representation immediately gives the amplitude of nonflipped reflected component to be $A + B_z$, and that of flipped components to be $B_x \mp iB_y$.

To get such a representation we use (29) and the following relation, which is valid for arbitrary function $f(x)$:

$$f^{-1}(\boldsymbol{p}\boldsymbol{\sigma}) = \left[f_+ + \frac{\boldsymbol{p}\boldsymbol{\sigma}}{p}f_-\right]^{-1} = \frac{1}{f_+^2 - f_-^2}\left[f_+ - \frac{\boldsymbol{p}\boldsymbol{\sigma}}{p}f_-\right] = \frac{f(-\boldsymbol{p}\boldsymbol{\sigma})}{f_+^2 - f_-^2}. \tag{41}$$

From the last relation we can find another one, which is valid for arbitrary functions $f(x)$ and $g(x)$:

$$[f(\boldsymbol{p\sigma}) + g(\boldsymbol{q\sigma})]^{-1} = \frac{1}{N}[f(-\boldsymbol{p\sigma}) + g(-\boldsymbol{q\sigma})], \tag{42}$$

where

$$f_\pm = \frac{1}{2}[f(p) \pm f(-p)], \quad g_\pm = \frac{1}{2}[g(q) \pm g(-q)],$$

$$N = f(p)f(-p) + g(q)g(-q) + 2f_+g_+ - 2f_-g_-\frac{\boldsymbol{pq}}{pq}.$$

Using all these relations we represent (40) in the form

$$\hat{\rho}_{u,d} =$$

$$\frac{k_{u,d}(\omega)k_{u,d}(-\omega) - k'_{u,d}(\Omega)k'_{u,d}(-\Omega) + 2k'^{+}_{u,d}k^{-}_{u,d}\sigma_z - 2k'^{-}_{u,d}k^{+}_{u,d}\frac{\boldsymbol{\sigma\Omega}}{\Omega} + 2i\sigma_y k'^{-}_{u,d}k^{-}_{u,d}\frac{\omega_1}{\Omega}}{k_{u,d}(\omega)k_{u,d}(-\omega) + k'_{u,d}(\Omega)k'_{u,d}(-\Omega) + 2k'^{+}_{u,d}k^{+}_{u,d} - 2k'^{-}_{u,d}k^{-}_{u,d}\frac{\omega_0-\omega}{\Omega}}, \tag{43}$$

where

$$k^{\pm}_{u,d} = \frac{1}{2}\left[\sqrt{\tilde{k}^2_{1,2} + 2\omega} \pm \sqrt{\tilde{k}^2_{1,2} - 2\omega}\right],$$

$$k'^{\pm}_{u,d} = \frac{1}{2}\left[\sqrt{\tilde{k}^2_{1,2} - u - 2\Omega} \pm \sqrt{\tilde{k}^2_{1,2} - u + 2\Omega}\right], \quad k^2_{1,2} = k_0^2 \mp 2\omega.$$

From (43) it is easy to find reflected nonflipped, ψ^{rf}_{nfl}, and flipped ψ^{rf}_{fl} parts of the wave function in the same way as in (32) and (34):

$$\psi^{rf}_{nfl} = \exp(-ik_0x - iE_0t)\xi_{nfl}, \quad \xi_{nfl} = \alpha^r_u\xi_u + \alpha^r_d\xi_d, \quad \alpha^r_{u,d} = \beta^r_{u,d}\alpha_{u,d}, \tag{44}$$

where

$$\beta^r_{u,d} = \frac{k_{u,d}(\omega)k_{u,d}(-\omega) - k'_{u,d}(\Omega)k'_{u,d}(-\Omega) \pm 2k'^{+}_{u,d}k^{-}_{u,d} \mp 2k'^{-}_{u,d}k^{+}_{u,d}\frac{\omega_0-\omega}{\Omega}}{k_{u,d}(\omega)k_{u,d}(-\omega) + k'_{u,d}(\Omega)k'_{u,d}(-\Omega) + 2k'^{+}_{u,d}k^{+}_{u,d} - 2k'^{-}_{u,d}k^{-}_{u,d}\frac{\omega_0-\omega}{\Omega}}, \tag{45}$$

where upper sign in the numerator is related to β^r_u, and lower one to β^r_d.

The spinor ξ_{nfl} of the wave (44) differs from ξ_0, but energy E_0 and velocity k_0 are the same as in the incident wave.

The wave function with flipped spinor is

$$\psi^{rf}_{fl} = \gamma^r_u\alpha_u e^{-iE_0t - i\omega(\sigma_z-1)t}e^{-ik_ux}\xi_d + \gamma^r_d\alpha_d e^{-iE_0t - i\omega(\sigma_z+1)t}e^{-ik_dx}\xi_u, \tag{46}$$

where

$$\gamma^{r}_{u,d}=\frac{-\frac{\omega_1}{\Omega}[k'_{u,d}(\Omega)-k'_{u,d}(-\Omega)]k_0}{k_{u,d}(\omega)k_{u,d}(-\omega)+k'_{u,d}(\Omega)k'_{u,d}(-\Omega)+2k'^{+}_{u,d}k^{+}_{u,d}-2k'^{-}_{u,d}k^{-}_{u,d}\frac{\omega_0-\omega}{\Omega}}, \tag{47}$$

where $k_0=\sqrt{2E_0}$. The wave (46) is of the form (25), and everything, which was said there, is valid here too. Flipped wave function consists of two plane waves with different energies and precisely defined polarizations. The flipped amplitudes are proportional to ω_1, and they naturally disappears, if RF magnetic field is absent.

At resonance, $\omega=\omega_0$, when $\Omega=\omega_1$, and ω_1 is negligeable comparing to other parameters all the formulas simplify. From (45,47) we obtain

$$\beta_{u,d}=\frac{k_0-k'_{\mp}}{k_0+k'_{\mp}},\quad \gamma_{u,d}=\frac{2\omega_1}{u\pm 2\omega}\frac{k_0(k_0-k'_{\mp})}{k'_{\mp}(k_{\mp}+k'_{\mp})}, \tag{48}$$

where $k_{\pm}=\sqrt{2(E_0\pm 2\omega)}$, and $k'_{\pm}=\sqrt{2(E_0\pm\omega)-u}$. If one or both $2(E_0\pm\omega)<u$, we have to replace $k'_{\pm}$ by $ik''_{\pm}$, where $k''_{\pm}=\sqrt{u-2(E_0\pm\omega)}$. In that case we have total reflection of one or both spin components from the interface.

Because of inelastic interaction with RF field, and conservation of the momentum of the incident particle parallel to interface, the reflected beam is splitted in three beams, which have different energy, angle of reflection and polarization. The specularly reflected beam has the same energy as the incident one, and a little bit different polarization. The bean, which is reflected at larger angle has higher energy and is polarized along internal magnetic field, and the beam reflected at lower angle has lower energy and is polarized oppositely to the internal field. It is important to tell, that angular splitting is determined by the magnetic field in the matter, and can be mach larger than angular splitting, which takes place at elastic reflection from a magnetic mirror, when magnetization is not collinear to the external field [6]. In the last case the angular splitting is determined by external field, which, as a rule, is lower, than the internal field.

5 Conclusion

We have demonstrated an analytical method of solution of the Krüger problem. This problem is equivalent to one dimensional scattering of neutron on RF field confined in a limited space region. The scattering produces three components: elastic one, and two inelastic components on both sides of it. Inelastic components are produced by absorption and emission of quantum of RF field, and they are completely polarized in opposite directions. The well known Rabi formula for RF reverse of polarization is reproduced in the limit of high energies, when energy of RF quanta can be neglected.

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Received on May 21, 2002.