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# SIMPLEST SOURCES OF ELECTROMAGNETIC FIELDS AS A TOOL FOR TESTING THE RECIPROCITY-LIKE THEOREMS

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# 1 Introduction

Toroidal solenoids (TS) play an important role in physics and technology. As the simplest 3-dimensional topologically nontrivial objects, they have been used for the experimental verification of the Aharonov-Bohm effect [1]. The corresponding calculations have been performed in [2]. They possess a number of nontrivial characteristics such as toroidal ([3,4]) and "hidden" ([5]) moments. Exact vector potentials (VP) of finite static TS were evaluated by Luboshitz and Smorodinsky ([6]), in a non-standart gauge, and in [7], in a Coulomb gauge. Similarly to the static magnetic toroidal solenoids outside which the electromagnetic field (EMF) strengths disappear, but magnetic VP differs from zero, there are electric TS outside which EMF strengths are zero, but nontrivial *electric* VP differs from zero ([8,9]). Further, there exists the toroidal Aharonov-Casher effect which describes quantum (not classical) scattering of toroidal dipoles by the electric charge ([10]). Turning to TS with time-dependent currents, one should mention two Page papers ([11]). Yet, his EMF strengths were presented in the integral form, unsuitable for practical applications. EMF of TS for a number of time dependences have been studied in [12]. Unfortunately, the most interesting case of periodical current was considered for a very special case of infinitely small TS. The multipole expansion of EMF for TS with periodical current has been given in ([13, 14]). However, that presentation was too schematic, without practical applications. The EMF of infinitely small TS with periodical current has earlier been obtained by Nevessky ([15]). His results were generalized to arbitrary time dependences in [16]. In the same reference, as well as in [9], the nontrivial charge-current toroidal configurations were found outside which nontrivial time-dependent electromagnetic potentials were different from zero despite the vanishing of EMF strengths. This makes possible the performance of experiments investigating the time-dependent Aharonov-Bohm effect. All these studies are summarized in [17].

The reciprocity theorem has a long history in physics. It originates from the third Newtonian law stating equality of action and reaction. Later, Rayleigh, in the 1-st volume of his encyclopedic treatise "Theory of Sound" ([18]) proved certain relations between the forces acting between two physical systems and the displacements induced by them. Since there is no time retardation in the Newtonian mechanics, this statement looks almost trivial. Further, Rayleigh applied reciprocity theorem to optics ([19]). We quote him: "Suppose that in any direction ($i$) and at any distance $r$ from a small surface ($S$) reflecting in any manner there be situated a radiant point ($A$) of given intensity, and consider the intensity of reflected vibrations at any point $B$ situated in direction $\epsilon$ and at distance $r'$ from $S$. The theorem is to the effect that the intensity is the same as it would be at $A$ if the radiant point were transferred to $B$". He gave no proof of this statement referring to the analogy with mechanical systems treated in the "Theory of Sound" and to the optical Lambert law. Helmholtz [20] and Lorentz [21] formulated the electric part of reciprocity theorem in its modern form. This theorem has numerous applications in the electric circuits theory [22], optics [23,24], electron diffraction [25] and in the radiophysics science (see, e.g.[26, 27]). The magnetic part of the reciprocity theorem was obtained by Feld [28] and Tai [29] in the same 1992 year. It was rederived by Monzon [30] in 1996 who, without knowing the above papers, pointed out numerous applications of this theorem . Other applications of the Feld-Tai lemma were given by Lakhtakia in his lucid book [31].

The aim of this consideration is to use EMFs of simplest sources for the studying the reciprocity-like theorems. The plan of our exposition is as follows. In section 2, we present



the formalism of elementary vector potentials (EVP). Although they are exposed in many text-books and treatises (see, e.g.,[32-34]), the lack of co-ordination between them is so large that we prefer to give self-consistent one-page exposition. In section 3, we apply EVP to the pure current time-dependent sources of EMF. Special attention is paid to the current loop and TS as well as to their interaction with external EMF. Various limiting cases of TS with periodical current are investigated. The EMF of the time-dependent electric dipole and its interaction with external EMF are studied in section 4. More complicated point-like EMF sources are treated in section 5. In section 6, by applying the Lorentz and Feld-Tai lemmas to the charge-current sources studied in previous sections, we find that these lemmas are fulfilled under more general conditions than it was known up to now. This obliges us to consider the derivation of the Lorentz and Feld-Tai lemmas more carefully. This is done in section 7 where it is shown that the reciprocity-like theorems are satisfied in the same cases when the equality of action and reaction is fulfilled. New reciprocity-like theorems are obtained in the same section, yet, their physical meaning remains unclear to us. Short resume of the results obtained is given in section 8.

# 2 Elementary vector potentials

Consider charge $\rho(\vec{r},t)$ and current $\vec{j}(\vec{r},t)$ densities confined to the finite volume $V$. Let their time dependence be periodical:

$$\rho = \rho_0 \exp(i\omega t), \quad \vec{j} = \vec{j}_0 \exp(i\omega t). \tag{2.1}$$

When presenting $\rho$ and $\vec{j}$ in such a complex form, one should keep in mind the static limit of the treated problem. For example, if one operates with pure current densities and wants to have in a static limit the time-independent current, then one puts

$$\vec{j} = \vec{j}_0 \exp(i\omega t), \quad \rho = 0$$

and, after all calculations, takes the real parts of EMF strengths (see section 3, where the EMF of a current loop and TS are considered). On the other hand, if one desires to obtain in a static limit the time-indepedent charge distribution, then one puts

$$\vec{j} = \omega\vec{j}_0 \exp(i\omega t), \quad \rho = i\rho_0 \exp(i\omega t), \quad \rho_0 = div\vec{j}_0$$

and, after all calculations, takes the imaginary parts of EMF strengths (see section 4, where the EMF of oscillating electric dipole is treated).
The electromagnetic potentials outside the space region $V$, to which the charge-current densities are confined, are given by

$$\Phi(\vec{r},t) = -4\pi ik \sum h_l(kr) Y_{lm}(\theta,\phi) q_{lm}, \quad \vec{A}(\vec{r},t) = -\frac{4\pi ik}{c} \sum \vec{A}_{lm}(\tau,\vec{r}) a_{lm}(\tau), \tag{2.2}$$

where $h_l(kr) \equiv h_l^{(2)}(kr) = j_l(kr) - in_l(kr)$ is the spherical Hankel function of the second kind, $j_l$ and $n_l$ are the spherical Bessel and Neumann functions ($j_l = J_{l+1/2}\sqrt{\pi/2x}$, $n_l = N_{l+1/2}\sqrt{\pi/2x}$); $Y_{lm}(\theta,\phi)$ are the usual spherical harmonics; $\vec{A}_{lm}(\tau,\vec{r})$ are the elementary vector potentials (EVP). Values $\tau = E, L$ and $M$ correspond to the electric, longitudinal and magnetic EVP, resp. Their manifest form is given by

$$\vec{A}_{lm}(L) = \frac{1}{k}\vec{\nabla} h_l Y_{lm}, \quad \vec{A}_{lm}(E) = -\frac{1}{k\sqrt{l(l+1)}} curl(\vec{r}\times\vec{\nabla}) h_l Y_{lm},$$

$$\vec{A}_{lm}(M) = -\frac{i}{\sqrt{l(l+1)}} h_l (\vec{r}\times\vec{\nabla}) Y_{lm}. \tag{2.3}$$

If not indicated, the arguments of the spherical Bessel functions ($j_l, n_l$) will be $kr$, and $\cos\theta$ will be the argument of the adjoint Legendre polynomials ($P_l^m$). In what follows, we closely follow Rose treatise [32] with the exception that instead of his non-standard radial functions, the usual spherical Bessel functions are used. EVP satisfy the following equations:

$$curl\vec{A}_{lm}(M) = ik\vec{A}_{lm}(E) \quad curl\vec{A}_{lm}(E) = -ik\vec{A}_{lm}(M).$$

It is useful to write out the spherical components of EVP in a manifest form

$$[\vec{A}_l^m(E)]_\theta = \frac{1}{\sqrt{l(l+1}}\frac{(l+1)h_{l-1} - lh_{l+1}}{2l+1}\frac{d}{d\theta}Y_{lm},$$

$$[\vec{A}_l^m(E)]_\phi = \frac{m}{\sin\theta\sqrt{l(l+1}}\frac{(l+1)h_{l-1} - lh_{l+1}}{2l+1}Y_{lm}, \quad [\vec{A}_l^m(E)]_r = \sqrt{l(l+1}\frac{1}{kr}h_l Y_{lm},$$

$$[\vec{A}_l^m(M)]_\theta = \frac{im}{\sin\theta\sqrt{l(l+1}}h_l Y_{lm}, \quad [\vec{A}_l^m(M)]_\phi = -\frac{ih_l}{\sqrt{l(l+1}}\frac{\partial Y_{lm}}{\partial\theta}, \quad [\vec{A}_l^m(M)]_r = 0. \tag{2.4}$$

The multipole coefficients (or formfactors) $a_{lm}(\tau)$ entering into (2.2) are defined as

$$q_{lm} = \int j_l Y_{lm}^* \rho dV, \quad a_{lm}(L) = -\frac{1}{k}\int j_l Y_{lm}^* div\vec{j} dV = \frac{1}{k}\int j_l Y_{lm}^*\dot{\rho} = icq_{lm},$$

$$a_{lm}(E) = -\frac{1}{k\sqrt{l(l+1)}}\int curl(\vec{r}\times\vec{\nabla}) j_l Y_{lm}^*\vec{j} dV =$$

$$= \frac{k}{\sqrt{l(l+1)}}\int j_l Y_{lm}^*(\vec{r}\vec{j})dV + \frac{1}{k\sqrt{l(l+1)}}\int[(l+1)j_l - krj_{l+1}]Y_{lm}^*\dot{\rho}dV,$$

$$a_{lm}(M) = -\frac{i}{\sqrt{l(l+1)}}\int j_l Y_{lm}^*(\vec{r} curl\vec{j})dV = \frac{i}{\sqrt{l(l+1)}}\int j_l Y_{lm}^* div(\vec{r}\times\vec{j})dV. \tag{2.5}$$

To escape ambiguities, under $\dot{\rho}$ we mean $-div\vec{j}$. The EMF strengths are given by

$$\vec{H} = \frac{4\pi k^2}{c}\sum[\vec{A}_{lm}(E)a_{lm}(M) - \vec{A}_{lm}(M)a_{lm}(E)],$$

$$\vec{E} = -\frac{4\pi k^2}{c}\sum[\vec{A}_{lm}(E)a_{lm}(E) + \vec{A}_{lm}(M)a_{lm}(M)]. \tag{2.6}$$



For the axial-symmetrical charge-current distributions, only $m = 0$ components survive

$$q_{lm} = \delta_{mo}q_l, \quad a_{lm}(E) = \delta_{mo}a_l(E), \quad a_{lm}(M) = \delta_{mo}a_l(M),$$

$$[\vec{A}_l(E)]_\theta \equiv [\vec{A}_l^0(E)]_\theta = \frac{(l+1)h_{l-1} - lh_{l+1}}{[l(l+1)(2l+1)4\pi]^{1/2}}P_l^1,$$

$$[\vec{A}_l(E)]_r \equiv [\vec{A}_l^0(E)]_r = \frac{1}{kr}[\frac{l(l+1)(2l+1)}{4\pi}]^{1/2}h_l P_l,$$

$$[\vec{A}_l(M)]_\phi \equiv [\vec{A}_l^0(M)]_\phi = -i[\frac{2l+1}{4\pi l(l+1)}]^{1/2}h_l P_l^1. \tag{2.7}$$

## 3 Pure current densities

When only current densities are present ($\rho = 0$), then $q_{lm} = 0$, $A_{lm}(L) = 0$, and

$$a_{lm}(E) = \frac{k}{\sqrt{l(l+1)}}\int j_l Y_{lm}^*(\vec{r}\vec{j})dV, \tag{3.1}$$

while $a_{lm}(M)$ has the same form (2.5). Taking into account that

$$j_l(x) \sim (x)^l/(2l+1)!!, \quad n_l(x) \sim -(2l-1)!!/(x)^{l+1} \quad \text{for} \quad x \to 0,$$

one gets in the static limit ($k \to 0$)

$$a_{lm}(E) \to \frac{k^{l+1}}{\sqrt{l(l+1)}}\frac{1}{(2l+1)!!}\int r^l Y_{lm}^*(\vec{r}\vec{j})dV,$$

$$a_{lm}(M) \to \frac{i}{\sqrt{l(l+1)}}\frac{k^l}{(2l+1)!!}\int r^l Y_{lm}^* div(\vec{r}\times\vec{j})dV. \tag{3.2}$$

The integrals entering into these equations are usually called electric and magnetic moments, resp. On the other hand, the toroidal moment corresponding to the current density $\vec{j}$ was defined in [4] as

$$T_{lm} = -\frac{\sqrt{\pi}l}{c(2l+1)}\int r^{l+1}[\vec{Y}_{l,-1,m}^* + \sqrt{\frac{l}{l+1}}\frac{1}{l+3/2}\vec{Y}_{l,l+1,m}^*]\vec{j}dV, \tag{3.3}$$

where $\vec{Y}_{j,l,m}^*$ are the so-called vector spherical harmonics (see, e.g., [32] for their definition). In view of the identities

$$\int r^{l+1}[\vec{Y}_{l,l-1,m}^* + \sqrt{\frac{l}{l+1}}\frac{1}{l+3/2}\vec{Y}_{l,l+1,m}^*]\vec{j}dV =$$

$$= -\sqrt{\frac{2l+1}{l}}\frac{1}{(l+1)(2l+3)}\int curl(\vec{r}\times\vec{\nabla})r^{l+2}Y_{lm}^*\vec{j}dV =$$

$$= -\frac{1}{(l+1)(2l+3)}\sqrt{\frac{2l+1}{l}}[(l+3)\int r^{l+2}Y_{lm}^* div\vec{j}dV + 2(2l+3)\int r^l Y_{lm}^*(\vec{r}\vec{j})dV \tag{3.4}$$

established in [14], one gets for the pure current densities

$$T_{lm} = \frac{2\sqrt{\pi}}{c(l+1)}\sqrt{\frac{l}{2l+1}}\int r^l Y^*_{lm}(\vec{r}\vec{j})dV. \tag{3.5}$$

Therefore, a toroidal moment $T_{lm}$, in the absence of charge density ($\rho = 0$), up to a factor independent of geometric parameters of current distribution, coincides with the electric moment (2.5) of this distribution..

## 3.1 Electromagnetic field of a current loop

Let the current loop lie in the $z = 0$ plane with its symmetry axis along the $z$ axis. Then, its current density is given by

$$\vec{j} = I_0\vec{n}_\phi\delta(\rho - d)\delta(z). \tag{3.6}$$

Since $\vec{r}\vec{j} = 0$, only the magnetic form factors differ from zero

$$a_l^m(M) = \delta_{m0}a_l(M), \quad a_l(M) = iI_0d[\frac{\pi(2l+1)}{l(l+1)}]^{1/2}j_l(kd)P_l^1(0). \tag{3.7}$$

Here $P_l^m(x)$ is the adjoint Legendre function. Since $P_l^1(0) = 0$ for $l$ even, only odd multipole coefficients contribute to EMF of the current loop ($P^1_{2n+1}(0) = (-1)^{n+1}(2n+1)!!/2^n n!$). Therefore, for the current loop

$$\vec{H} = \frac{4\pi k^2}{c}\sum\vec{A}_l(E)a_l(M), \quad \vec{E} = -\frac{4\pi k^2}{c}\sum\vec{A}_l(M)a_l(M). \tag{3.8}$$

From the facts that: (i) $\vec{r}\vec{E} = 0$ and (ii) $P\vec{A}_l(E) = (-1)^{l+1}\vec{A}_l(E)$ it follows [32] that the radiation field of the current loop is of the magnetic type ($P$ is the parity operator). When the time dependence of $\rho$ and $\vec{j}$ is $\cos\omega t$, the nonvanishing EMF strengths are given by

$$E_\phi = -\frac{2\pi I_0dk^2}{c}\sum_{l=odd}\frac{2l+1}{l(l+1)}(\cos\omega t j_l + \sin\omega t n_l)P_l^1 j_l(kd)P_l^1(0),$$

$$H_\theta = \frac{2\pi I_0dk^2}{c}\sum_{l=odd}\frac{1}{l(l+1)}P_l^1 j_l(kd)P_l^1(0)\times$$

$$\times\{\cos\omega t[(l+1)n_{l-1} - ln_{l+1}] - \sin\omega t[(l+1)j_{l-1} - lj_{l+1}]\},$$

$$H_r = \frac{2\pi I_0kd}{cr}\sum_{l=odd}(2l+1)(n_l\cos\omega t - j_l\sin\omega t)P_l j_l(kd)P_l^1(0). \tag{3.9}$$

Consider particular cases.
1. In the static case ($k \to 0$), one gets

$$j_l(kd) \sim (kd)^l/(2l+1)!!, \quad n_l(kr) \sim -(2l-1)!!/(kr)^{l+1},$$

$$E_\phi = 0, \quad H_\theta = \frac{2\pi I_0d}{cr^2}\sum\frac{1}{l+1}\frac{d^l}{r^l}P_l^1P_l^1(0), \quad H_r = -\frac{2\pi I_0d}{cr^2}\sum\frac{d^l}{r^l}P_lP_l^1(0). \tag{3.10}$$



The term $l = 1$ of this sum

$$H_\theta = \frac{\pi I_0d^2}{cr^3}\sin\theta, \quad H_r = \frac{2\pi I_0d^2}{cr^3}\cos\theta,$$

corresponds to the field of magnetic dipole of the power $m = \pi I_0d^2/c$ oriented along the $z$ axis.
2. When the radius $d$ of the loop is so small that $kd << 1$, only the $l = 1$ term contributes to (3.9). Then, EMF strengths are equal to

$$E_\phi = \frac{\pi I_0d^2k^2}{cr}\sin\theta(\cos\psi - \frac{1}{kr}\sin\psi), \quad H_r = \frac{2\pi I_0d^2k\cos\theta}{cr^2}(\sin\psi + \frac{1}{kr}\cos\psi),$$

$$H_\theta = -\frac{\pi I_0d^2k^2\sin\theta}{cr}[(1 - \frac{1}{k^2r^2})\cos\psi - \frac{1}{kr}\sin\psi)]. \tag{3.11}$$

Here $\psi = kr - \omega t$. These expressions are valid at arbitrary distances from the current loop.
3. For large distances ($kr >> 1$), spherical Bessel functions can be changed by their asymptotic values

$$j_l(kr) \approx \frac{1}{kr}\cos(kr - \frac{l+1}{2}\pi), \quad n_l(kr) \approx \frac{1}{kr}\sin(kr - \frac{l+1}{2}\pi).$$

Then,

$$E_\phi = -H_\theta = \frac{\pi I_0dk}{c}\frac{\cos\psi}{r}\sum_{n=0}^{\infty}(-1)^n\frac{4n+3}{(n+1)(2n+1)}P^1_{2n+1}j_{2n+1}(kd)P^1_{2n+1}(0),$$

$$H_r = -\frac{2\pi I_0d}{cr^2}\sin\psi\sum(-1)^n(4n+3)P_{2n+1}j_{2n+1}(kd)P^1_{2n+1}(0). \tag{3.12}$$

The energy flux through the sphere of the radius $r$ is

$$S_r = \frac{c}{4\pi}\int d\Omega E_\phi H_\theta = \frac{2}{c}(I_0kd\cos\psi)^2\sum\frac{4n+3}{(n+1)(2n+1)}[j_{2n+1}(kd)P^1_{2n+1}(0)]^2.$$

The energy lost for the period

$$S_r = \frac{1}{c}(I_0kd)^2\sum\frac{4n+3}{(n+1)(2n+1)}[j_{2n+1}(kd)P^1_{2n+1}(0)]^2.$$

These expressions are valid for arbitrary $kd$.

### 3.1.1 Interaction of current loop with external electromagnetic field

The interaction of current (3.6) with the external EMF is given by

$$U = -\frac{1}{c}\int\vec{j}_L\vec{A}_{ext}dV. \tag{3.13}$$

Since $div\vec{j}_L = 0$, the current density can be represented as

$$\vec{J}_L = curl\vec{M}_L, \quad \vec{M}_L = I_L\vec{n}_z\Theta(d-\rho)\delta(z). \tag{3.14}$$

Substituting this into (3.13) and integrating by parts, one gets

$$U = -\frac{1}{c}\int \vec{M}_L \vec{H}_{ext} dV.$$

For the distances large compared with the loop radius $d$

$$U = -\frac{1}{c}\vec{H}_{ext}\int \vec{M}_L dV = -\vec{\mu}\vec{H}_{ext},$$

where

$$\vec{\mu} = \frac{1}{c}\int \vec{M} dV = \frac{1}{2c}\int \vec{r}\times\vec{j} dV = \frac{I_L \pi d^2}{c}\vec{n}_z$$

coincides with the usual magnetic moment. These equations illustrate Ampere's hypothesis according to which the current loop is equivalent to the magnetic moment normal to it. When the radius $d$ of the loop tends to zero,

$$\vec{M}_L \to I_L \pi d^2 \vec{n}\delta^3(\vec{r}), \quad \vec{J}_L = curl \vec{M}_L, \quad \delta^3(\vec{r}) = \delta(\rho)\delta(z)/2\pi\rho. \qquad (3.15)$$

Let now the dependence of this current flowing in the loop be $f_L(t)$, i.e.,

$$\vec{J}_L = f_L(t) curl \vec{n}\delta^3(\vec{r}) \qquad (3.16)$$

(the factor $\pi I_L d_L^2$ is absorbed into $f_L(t)$). Then, the EMF potentials and field strengths are given by

$$\vec{A}_L = -\frac{1}{c^2 r^2} D_L(\vec{r}\times\vec{n}_L), \quad \vec{E}_L = \frac{1}{c^3 r^2}\dot{D}_L(\vec{r}\times\vec{n}_L), \quad \vec{H}_L = \frac{1}{c^3 r}[\frac{(\vec{r}\vec{n}_L)}{r^2}\vec{r}F_L - \vec{n}_L G_L], \qquad (3.17)$$

where we put

$$D_L = D(f_L) = \dot{f}_l + \frac{c}{r} f_L, \quad F_L = F(f_L) = \ddot{f}_l + \frac{3c}{r}\dot{f}_L + \frac{3c^2}{r^2} f_L,$$

$$G_L = G(f_L) = \ddot{f}_l + \frac{c}{r}\dot{f}_L + \frac{c^2}{r^2} f_L. \qquad (3.18)$$

The arguments of $f_L$ functions entering into $D_L$, $F_L$ and $G_L$ are $t_r = t - r/c$; dots above the $f_L$, $D_L$, $F_L$ and $G_L$ functions mean time derivatives. When $f_L$ does not depend on time, one obtains the field of elementary magnetic dipole

$$\vec{H}_L = \frac{p}{r^3}[3\vec{r}\frac{(\vec{r}\vec{n}_L)}{r^2} - \vec{n}_L]$$

of the power $p = f_L/c$. Obviously, Eqs.(3.15)-(3.18) generalize (3.11) to arbitrary time-dependences and orientations.



## 3.2 Electromagnetic field of the toroidal solenoid

Consider the poloidal current flowing on the torus surface (Fig.1 )

$$\vec{j}_0 = -\frac{gc}{4\pi}\vec{n}_\psi \frac{\delta(R-\tilde{R})}{d+\tilde{R}\cos\psi}, \quad \vec{n}_\psi = \vec{n}_z\cos\psi - \vec{n}_\rho\sin\psi. \qquad (3.19)$$

The coordinates $\tilde{R}$, $\psi$ and $\phi$ are related to the Cartesian ones as follows:

$$x = (d+\tilde{R}\cos\psi)\cos\phi, \quad y = (d+\tilde{R}\cos\psi)\sin\phi, \quad z = \tilde{R}\sin\psi. \qquad (3.20)$$

The condition $\tilde{R} = R$ defines the surface of a particular torus (Fig.2). For $\tilde{R}$ fixed and $\psi, \phi$ varying, the points $x, y, z$ given by (3.20) fill the surface of torus $(\rho - d)^2 + z^2 = R^2$. The choice $\vec{j}_0$ in the form (3.19) is convenient, because in the static case a magnetic field $H$ equals $g/\rho$ inside the torus and vanishes outside it. In this case, $g$ may also be expressed through either the magnetic flux $\Phi$ penetrating the torus or the total number $N$ of turns in toroidal winding and the current $I$ in a particular turn;

$$g = \frac{\Phi}{2\pi(d - \sqrt{d^2 - R^2})} = \frac{2NI}{c}.$$

Let the current in TS winding periodically changes with time: $\vec{j} = \vec{j}_0\exp(i\omega t)$. Since

$$\vec{r}\times\vec{j}_0 = \frac{gc}{4\pi}\delta(\tilde{R}-R)\vec{n}_\phi, \quad \text{and} \quad \vec{r}\vec{j}_0 = \frac{gcd\sin\psi}{4\pi}\frac{\delta(\tilde{R}-R)}{d+R\cos\psi},$$

one has

$$div(\vec{r}\times\vec{j}) = 0, \quad a_{lm}(M) = 0, \quad a_{lm}(E) \neq 0.$$

Therefore,

$$\vec{A} = -\frac{4\pi ik}{c}\sum \vec{A}_l(E)a_l(E), \quad \vec{H} = -\frac{4\pi k^2}{c}\sum \vec{A}_l(M)a_l(E), \quad \vec{E} = -\frac{4\pi k^2}{c}\sum \vec{A}_l(E)a_l(E) \qquad (3.21)$$

($\vec{A}$ is the vector-potential). From the facts that: (i) $\vec{r}\vec{H} = 0$ and (ii) $P\vec{A}_l(M) = (-1)^l\vec{A}_l(M)$, it follows [32,34] that the radiation field of TS is of electric type.
The electric form factor $a_l(E)$ for the radiating TS is equal to

$$a_l(E) = \frac{1}{4}gcdRk\sqrt{\frac{2l+1}{\pi l(l+1)}}I_l, \quad I_l = \int_0^{2\pi} j_l(ky)P_l(x)\sin\psi d\psi, \qquad (3.22)$$

where $y = [d^2 + R^2 + 2dR\cos\psi]^{1/2}$ and $x = R\sin\psi/y$. It easy to check that $a_l(E) = 0$ for $l$ even. Let the current time dependence be $\cos\omega t$. Then, EMF is given by the real parts of $\vec{A}$, $\vec{E}$, $\vec{H}$:

$$A_\theta = \frac{gdRk^2}{2}\sum\frac{1}{l(l+1)}I_l P_l^1\{[(l+1)j_{l-1} - lj_{l+1}]\sin\omega t - [(l+1)n_{l-1} - ln_{l+1}]\cos\omega t\},$$

$$A_r = \frac{gdRk}{2r}\sum(2l+1)I_l P_l(j_l\sin\omega t - n_l\cos\omega t),$$

$$H_\phi = \frac{gdRk^3}{2}\sum \frac{2l+1}{l(l+1)} I_l P_l^1 (n_l \cos\omega t - j_l \sin\omega t),$$

$$E_\theta = -\frac{gdRk^3}{2}\sum \frac{1}{l(l+1)} I_l P_l^1 \{[(l+1)j_{l-1} - l j_{l+1}]\cos\omega t + [(l+1)n_{l-1} - l n_{l+1}]\sin\omega t\},$$

$$E_r = -\frac{gdRk^2}{2r}\sum (2l+1) I_l P_l (j_l \cos\omega t + n_l \sin\omega t). \quad (3.23)$$

Consider particular cases.

1. In the static limit ($k \to 0$) one gets

$$I_l \to \frac{k^l}{(2l+1)!!} C_l, \quad a_l(E) \to \frac{gcdRk^{l+1}}{4(2l+1)!!}\sqrt{\frac{2l+1}{\pi l(l+1)}} C_l,$$

$$T_{lm} = \delta_{m0} T_l, \quad T_l = \frac{gdR\sqrt{l}}{2(l+1)} C_l, \quad C_l = \int_0^{2\pi} y^l P_l\left(\frac{R\sin\psi}{y}\right)\sin\psi d\psi, \quad (3.24)$$

where $T_{lm}$ is the same as in (3.5). This integral can be taken in a closed form. We give its value only for $l = 1$

$$C_1 = \pi R, \quad a_1(E) = \frac{\pi g d R^2 k^2 c}{4\sqrt{6\pi}}.$$

EMF strengths of TS decrease like $k^2$

$$H_\phi \sim -gdRk^2 \sum \frac{1}{l(l+1)}\frac{1}{r^{l+1}} C_l P_l^1, \quad E_\theta \sim -\frac{gdRk^2}{2} ct \sum \frac{C_l}{l+1}\frac{1}{r^{l+2}} P_l^1,$$

$$E_r \sim \frac{gdRk^2}{2} ct \sum C_l P_l \frac{1}{r^{l+2}}. \quad (3.25)$$

On the other hand, the vector potential of TS does not vanish in the static limit

$$A_\theta \to -\frac{gdR}{2}\sum \frac{C_l}{l+1} P_l^1 \frac{1}{r^{l+2}}, \quad A_r \to \frac{gdR}{2}\sum \frac{1}{r^{l+2}} C_l P_l. \quad (3.26)$$

The linear time dependence in $\vec{E}$ (for $\omega t << 1$) arises when one differentiates the $\cos\omega t$ term in $\vec{A}$, and then let $\omega$ go to zero. For the infinitely thin TS ($R << d$), $C_l$ is reduced to

$$C_{2n+1} = \pi R d^{2n} (-1)^n \frac{(2n+1)!!}{2^n n!}.$$

2. Infinitely small toroidal solenoid ($kd << 1$).
Obviously, only the $l = 1$ term contributes to sums in (3.23)

$$I_1 = \frac{\pi k R}{3}, \quad a_1(E) = \frac{\pi g d R^2 k^2 c}{4\sqrt{6\pi}}, \quad E_r = \frac{\pi g d R^2 k^2}{2r^2}\cos\theta[\cos\psi - \frac{1}{kr}\sin\psi],$$

$$E_\theta = \frac{\pi g d R^2 k^3}{4r}\sin\theta[\sin\psi(1 - \frac{1}{k^2 r^2}) + \frac{1}{kr}\cos\psi], \quad H_\phi = \frac{\pi g d R^2 k^3}{4r}\sin\theta[\sin\psi + \frac{1}{kr}\cos\psi]. \quad (3.27)$$



For estimations, let the major radius $d$ of TS be $10cm$. We rewrite the condition $kd << 1$ in the wavelength language

$$\frac{2\pi d}{\lambda} \approx \frac{60}{\lambda} << 1.$$

This means that Eqs, (3.27) will work for $\lambda \geq 5m$.

3. Infinitely thin toroidal solenoid ($R << d$).
Taking into account that

$$P_{2n+1}(x) \to -P^1_{2n+1}(0)x \quad \text{for} \quad x \to 0,$$

one gets

$$I_{2n+1} = -\frac{R}{d} P^1_{2n+1}(0) D_{2n+1}, \quad D_{2n+1} = \int_0^{2\pi} j_{2n+1}(ky)\sin^2\psi d\psi,$$

$$a_{2n+1}(E) = -\frac{1}{4} g c R^2 k \sqrt{\frac{4n+3}{\pi(2n+1)2(n+1)}} P^1_{2n+1}(0) D_{2n+1}. \quad (3.28)$$

For $R << d$ (but for arbitrary $kd$ and $kR$) $D_{2n+1}$ can be taken in a closed form (see Appendix):

$$D_{2n+1} = \pi\{J_0(kR) j_{2n+1}(kd) - \frac{1}{2} J_2(kR)[j_{2n+3}(kd) + j_{2n-1}(kd)]\}. \quad (3.29)$$

If, in addition, $kR << 1$, then

$$D_{2n+1} = \pi j_{2n+1}(kd)$$

and

$$a_{2n+1}(E) = -\frac{\pi}{4} g c R^2 k \sqrt{\frac{4n+3}{\pi(2n+1)2(n+1)}} P^1_{2n+1}(0) j_{2n+1}(kd). \quad (3.30)$$

On the other hand, if $kR >> 1$, then

$$D_{2n+1} = \frac{2}{kd}\sqrt{\frac{2\pi}{kR}}\cos(kR - \frac{\pi}{4})[(n+1) j_{2n+2}(kd) + n j_{2n}(kd)]. \quad (3.31)$$

For $kd >> 1$, Eqs. (3.27) are not applicable. For example, for $d = 10cm$ and $\lambda = 1cm$, $kd \approx 60$. The possible outcome is to take the minor radius of $TS$ as small as possible. Equations (3.23) with $a_l(E)$ given by (3.28) and (3.29) are valid for arbitrary frequencies if $R \leq 2cm$ (for $d = 10cm$). The advantage of electric formfactors (3.28) and (3.29) is that they do not involve integration that is very cumbersome for high frequencies.
To estimate the number of $a_l(E)$ contributing to sums in (3.23), we need the asymptotic behavior of $J_\nu(x)$ for $x$ fixed and $\nu >> 1$. It is given by (see [35], Chapter 8)

$$J_\nu(x) \sim \frac{1}{\sqrt{2\pi\nu}}\left(\frac{xe}{2\nu}\right)^\nu. \quad (3.32)$$

For $x = kd$ the same as above ($kd \approx 60$), $J_\nu(kd) \approx 10^{-10}$ for $\nu = 100$, that corresponds to $n \approx 50$. It follows from (3.32) that the number of terms contributing to (3.23) with $a_l(E)$ given by (3.28) and (3.29) should be slightly greater than $0.7kd$.

4. Large distances ($kr >> 1$).
Then,

$$E_\theta = H_\phi = -\frac{gdRk^2}{4r}\sin\psi \sum(-1)^n \frac{4n+3}{(2n+1)(n+1)} I_{2n+1} P^1_{2n+1},$$

$$E_r = \frac{gdRk}{2r^2}\cos\psi \sum(4n+3)(-1)^n I_{2n+1} P_{2n+1}. \qquad (3.33)$$

The energy flux through the sphere of the radius $r$ is

$$S_r = \frac{c}{4\pi} r^2 \int d\Omega E_\theta H_\phi = c(\frac{gdRk^2 \sin\psi}{2})^2 \sum \frac{4n+3}{2(n+1)(2n+1)} I^2_{2n+1}.$$

Correspondingly, the energy lost for the period is

$$S_r = \frac{c}{2}(\frac{gdRk^2}{2})^2 \sum \frac{4n+3}{2(n+1)(2n+1)} I^2_{2n+1}.$$

### 3.2.1 Interaction of toroidal solenoid with external electromagnetic field

The interaction of TS with external EMF is given by

$$U = -\frac{1}{c}\int \vec{j}_T \vec{A}_{ext} dV. \qquad (3.34)$$

Since $div\vec{j}_T = 0$, the polodal current (3.19) flowing on the torus surface can be represented in the form ([9])

$$\vec{j}_T = curl\vec{M}, \quad div\vec{M} = 0, \quad \vec{M} = \vec{n}_\phi \frac{gc}{4\pi\rho}\Theta(R - \sqrt{(\rho-d)^2+z^2}), \quad div\vec{M} = 0. \qquad (3.35)$$

That is, the magnetization $\vec{M}$ has only the azimuthal component and differs from zero only inside the torus (middle part of Fig. 3). Since $div\vec{M} = 0$, the magnetization $\vec{M}$, in its turn, can be written as

$$\vec{M} = curl\vec{T}, \quad div\vec{T} \neq 0, \qquad (3.36)$$

where

$$\vec{T} = \vec{n}_z T, \quad T = \frac{gc}{4\pi}[\Theta(d - \sqrt{R^2-z^2} - \rho)\ln\frac{d+\sqrt{R^2-z^2}}{d-\sqrt{R^2-z^2}} +$$

$$+\Theta(d+\sqrt{R^2-z^2} - \rho)\Theta(\rho - d + \sqrt{R^2-z^2})\ln\frac{d+\sqrt{R^2-z^2}}{\rho}]. \qquad (3.37)$$

Thus, $T$ differs from zero in two space regions (see the lower part of Fig.3):
a) Inside the torus hole defined as $0 \le \rho \le d - \sqrt{R^2-z^2}$, where $T$ does not depend on $\rho$

$$T_z = \frac{gc}{4\pi}\ln\frac{d+\sqrt{R^2-z^2}}{d-\sqrt{R^2-z^2}}. \qquad (3.38)$$

b) Inside the torus itself ($d - \sqrt{R^2-z^2} \le \rho \le d + \sqrt{R^2-z^2}$) where

$$T = \frac{gc}{4\pi}\ln\frac{d+\sqrt{R^2-z^2}}{\rho}. \qquad (3.39)$$



In other space regions, $T = 0$. Therefore,

$$\vec{j}_T = curlcurl\vec{T}, \quad div\vec{T} \neq 0. \qquad (3.40)$$

Substituting (3.40) into (3.34), one gets

$$U = -\frac{1}{c^2}\int \dot{\vec{E}}\vec{T} dV$$

(dot above $\vec{E}$ means time derivative). For the distances large compared with TS' large radius

$$U = -\frac{1}{c^2}\dot{\vec{E}}\int \vec{T} dV \qquad (3.41)$$

Despite the fact that $T$ is rather complicated, the volume integral looks very simple

$$\int \vec{T} dV = \vec{n}_z \frac{\pi c g d R^2}{4} \qquad (3.42)$$

Physically, Eqs. (3.35), (3.36) and (3.40) mean that the poloidal current $\vec{j}$ given by Eq.(3.35) is equivalent (i.e., produces the same magnetic field) to the toroidal tube with the magnetization $\vec{M}$ defined by (3.36) and to the toroidization $\vec{T}$ given by (3.37). This is illustrated in Fig. 3. Obviously, these equations generalize Ampere's hypothesis.
Now let the minor radius $R$ of a torus tend to zero (this corresponds to an infinitely thin torus). Then, the second term in (3.37) drops out, while the first one reduces to

$$T \to \frac{gc}{2\pi d}\Theta(d-\rho)\sqrt{R^2-z^2}. \qquad (3.43)$$

For infinitesimal $R$

$$\sqrt{R^2-z^2} \to \frac{1}{2}\pi R^2 \delta(z).$$

Therefore, in this limit,

$$\vec{j} = curlcurl\vec{T}, \quad \vec{T} = \vec{n}_z \frac{gcR^2}{4d}\delta(z)\Theta(d-\rho). \qquad (3.44)$$

i.e., the vector $\vec{T}$ is confined to the equatorial plane of a torus and is perpendicular to it.
Let now $d \to 0$ (in addition to $R \to 0$). Then,

$$\frac{1}{d}\Theta(d-\rho) \to \frac{d}{2\rho}\delta(\rho)$$

and the current of an elementary ( i.e., infinitely small) TS is

$$\vec{j} = curlcurl\vec{T}, \quad \vec{T} = \frac{1}{4}\pi c g d R^2 \delta^3(\vec{r})\vec{n}_z. \qquad (3.45)$$

Let now the dependence of the current flowing in the toroidal solenoid be $f_T(t)$, i.e.,

$$\vec{J}_T = f_T(t) curl \vec{n}_T \delta^3(\vec{r}). \qquad (3.46)$$

(the factor $\frac{1}{4}\pi c g d_T R^2$ is included into $f_T(t)$). Then, EMF potentials and field strengths are given by

$$\vec{A}_T = \frac{1}{c^3 r}[-\vec{n}_T G_T + \frac{1}{r^2}\vec{r}(\vec{r}\vec{n}_T)F_T], \quad \vec{E}_T = \frac{1}{c^4}r[\vec{n}_T \dot{G}_T - \frac{1}{r^2}\vec{r}(\vec{r}\vec{n}_T)\dot{F}_T],$$

$$\vec{H}_T = \frac{1}{4c^3 r}(\vec{r}\times\vec{n}_T)\ddot{D}_T, \tag{3.47}$$

where $D_T = D(f_T)$, $F_T = F(f_T)$, $G_T = G(f_T)$. When $f_T$ is independent of $t$, only the vector potential survives

$$\vec{A}_T = -\frac{1}{4cr^3}f_T[\vec{n}_T - \frac{3}{r^2}\vec{r}(\vec{r}\vec{n}_T)].$$

Clearly, Eqs. (3.47) generalize (3.27) for arbitrary time dependences and orientations.

# 4 Electromagnetic field of electric dipole

Consider two point charges at the points $\pm a_d\vec{n}$. Its charge density is given by

$$\rho_d = e[\delta^3(\vec{r} - a_d\vec{n}) - \delta^3(\vec{r} + a_d\vec{n})].$$

For an infinitely small dipole, this takes the form

$$\rho_d = -2ea(\vec{n}\vec{\nabla})\delta^3(\vec{r}), \quad \vec{\nabla}_i = \frac{\partial}{\partial x_i}.$$

Now let the charge density depend on time

$$\rho_d = f(t)(\vec{n}\vec{\nabla})\delta^3(\vec{r})$$

(factor $-2ea$ is included in $f(t)$). The corresponding current density is given by

$$\vec{j}_d = -\dot{f}(t)\vec{n}\delta^3(\vec{r}). \tag{4.1}$$

The following EMF strengths correspond to these densities:

$$\vec{H}_d = \frac{1}{c^2 r^2}(\vec{r}\times\vec{n})\dot{D}_d, \quad \vec{E}_d = \frac{1}{c^2 r}[\vec{n}G_d - \frac{1}{r^2}(\vec{n}\vec{r})\vec{r}F_d]. \tag{4.2}$$

Now let the time dependence of charge density be $\cos\omega t$:

$$\rho_d = -2ea_d\cos\omega t(\vec{n}\vec{\nabla})\delta^3(\vec{r}), \quad \vec{j}_d = -2ea_d\omega\sin\omega t\vec{n}\delta^3(\vec{r}). \tag{4.3}$$

For the unit vector $\vec{n}$ along the $z$ axis, one gets

$$H_{d\phi} = -\frac{2ea_d k^2}{r}\sin\theta(\cos\psi - \frac{\sin\psi}{kr}), \quad E_{d\theta} = -\frac{2ea_d k^2}{r}\sin\theta[\cos\psi(1 - \frac{1}{k^2 r^2}) - \frac{\sin\psi}{kr}],$$

$$E_d^r = \frac{4ea_d k}{r^2}\cos\theta(\sin\psi + \frac{1}{kr}\cos\psi), \quad \psi = kr - \omega t. \tag{4.4}$$



In the static limit ($k \to 0$) one gets the field of electric dipole

$$E_{d\theta} \to \frac{2a_d e}{r^3}\sin\theta, \quad E_{dr} \to \frac{4ea_d}{r^3}\cos\theta, \quad H_{d\phi} \to 0.$$

For the oscillating electric dipole with a finite $a_d$, oriented along the $z$ axis

$$\rho_d = i\exp(i\omega t)\rho_{d0}, \quad \rho_{d0} = \frac{e}{2\pi a_d^2\sin\theta}\delta(r - a_d)[\delta(\theta) - \delta(\pi - \theta)],$$

$$\vec{j}_d = \vec{n}_r j_d, \quad j_d = -\omega\exp(i\omega t)j_{d0}, \quad j_{d0} = \frac{e}{2\pi r^2\sin\theta}\Theta(a_d - r)[\delta(\theta) - \delta(\pi - \theta)] \tag{4.5}$$

If we desire to obtain, in the static limit, the static electric density $\rho_{d0}$, we should take,at the end of all calculations, the imaginary parts of the $EMF$ strengths (since $\rho_d$ in (4.5) contains the imaginary unit factor $i$). It turns out that $a_l^m(M) = 0$, i.e., only the electric formfactors with $l$ odd contribute to the EMF strengths

$$a_l^m(E) = \delta_{m0}a_l(E), \quad a_l(E) = -2ec\sqrt{\frac{1}{l(l+1)}}F_l(ka_d), \tag{4.6}$$

$$F_l(ka_d) = \int_0^{ka_d} j_l(x)xdx + ka_d\frac{(l+1)j_{l-1}(ka_d) - lj_{l+1}(ka_d)}{2l+1}.$$

For $ka_d \to 0$ this reduces to

$$F_l \to \frac{l+1}{(2l+1)!!}(ka_d)^l.$$

Taking the imaginary parts of the EMF strengths (3.18) with $a_l(E)$ given by (4.6), one obtains

$$H_\phi = -2ek^2\sum\frac{2l+1}{l(l+1)}(\cos\omega t j_l + \sin\omega t n_l)P_l^1 F_l(ka_d),$$

$$E_\theta = -2ek^2\sum\frac{1}{l(l+1)}\{\cos\omega t[(l+1)n_{l-1} - ln_{l+1}] - \sin\omega t[(l+1)j_{l-1} - lj_{l+1}]\}P_l^1 F_l(ka_d),$$

$$E_r = -\frac{2ek}{r}\sum(2l+1)(\cos\omega t n_l - \sin\omega t j_l)P_l F_l(ka_d). \tag{4.7}$$

We evaluate the square bracket entering into the definition of toroidal moment (see the last line in (3.4)) for the electric dipole charge-current density given by (4.5):

$$(l+3)\int r^{l+2}Y_{lm}^* div\vec{j}_d dV + 2(2l+3)\int r^l Y_{lm}^*(\vec{r}\vec{j}_d)dV = \delta_{m0}\frac{2e\omega l(l+1)}{(l+2)}a_d^{l+2}, \tag{4.8}$$

(factor $\exp(i\omega t)$ is omitted).

### 4.1 Interaction of electric dipole with external EMF

Substituting the charge-current of densities of the elementary electric dipole

$$\rho_d = f(t)(\vec{n}\vec{\nabla})\delta^3(\vec{r}-\vec{r}_d), \quad \vec{j}_d = -\dot{f}(t)\vec{n}\delta^3(\vec{r}-\vec{r}_d)$$

into the expression for interaction energy

$$U = \int[\rho_d(\vec{r})\Phi_{ext}(\vec{r}) - \frac{1}{c}\vec{j}_d(\vec{r})\vec{A}_{ext}(\vec{r})]dV,$$

one gets

$$U = -f_d(t)(\vec{n}\vec{\nabla})\Phi_{ext}(\vec{r}_d) + \frac{1}{c}\dot{f}_d(t)\vec{n}\vec{A}_{ext}(\vec{r}_d). \tag{4.9}$$

Let the external EMF be the field of TS with a constant current in its winding. Then, outside the TS, $\Phi_{ext} = 0$, $\vec{E}_{ext} = 0$, $\vec{H}_{ext} = 0$, $\vec{A}_{ext} \neq 0$ and

$$U = -\frac{1}{c}\dot{f}_d(t)\vec{n}\vec{A}_{ext}(\vec{r}_d). \tag{4.10}$$

It is surprising enough that the interaction energy differs from zero in the space region where $\vec{E}_{ext} = \vec{H}_{ext} = 0$. Despite the fact that EMF strengths vanish outside the static TS, the VP $\vec{A}$ cannot be eliminated by a gauge transformation everywhere in this region. This is due to the fact that $\int \vec{A}d\vec{s}$ along any closed path passing through the TS hole, is equal to the magnetic flux inside TS. However, the space region where $\vec{A}$ differs from zero, depends on the gauge choice (see, e.g., [17]). On the other hand, the interaction energy (4.10) should not depend on the gauge choice. The origin of this inconsistency is unclear for us.

## 5 More complicated elementary toroidal sources

In this section we give without derivation EMFs of more complicated toroidal souces obtained earlier in [16]. They are needed for the evaluation of integrals entering in the Lorentz and Feld-Tai theorems. Unfortunately, their omission makes the text to be unreadable. Consider the hierarchy of TS each turn of which is again TS. The simplest of them is the usual TS obtained by the replacement of a single turn, representing the current loop, by the infinitely thin TS. We denote this TS by $TS_1$ ( the initial current loop will be denoted by $TS_0$ ). The next-in-complexity case is obtained when each turn of $TS_1$ is replaced by an infinitely thin toroidal solenoid $ts_1$ with the time-dependent current in its winding. Thus obtained current configuration denoted by $TS_2$ is shown in Fig. 4. We see on it the poloidal current $\vec{j}$ flowing on the surface of a particular torus $ts_1$. Only one particular turn with the current $\vec{j}$ and only the central line of $ts_1$ are shown (for the torus $(\rho-d)^2 + z^2 = R$, the central line is defined as $\rho = d$, $z = 0$). The arising time-dependent magnetization ( due to the current $\vec{j}$ flowing in $ts_1$) coincides with the central line of $ts_1$ and lies on the surface of $TS_1$, in its meridional plane. Since there are many turns in $TS_1$, (each of them is the same as $ts_1$), the superposition of their magnetizations gives the overall magnetization $\vec{M}$, filling the surface of $TS_1$ (see Fig. 1 or upper part of Fig.3, where $\vec{j}$ now means $\vec{M}$). This distribution of magnetization is equivalent to the closed chain of toroidal moments $\vec{T}$ aligned along the central line of $TS_1$ (see the middle part of Fig.3, where $\vec{M}$ now means $\vec{T}$). The closed chain of toroidal moments leads to the appearance of higher order toroidal moment shown in Fig.4 by the vertical arrow. When the dimensions of just obtained configuration $TS_2$ tend to zero, we get (see [10,16]):

$$\vec{j}_2 = f_2(t)curl^{(3)}(\vec{n}\delta^3(\vec{r})), \quad curl^{(3)} = curl \cdot curl \cdot curl. \tag{5.1}$$

The corresponding VP and field strengths are given by

$$\vec{A}_2 = \frac{1}{c^4r^2}D_2^{(2)}(\vec{r}\times n), \quad \vec{E}_2 = -\frac{1}{c^5r^2}D_2^{(3)}(\vec{r}\times n), \quad \vec{H}_2 = \vec{n}\frac{1}{c^5r}G_2^{(2)} - \frac{1}{c^5r^3}\vec{r}(\vec{r}\vec{n})F_2^{(2)}. \tag{5.2}$$

Here subscripts at $D, F$ and $G$ functions mean that they depend on the $f$ function with this index, while the superscript means the time derivative of the order equal to this superscript. For example,

$$D_m^{(n)} = \frac{d^n}{dt^n}D(f_m).$$

By comparing Eqs.(5.1),(5.2) with (3.16),(3.17) we conclude that for the current configurations $TS_0$ and $TS_2$ the electromagnetic fields coincide everywhere except for the origin if the following relation between time-dependent intensities is fulfilled: $f_2^{(2)} = -f_0/c^2$. This means, in particular, that the EMF of the static magnetic dipole ($f_0 = const$) coincides with that of the current configuration $TS_2$ if the current in it quadratically varies with time ( $f_2 = -f_0c^2t^2/2$). It follows from this that the magnetic field of the usual magnetic dipole can be compensated everywhere (except for the origin) by the time-dependent current flowing in $TS_2$.

Now we are able to write out the electromagnetic field for the point-like toroidal configuration of the arbitrary order. Let

$$\vec{j}_m = f_m(t)curl^{(m+1)}(\vec{n}\delta^3(\vec{r})). \tag{5.3}$$

We consider even and odd $m$ separately.

#### Toroidal configurations of even order

Let $m$ be even ( $m = 2k, k \geq 0$ ). Then

$$\vec{A}_{2k} = (-1)^{k+1}\frac{1}{c^{2k+2}r^2}D_{2k}^{(2k)}(\vec{r}\times n), \quad \vec{E}_{2k} = (-)^k\frac{1}{c^{2k+3}r^2}D_{2k}^{(2k+1)}(\vec{r}\times n)$$

$$\vec{H}_{2k} = (-1)^k\frac{1}{c^{2k+3}}[\frac{1}{r^3}\vec{r}(\vec{r}\vec{n})F_{2k}^{(2k)} - \vec{n}\frac{1}{r}G_{2k}^{(2k)}]. \tag{5.4}$$

The distribution of the radial energy flux on the sphere of the radius $r$ is given by

$$S_r = \frac{c}{4\pi}(\vec{E}\times\vec{H})_r = \frac{\sin^2\theta}{4\pi c^{4k+5}r^2}D_{2k}^{2k+1}G_{2k}^{2k}.$$

Here $\theta$ is the angle between the symmetry axis $\vec{n}$ and a particular point on the sphere. The total energy flux through this sphere is

$$r^2\int S_r d\Omega = \frac{2}{3c^{4k+5}}D_{2k}^{2k+1}G_{2k}^{2k}.$$

The interaction of the even toroidal source with the external EMF is given by

$$U = -\frac{f_{2k}}{c}\int dV \vec{A}_{ext} curl^{2k+1}(\vec{n}\delta^3(\vec{r}-\vec{r}_s)) = (-1)^{k+1}\frac{f_{2k}}{c^{2k+1}}(\vec{n}\vec{H}^{(2k)}_{ext}),$$

where the external magnetic field is taken at the position of a point-like toroidal source.

### Toroidal configurations of odd order

On the other hand, for $m$ odd ( $m = 2k+1, k \geq 0$)

$$\vec{A}_{2k+1} = (-1)^k \frac{1}{c^{2k+3}}[\frac{1}{r^3}\vec{r}(\vec{r}\vec{n})F^{(2k)}_{2k+1} - \vec{n}\frac{1}{r}G^{(2k)}_{2k+1}],$$

$$\vec{E}_{2k+1} = (-1)^{k+1}\frac{1}{c^{2k+4}}[\frac{1}{r^3}\vec{r}(\vec{r}\vec{n})F^{(2k+1)}_{2k+1} - \vec{n}\frac{1}{r}G^{(2k+1)}_{2k+1}],$$

$$\vec{H}_{2k+1} = (-1)^k \frac{1}{c^{2k+4}r^2}D^{(2k+2)}_{2k+1}(\vec{r}\times n) \quad S = \frac{2}{3c^{4k+7}}G^{(2k+1)}_{2k+1}D^{(2k+2)}_{2k+1}. \quad (5.5)$$

The distribution of the radial energy flux on the sphere of the radius $r$ is given by

$$S_r = \frac{c}{4\pi}(\vec{E}\times\vec{H})_r = \frac{\sin^2\theta}{4\pi c^{4k+7}r^2}D^{2k+2}_{2k+1}G^{2k+1}_{2k+1}.$$

The total energy flux through this sphere is

$$r^2\int S_r d\Omega = \frac{2}{3c^{4k+7}}D^{2k+2}_{2k+1}G^{2k+1}_{2k+1}.$$

The interaction of the even toroidal source with the external EMF is given by

$$U = -\frac{f_{2k+1}}{c}\int dV \vec{A}_{ext} curl^{2k+2}(\vec{n}\delta^3(\vec{r}-\vec{r}_s)) = (-1)^{k+1}\frac{f_{2k+1}}{c^{2k+2}}(\vec{n}\vec{E}^{(2k+1)}_{ext}).$$

Again, the external electric field is taken at the position of a point-like toroidal source.

### Short resume of this section

We see that there are two branches of toroidal point-like currents generating essentially different electromagnetic fields. A representative of the first branch is the usual magnetic dipole. The electromagnetic field of the k-th member of this family reduces to that of the circular current if the time dependences of these currents are properly adjusted

$$f^{(2k)}_{2k} = (-1)^k f_0(t)/c^{2k}, \quad (k \geq 0). \quad (5.6)$$

We remember that the lower index of the $f$ functions selects a particular member of the first branch, while the upper one means the time derivative.
The representative of the second branch is the elementary TS. Again, the electromagnetic fields of this family are the same if the time dependences of currents are properly adjusted

$$f^{(2k)}_{2k+1} = (-1)^k f_1(t)/c^{2k}, \quad (k \geq 0). \quad (5.7)$$



From the equations defining the energy flux it follows that for high frequencies, the toroidal emitters of the higher order are more effective ( as the time derivatives of higher orders contribute to the energy flux). They may be used in the same way as usual FM transmitters. Namely, the EMF of high frequency carries the energy. It is modulated by the low frequency EMF carrying the information. The resulting signal is decoded in the receiver, its high-frequency is removed, while its low-frequency part comes to our ears.
From the classical electrodynamics it is known [32, 34] that there are two types of radiation. For the multipole radiation of magnetic type $\vec{r}\vec{E} = 0, \vec{r}\vec{H} \neq 0$, while for the radiation of electric type should be $\vec{r}\vec{H} = 0, \vec{r}\vec{E} \neq 0$. It follows from (5.4) that $\vec{r}\vec{E}_{2k} = 0, \vec{r}\vec{H}_{2k} \neq 0$. Thus, radiation fields of the time-dependent currents flowing in a circular turn and in toroidal emitters of the even order are of magnetic type. It follows from (5.5) that $\vec{r}\vec{H}_{2k} = 0, \vec{r}\vec{E}_{2k} \neq 0$. Correspondingly, radiation fields of the time-dependent currents flowing in a toroidal coil and in toroidal emitters of the odd order are of electric type.

# 6 The Lorentz and Feld-Tai lemmas

## 6.1 Standard derivation of the Lorentz lemma

We write out Maxwell's equations for two current sources $\vec{j}_1$ and $\vec{j}_2$:

$$curl\vec{E}_1 = -\frac{1}{c}\dot{\vec{H}}_1, \quad curl\vec{H}_1 = \frac{1}{c}\dot{\vec{E}}_1 + \frac{4\pi}{c}\vec{j}_1,$$

$$curl\vec{E}_2 = -\frac{1}{c}\dot{\vec{H}}_2, \quad curl\vec{H}_2 = \frac{1}{c}\dot{\vec{E}}_2 + \frac{4\pi}{c}\vec{j}_2. \quad (6.1)$$

From this one easily obtains

$$div(\vec{E}_1\times\vec{H}_2) = \vec{H}_2 curl\vec{E}_1 - \vec{E}_1 curl\vec{E}_2 = -\frac{1}{c}\vec{H}_2\dot{\vec{H}}_1 - \frac{1}{c}\vec{E}_1\dot{\vec{E}}_2 - \frac{4\pi}{c}\vec{j}_2\vec{E}_1,$$

$$div(\vec{E}_2\times\vec{H}_1) = \vec{H}_2 curl\vec{E}_1 - \vec{E}_1 curl\vec{E}_2 = -\frac{1}{c}\vec{H}_1\dot{\vec{H}}_2 - \frac{1}{c}\vec{E}_2\dot{\vec{E}}_1 - \frac{4\pi}{c}\vec{j}_1\vec{E}_2.$$

Subtracting these equations from each other, one gets

$$div(\vec{E}_1\times\vec{H}_2 - \vec{E}_2\times\vec{H}_1) = \frac{1}{c}(\vec{H}_1\dot{\vec{H}}_2 - \vec{H}_2\dot{\vec{H}}_1) - \frac{1}{c}(\vec{E}_1\dot{\vec{E}}_2 - \vec{E}_2\dot{\vec{E}}_1) + \frac{4\pi}{c}(\vec{j}_1\vec{E}_2 - \vec{j}_1\vec{E}_2). \quad (6.2)$$

When the time dependence of field strengths is given by $\exp(i\omega t)$, i.e.,

$$\vec{E}_1 = \exp(i\omega t)\vec{E}^0_1, \quad \vec{E}_2 = \exp(i\omega t)\vec{E}^0_2, \quad \vec{H}_1 = \exp(i\omega t)\vec{H}^0_1, \quad \vec{H}_2 = \exp(i\omega t)\vec{H}^0_2, \quad (6.3)$$

then

$$\vec{E}_1\dot{\vec{E}}_2 = \vec{E}_2\dot{\vec{E}}_1, \quad \vec{H}_1\dot{\vec{H}}_2 = \vec{H}_2\dot{\vec{H}}_1 \quad (6.4)$$

and

$$div(\vec{E}_1\times\vec{H}_2 - \vec{E}_2\times\vec{H}_1) = \frac{4\pi}{c}(\vec{j}_1\vec{E}_2 - \vec{j}_1\vec{E}_2).$$

Integrate this relation over the sphere of the radius $R_0$ and apply the Gauss theorem

$$R_0^2\int(\vec{E}_1\times\vec{H}_2-\vec{E}_2\times\vec{H}_1)_r d\Omega=\frac{4\pi}{c}\int(\vec{j}_1\vec{E}_2-\vec{j}_1\vec{E}_2)dV. \quad (6.5)$$

For $R_0\to\infty$, the LHS of this equation disappears and one gets the famous Lorentz lemma

$$\mathcal{E}_{12}=\mathcal{E}_{21}, \quad (6.6)$$

where we put $\mathcal{E}_{12}=\int\vec{j}_1\vec{E}_2dV,\quad \mathcal{E}_{21}=\int\vec{j}_2\vec{E}_1dV.$

## 6.2 The Feld-Tai lemma

The Feld-Tai lemma states that

$$\mathcal{H}_{12}=\mathcal{H}_{21}, \quad (6.7)$$

where $\mathcal{H}_{12}=\int\vec{j}_1\vec{H}_2dV,\quad \mathcal{H}_{21}=\int\vec{j}_2\vec{H}_1dV.$

It is proved along the same lines as the Lorentz lemma. From (6.1) one easily obtains

$$div(\vec{H}_1\times\vec{H}_2)=\frac{1}{c}(\vec{H}_2\dot{\vec{E}}_1-\vec{H}_1\dot{\vec{E}}_2)+\vec{j}_1\vec{H}_2-\vec{j}_1\vec{H}_2,\quad div(\vec{E}_1\times\vec{E}_2)=\frac{1}{c}(\vec{E}_1\dot{\vec{H}}_2-\vec{E}_2\dot{\vec{H}}_1). \quad (6.8)$$

Subtracting these equations from each other, one gets

$$div(\vec{E}_1\times\vec{E}_2-\vec{H}_1\times\vec{H}_2)=\frac{1}{c}(\vec{E}_1\dot{\vec{H}}_2-\vec{E}_2\dot{\vec{H}}_1)-\frac{1}{c}(\vec{H}_2\dot{\vec{E}}_1-\vec{H}_1\dot{\vec{E}}_2)-\vec{j}_1\vec{H}_2+\vec{j}_2\vec{H}_1. \quad (6.9)$$

If the time dependences of $\vec{E}$ and $\vec{H}$ are $\exp(i\omega t)$, then the first two terms in the RHS of (6.9) cancel each other. Integrating the remaining ones over the whole volume, one gets

$$r^2\int d\Omega(\vec{E}_1\times\vec{E}_2-\vec{H}_1\times\vec{H}_2)_r=\int dV(-\vec{j}_1\vec{H}_2+\vec{j}_2\vec{H}_1). \quad (6.10)$$

Since $\vec{E}=\vec{H}\times\vec{n},\quad \vec{n}=\vec{r}/r$ on the sphere of infinite radius, the LHS of (6.10) disappears and one obtains the Feld-Tai lemma (6.7).

## 6.3 Lorentz and Feld-Tai lemmas for real time-dependences

The crucial point in obtaining (6.6) and (6.7) is Eq.(6.3). However, the real current densities should be real. The possibility of operating with complex quantities like

$$\exp(i\omega t)\vec{j},\quad \exp(i\omega t)\vec{E},\quad \exp(i\omega t)\vec{H}$$

is valid as far as we deal with the quantities linear in field strengths. For example, if the actual dependence of the current density is $\cos\omega t$, then we may solve Maxwell equations with $\exp(i\omega t)\vec{j}, \exp(i\omega t)\vec{E}$ and $\exp(i\omega t)\vec{H}$ and at the end take the real parts of these quantities. However, one should be very careful dealing with quadratic combinations like (6.2) and (6.9). To avoid mistakes, one should first take real parts of EMF strengths and substitute them into quadratic combinations of field strengths. Consider two equalities



(6.4) obtained under assumption (6.3). Equations (3.9),(3.23) and (4.4) show that actual field strengths contain both $\cos\omega t$ and $\sin\omega t$

$$\vec{E}_1=\cos\omega t\vec{E}_1^c+\sin\omega t\vec{E}_1^s,\quad \vec{E}_2=\cos\omega t\vec{E}_2^c+\sin\omega t\vec{E}_2^s,$$

$$\vec{H}_1=\cos\omega t\vec{H}_1^c+\sin\omega t\vec{H}_1^s,\quad \vec{H}_2=\cos\omega t\vec{H}_2^c+\sin\omega t\vec{H}_2^s. \quad (6.11)$$

Substituting (6.11) into (6.4), we find that the latter are satisfied if

$$\vec{E}_1^c\vec{E}_2^s=\vec{E}_1^s\vec{E}_2^c,\quad \vec{H}_1^c\vec{H}_2^s=\vec{H}_1^s\vec{H}_2^c \quad (6.12)$$

It is not evident that these equations are fulfilled for the real time dependences $\cos\omega t$ and $\sin\omega t$. We show below (section 6.5) that they are not satisfied for the simplest EMF sources.

## 6.4 The Lorentz and Feld-Tai lemmas for elementary electromagnetic sources

We apply now the Lorentz and Feld-Tai lemmas to the simplest electromagnetic sources.

### 6.4.1 Interacting electric dipole and current loop

Equations (3.16) and (3.17) define the current density and EMF strengths of the current loop, resp. Correspondingly, Eqs. (4.1) and (4.2) define the same quantities for the electric dipole. Combining them, we evaluate the integrals entering into the Lorentz and Feld-Tai lemmas:

$$\mathcal{E}_{Ld}=f_L\int curl(\vec{n}_L\delta^3(\vec{r}-\vec{r}_L))\vec{E}_d dV=-\frac{1}{c}f_L\vec{n}_L\int\delta^3(\vec{r}-\vec{r}_L)\dot{\vec{H}}_d dV=$$

$$=\frac{1}{c^3R_{dL}^2}f_L(t)(\vec{R}_{Ld}(\vec{n}_d\times\vec{n}_L)\ddot{D}_d,\quad \mathcal{E}_{dL}=\frac{1}{c^3R_{dL}^2}\dot{f}_d(t)(\vec{R}_{dL}(\vec{n}_L\times\vec{n}_d)\dot{D}_L,$$

$$\mathcal{H}_{Ld}=-\frac{1}{c^3R_{dL}}f_L(t)[(\vec{n}_d\vec{n}_L)\dot{G}_d-\frac{1}{R_{dL}^2}(\vec{n}_d\vec{R}_{dL})(\vec{n}_L\vec{R}_{dL})\dot{F}_d],$$

$$\mathcal{H}_{dL}=-\frac{1}{c^3R_{dL}}\dot{f}_d(t)[(\vec{n}_d\vec{n}_L)G_L-\frac{1}{R_{dL}^2}(\vec{n}_d\vec{R}_{dL})(\vec{n}_L\vec{R}_{dL})F_L]. \quad (6.13)$$

Here $\vec{R}_{Ld}=-\vec{R}_{dL}=\vec{r}_L-\vec{r}_d$. We see that

$$\mathcal{E}_{Ld}=\mathcal{E}_{dL}\quad\text{and}\quad\mathcal{H}_{Ld}=\mathcal{H}_{dL} \quad (6.14)$$

if $f_L=\dot{f}_d$.

#### 6.4.2 Interacting electric dipole and toroidal solenoid

Combining Eqs. (3.46),(3.47) defining current density and EMF strengths of TS and Eqs. (4.1), (4.2) defining the same quantities for the electric dipole, we evaluate the integrals entering into the Lorentz and Feld-Tai lemmas:

$$\mathcal{E}_{Td} = f_T(t)\int curl^{(2)}(\vec{n}_T\delta^3(\vec{r}-\vec{r}_T)\vec{E}_d dV = -\frac{1}{c^2}f_T(t)\ddot{\vec{E}}_d(\vec{R}_{TD}) =$$

$$= \frac{1}{c^4 R_{dT}}[(\vec{n}_T\vec{n}_d)\ddot{G}_d - \frac{1}{R^2_{dT}}(\vec{n}_d\vec{R}_{Td})(\vec{n}_T\vec{R}_{Td})\ddot{F}_d],$$

$$\mathcal{E}_{dT} = \dot{f}_d(t)\frac{1}{c^4 R_{dT}}[(\vec{n}_T\vec{n}_d)\dot{G}_T - \frac{1}{R^2_{dT}}(\vec{n}_d\vec{R}_{Td})(\vec{n}_T\vec{R}_{Td})\dot{F}_T],$$

$$\mathcal{H}_{Td} = -\frac{1}{c^2}f_T(\vec{n}_T\ddot{\vec{H}}_d(\vec{R}_{TD}) = -\frac{1}{c^4 R^2_{Td}}f_T\vec{R}_{Td}(\vec{n}_d\times\vec{n}_T)D^{(3)}_d,$$

$$\mathcal{H}_{dT} = \frac{1}{c^4 R^2_{Td}}\dot{f}_d\vec{R}_{dT}(\vec{n}_d\times\vec{n}_T)D^{(2)}_T. \quad (6.15)$$

The dots above the field strengths mean time derivatives. Again, we see that these integrals coincide when $f_T = \dot{f}_d$.

#### 6.4.3 Interacting current loop and toroidal solenoid

Finally, using Eqs.(3.16), (3,17) and (3.46),(3.47) we get for the integrals entering into the Lorentz and Feld-Tai lemmas

$$\mathcal{E}_{LT} = f_L\int curl(\vec{n}_L\delta^3(\vec{r}-\vec{r}_L))\vec{E}_T(\vec{r}-\vec{r}_T)dV = -\frac{1}{c}f_L\vec{n}_L\int\delta^3(\vec{r}-\vec{r}_L)\dot{\vec{H}}_T(\vec{r}-\vec{r}_T)dV =$$

$$= -\frac{1}{c^5 R^2_{LT}}f_L\vec{R}_{LT}(\vec{n}_T\times\vec{n}_L)D^{(3)}_T,$$

$$\mathcal{E}_{TL} = f_T(t)\int curl^{(2)}(\vec{n}_T\delta^3(\vec{r}-\vec{r}_T))\vec{E}_L(\vec{r}-\vec{r}_L)dV =$$

$$= -f_T(t)\frac{1}{c^2}\vec{n}_T\ddot{\vec{E}}_L(\vec{R}_{TL}) = -\frac{1}{c^5 R^2_{TL}}f_T(t)D^{(3)}_L\vec{R}_{TL}(\vec{n}_L\times\vec{n}_T),$$

$$\mathcal{H}_{LT} = f_L\int curl(\vec{n}_L\delta^3(\vec{r}-\vec{r}_L))\vec{H}_T(\vec{r}-\vec{r}_T)dV = \frac{1}{c}f_l\vec{n}_L\dot{\vec{E}}_T(\vec{R}_{LT}) =$$

$$= \frac{1}{c^5 R_{LT}}f_L[(\vec{n}_L\vec{n}_T)\ddot{G}_T - \frac{1}{R^2_{dT}}(\vec{n}_L\vec{R}_{LT})(\vec{n}_T\vec{R}_{LT})\ddot{F}_T],$$

$$\mathcal{H}_{TL} = f_T\int curl^{(2)}(\vec{n}_T\delta^3(\vec{r}-\vec{r}_T))\vec{H}_L dV = -f_T\frac{1}{c^2}\vec{n}_T\int\delta^3(\vec{r}-\vec{r}_T)\ddot{\vec{H}}_L(\vec{R}_L)dV =$$

$$= -\frac{f_T}{c^2}\vec{n}_T\ddot{\vec{H}}(\vec{R}_{TL}) = \frac{f_T}{c^5 R_{TL}}[(\vec{n}_L\vec{n}_T)\ddot{G}_L - \frac{1}{R^2_{dT}}(\vec{n}_L\vec{R}_{LT})(\vec{n}_T\vec{R}_{LT})\ddot{F}_L]. \quad (6.16)$$

We see that these integrals coincide when $f_T = f_L$.



### 6.5 Conditions for the validity of the Lorentz and Feld-Tai lemmas

We analyze the conditions (6.4) and (6.12) using the interacting current loop and toroidal solenoid as an example. As we have seen, the equalities

$$\mathcal{E}_{LT} = \mathcal{E}_{TL} \quad \text{and} \quad \mathcal{H}_{LT} = \mathcal{H}_{TL}$$

are satisfied $f_T = f_L$. However, it is easy to check that the conditions (6.4) and (6.12) under which the Lorentz and Feld-Tai lemmas were obtained are not satisfied for arbitrary $f_T = f_d$. More accurately, Eqs.(6.4) and (6.12) are valid if the time dependences $f_T$ and $f_L$ are of the following specific form: $f_T \sim \exp(i\omega t)$, $f_L \sim \exp(i\omega t)$. But how to reconcile the violation of (6.4) and (6.12) with the fulfillment of (6.6) and (6.7) proved in a previous section? The answer is that although Eqs.(6.4) are not satisfied, the space integrals from them do. This, in turn, means that the Lorentz and Feld-Tai lemmas have a greater range of applicability than it was suggested up to now. The same conclusions are valid for the interaction of electric dipole with the current loop and with the toroidal solenoid. The fact that the Lorentz lemma (6.6) may be fulfilled due to the equalities of the space integrals from (6.4), not to (6.4) itself, was earlier admitted by Ginzburg ([36]).

## 7 Alternative proof of the Lorentz and Feld-Tai lemmas

### 7.1 Digression on the energy exchange

At first we consider a simpler case corresponding to the energy exchange between two sources of electromagnetic energy. The energy transmitted from one charge-current source $\rho_2(\vec{r},t), \vec{j}_2(\vec{r},t)$ to the other source $\rho_1(\vec{r},t), \vec{j}_1(\vec{r},t)$ is given by

$$W_{12}(t) = \int[\rho_1(\vec{r}_1,t)\Phi_2(\vec{r}_1,t) - \frac{1}{c}\vec{j}_1(\vec{r}_1,t)\vec{A}_2(\vec{r}_1,t)]dV_1, \quad (7.1)$$

where $\Phi_2(\vec{r}_1,t)$ and $\vec{A}_2(\vec{r}_1,t)$ are the scalar and electric potentials induced by the charge-current density $(\rho_2,\vec{j}_2)$ at the position of the charge-current density $(\rho_1,\vec{j}_1)$. They are given by

$$\Phi_2(\vec{r}_1,t) = \int\frac{1}{R_{12}}\rho_2(\vec{r}_2,\tau)\delta(\tau - t + R_{12}/c)dV_2 d\tau,$$

$$\vec{A}_2(\vec{r}_1,t) = \frac{1}{c}\int\frac{1}{R_{12}}\vec{j}_2(\vec{r}_2,\tau)\delta(\tau - t + R_{12}/c)dV_2 d\tau. \quad (7.2)$$

Here $R_{12} = |\vec{r}_1 - \vec{r}_2|$ is the distance between the particular point of sources 1 and 2. Substituting this into (7.1), one gets

$$W_{12}(t) = \int[\rho_1(\vec{r}_1,t)\rho_2(\vec{r}_2,\tau) - \frac{1}{c^2}\vec{j}_1(\vec{r}_1,t)\vec{j}_2(\vec{r}_2,\tau)]\frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)dV_1 dV_2 d\tau. \quad (7.3)$$

In the same way,

$$W_{21}(t) = \int[\rho_2(\vec{r}_2,t)\rho_1(\vec{r}_1,\tau) - \frac{1}{c^2}\vec{j}_1(\vec{r}_1,\tau)\vec{j}_2(\vec{r}_2,t)]\frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)dV_1dV_2d\tau. \quad (7.4)$$

We see, that in general $W_{21}(t) \neq W_{12}(t)$. Let now time dependences in $\rho$ and $\vec{j}$ be separated

$$\rho_1(\vec{r}_1,t_1) = \rho_1(t_1)\rho_1(\vec{r}_1), \quad \rho_2(\vec{r}_2,t_2) = \rho_2(t_2)\rho_2(\vec{r}_2),$$
$$\vec{j}_1(\vec{r}_1,t_1) = j_1(t_1)\vec{j}_1(\vec{r}_1), \quad \vec{j}_2(\vec{r}_2,t_2) = j_2(t_2)\vec{j}_2(\vec{r}_2). \quad (7.5)$$

Then,

$$W_{12}(t) = \int[\rho_1(t)\rho_1(\vec{r}_1)\rho_2(\vec{r}_2)\rho_2(\tau) - \frac{1}{c^2}j_1(t)\vec{j}_1(\vec{r}_1)\vec{j}_2(\vec{r}_2)j_2(\tau)]\frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)dV_1dV_2d\tau, \quad (7.6)$$

$$W_{21}(t) = \int \rho_2(t)\rho_2(\vec{r}_2)\rho_1(\vec{r}_1)\rho_1(\tau) - \frac{1}{c^2}j_2(t)\vec{j}_2(\vec{r}_2)\vec{j}_1(\vec{r}_1)j_1(\tau)]\frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)dV_1dV_2d\tau. \quad (7.7)$$

It follows from this that $W_{12} = W_{21}$ if the time dependences of sources 1 and 2 coincide, i.e., when

$$\rho_1(t) = \rho_2(t), \quad j_1(t) = j_2(t), \quad (7.8)$$

that is the action and reaction coincide if the time-dependences of sources 1 and 2 are synchronized.
The violation of action and reaction due to the retarded nature of electromagnetic interaction was first recognized by H.A. Lorentz in 1895 [37]. As far as we know, the best exposition of these questions has been given in Cullwick's book [38] where the explicit violation of action and reaction equality was demonstrated the interacting for the interaction of a charge with TS. In a modern physical literature the violation of this equality is considered as almost obvious. We quote, e.g., French [39]: "The equality of action and reaction has almost no place in relativistic mechanics. It must be essentially a statement about the forces acting on two bodies, as a result of their mutual interaction at a given instant. And, because of the relativity of simultaneity, this phrase has no meaning."

The violation of action and reaction equality for the interaction between the moving current loop and charge and between two moving charges has been noted by O. Jefimenko [40] and P. Cornille [41], resp. However, this violation is not restricted only to the retardation effects. Even for the interacting static metallic currents there are known two interaction laws: Ampere law which agrees with Newton's third law (equality of action and reaction forces) and Lorentz law which violates it (see. e.g., [42, 43]). However, if the above currents are closed, the difference between these forces disappears: both of them satisfy Newton's third law [44]. Some experiments [45] seem to support only the Ampere law of force, while others [46] give the same result for both laws. These questions need the further consideration.



## 7.2 Concrete examples: the energy exchange between elementary toroidal sources

Let us have two toroidal sources $TS_1$ and $TS_2$ of an arbitrary order. The interaction energy is

$$\mathcal{E} = \mathcal{E}_{12} + \mathcal{E}_{21},$$

where $\mathcal{E}_{12}$ and $\mathcal{E}_{21}$ are the parts of $\mathcal{E}$ localized at the positions of $TS_1$ and $TS_2$. More accurately, $\mathcal{E}_{12}$ is the energy induced by the source 2 at the position of source 1. And, similarly, for $\mathcal{E}_{21}$. They are given by:

$$\mathcal{E}_{12} = -\frac{1}{c}\int dV\vec{j}_1(\vec{r}-\vec{r}_1)\vec{A}_2(\vec{r}-\vec{r}_2)dV \quad (7.9)$$

and

$$\mathcal{E}_{21} = -\frac{1}{c}\int dV\vec{j}_2(\vec{r}-\vec{r}_2)\vec{A}_1(\vec{r}-\vec{r}_1)dV, \quad (7.10)$$

resp.

### 7.2.1 The interaction of even toroidal sources

Let $\vec{j}_1$ and $\vec{j}_2$ be both of even order

$$\vec{j}_1 = f_1(t)curl^{2l_1+1}[\vec{n}_1\delta^3(\vec{r}-\vec{r}_1)], \quad \vec{j}_2 = f_2(t)curl^{2l_2+1}[\vec{n}_2\delta^3(\vec{r}-\vec{r}_2)].$$

Then,

$$\mathcal{E}_{12} = \frac{(-1)^{l_1+1}}{c^{2l_1+1}}f_1(t)\vec{n}_1\cdot\vec{H}_2^{(2l_1)}(\vec{R}_{12}), \quad \mathcal{E}_{21} = \frac{(-1)^{l_2+1}}{c^{2l_2+1}}f_2(t)\vec{n}_2\cdot\vec{H}_1^{(2l_2)}(\vec{R}_{21}), \quad (7.11)$$

where $\vec{H}_2^{(2l_1)}(\vec{R}_{12})$ is the $2l_1$ time-derivative of the magnetic field produced by $TS_2$ at the position of $TS_1$ and $\vec{H}_1^{(2l_2)}(\vec{R}_{21})$ is the $2l_2$ time-derivative of the magnetic field produced by $TS_1$ at the position of $TS_2$. Substituting them from (5.4), one gets

$$\mathcal{E}_{12} = f_1\frac{(-1)^{l_1+l_2+1}}{c^{2l_1+2l_2+4}R_{12}}[\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12})F_2^{(2l_1+2l_2)} - (\vec{n}_1\vec{n}_2)G_2^{(2l_1+2l_2)}],$$

$$\mathcal{E}_{21} = f_2\frac{(-1)^{l_1+l_2+1}}{c^{2l_1+2l_2+4}R_{12}}[\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12})F_1^{(2l_1+2l_2)} - (\vec{n}_1\vec{n}_2)G_1^{(2l_1+2l_2)}]. \quad (7.12)$$

We see that $\mathcal{E}_{12} = \mathcal{E}_{21}$ for arbitrary $f_1 = f_2$. Let $f_1$ and $f_2$ do not depend on time. Then, $\mathcal{E}_{12}$ and $\mathcal{E}_{21}$ differ from zero only for $l_1 = l_2 = 0$:

$$\mathcal{E}_{12} = \mathcal{E}_{21} = -\frac{f_1f_2}{c^2R_{12}^3}[3\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12}) - (\vec{n}_1\vec{n}_2)],$$

that coincides with an interaction of two magnetic dipoles.

### 7.2.2 The interaction of odd toroidal sources

Let $\vec{j}_1$ and $\vec{j}_2$ both be of odd order.

$$\vec{j}_1 = f_{2l_1+1}(t)curl^{2l_1+2}[\vec{n}_1\delta^3(\vec{r}-\vec{r}_1)], \quad \vec{j}_2 = f_{2l_2+1}(t)curl^{2l_2+2}[\vec{n}_2\delta^3(\vec{r}-\vec{r}_2)].$$

Then,

$$\mathcal{E}_{12} = \frac{(-1)^{l_1+1}}{c^{2l_1+1}} f_1(t)\vec{n}_1\cdot \vec{E}_2^{(2l_1+1)}(\vec{R}_{12}), \quad \mathcal{E}_{21} = \frac{(-1)^{l_2+1}}{c^{2l_2+1}} f_2(t)\vec{n}_2\cdot\vec{E}_1^{(2l_2+1)}(\vec{R}_{21}), \qquad (7.13)$$

where $\vec{E}_2^{(2l_1+1)}(\vec{R}_{12})$ is the $2l_1+1$ time-derivative of the electric field induced by $TS_2$ at the position of $TS_1$ and $\vec{E}_1^{(2l_2+1)}(\vec{R}_{21})$ is the $2l_2+1$ time-derivative of the electric field induced by $TS_1$ at the position of $TS_2$. Substituting them from (5.5), we get

$$\mathcal{E}_{12} = f_1\frac{(-1)^{l_1+l_2}}{c^{2l_1+2l_2+6}R_{12}}\left[\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12})F_2^{(2l_1+2l_2+2)} - (\vec{n}_1\vec{n}_2)G_2^{(2l_1+2l_2+2)}\right],$$

$$\mathcal{E}_{21} = f_2\frac{(-1)^{l_1+l_2}}{c^{2l_1+2l_2+6}R_{12}}\left[\frac{1}{R_{12}^2}(\vec{n}_1\vec{R}_{12})(\vec{n}_2\vec{R}_{12})F_1^{(2l_1+2l_2+2)} - (\vec{n}_1\vec{n}_2)G_1^{(2l_1+2l_2+2)}\right]. \qquad (7.14)$$

Again, we see that $\mathcal{E}_{12} = \mathcal{E}_{21}$ for arbitrary $f_1 = f_2$. Let $f_1$ and $f_2$ do not depend on time. Then, $\mathcal{E}_{12} = \mathcal{E}_{21} = 0$. This means that static toroidal sources of an odd order (and, in particular, usual static toroidal solenoids) do not interact.
It follows from (7.12) and (7.14) that toroidal sources of the same order do not interact when the following two conditions are fulfilled simultaneously:
i) The symmetry axes of toroidal sources are mutually orthogonal ; ii) The symmetry axes of toroidal sources are perpendicular to the vector $\vec{R}_{12}$ going from $TS_1$ to, $TS_2$ .
In particular, this is valid for two interacting current loops or toroidal solenoids.

### 7.2.3 The interaction of even and odd toroidal sources

Let one of the currents be of the even order and the other of the odd one:
$\vec{j}_1 = f_1(t)curl^{2l_1+1}[\vec{n}_1\delta^3(\vec{r}-\vec{r}_1)]$, $\vec{j}_2 = f_2(t)curl^{2l_2+2}[\vec{n}_2\delta^3(\vec{r}-\vec{r}_2)]$. Then,

$$\mathcal{E}_{12} = \frac{(-1)^{l_1+1}}{c^{2l_1+1}} f_1(t)\vec{n}_1\cdot\vec{H}_2^{(2l_1)}(\vec{R}_{12}), \quad \mathcal{E}_{21} = \frac{(-1)^{l_2+1}}{c^{2l_2+2}} f_2(t)\vec{n}_2\cdot\vec{E}_1^{(2l_2+1)}(\vec{R}_{21}). \qquad (7.15)$$

We observe a curious fact: $TS_1$ interacts with time derivatives of the magnetic field induced by $TS_2$ while $TS_2$ interacts with time derivatives of the electric field induced by $TS_1$ (under the words 'interacts with time derivative' we mean that the time derivative of the corresponding order enters into the interaction energy). Substitution of $\vec{E}_1$ from (5.4) and $\vec{H}_2$ from (5.5) gives

$$\mathcal{E}_{12} = f_1\frac{(-1)^{l_1+l_2+1}}{c^{2l_1+2l_2+5}}\frac{1}{R_{12}^2}\vec{n}_1(\vec{R}_{12}\times n_2)D_2^{(2l_1+2l_2+2)},$$

$$\mathcal{E}_{21} = f_2\frac{(-1)^{l_1+l_2+1}}{c^{2l_1+2l_2+5}}\frac{1}{R_{12}^2}\vec{n}_2(\vec{R}_{21}\times n_1)D_1^{(2l_1+2l_2+2)}. \qquad (7.16)$$

Again, we observe that $\mathcal{E}_{12} = \mathcal{E}_{21}$ for arbitrary $f_1 = f_2$.



From this one can see at once the violation of action and reaction equality. Take, for example the last equation. Let $f_1$ and $f_2$ depend and do not depend on time, resp. Then, $\mathcal{E}_{12} = 0$ and $\mathcal{E}_{21} \neq 0$. This means that $TS_1$ acts on $TS_2$ while $TS_2$ does not act on $TS_1$.
It follows from (7.16) that toroidal source of even order does not interact with that of odd order if one of the following two conditions is fulfilled:
i) When the symmetry axes of $TS_1$ and $TS_2$ are parallel ; ii) When at least one of two symmetry axes ($TS_1$ or $TS_2$) is parallel to the vector $\vec{R}_{12}$ going from $TS_1$ to $TS_2$.
In particular, this is valid for the interaction of a current loop with a toroidal solenoid.

### 7.2.4 Numerical estimations

To see explicitly at what level the equality of action and reaction is violated, consider an interacting current loop and TS with a constant current in its winding. Since, there is no EMF outside such TS it does not act on a current loop. On the other hand, the action of current loop on a TS is given by (7.16) where one should put $l_1 = l_2 = 0$. Then,

$$\mathcal{E}_{LT} = 0, \quad \mathcal{E}_{TL} = -f_T\frac{1}{c^5 R_{TL}^2}\vec{R}_{TL}(\vec{n}_L\times\vec{n}_T)D_L^{(2)}.$$

Let $f_L$ periodically changes with on time. Then,

$$f_L = \pi I_L d_L^2\cos\omega t, \quad D_L = -\pi I_L d_L^2\omega(\sin\omega t_r - \frac{1}{kr}\cos\omega t_r), \quad D_L^{(2)} = -\omega^2 D_L, \quad k = \frac{\omega}{c},$$

$$t_r = t - \frac{R_{TL}}{c}, \quad \mathcal{E}_{TL} = -f_T\frac{\pi I_L d_L^2}{c^5 R_{TL}^2}\vec{R}_{TL}(\vec{n}_L\times\vec{n}_T)D_L^{(2)}\omega^3(\sin\omega t_r - \frac{1}{kr}\cos\omega t_r).$$

Now we choose $f_T$. It is equal to $\pi c g d_T R^2/4$, where $g = 2NI_T/c$, $N$ is a number of coils in TS winding, $I_T$ ia a current in a particular coil. However, instead of TS winding, it is convenient to use the ferromagnetic ring magnetized in the azimuthal direction (see the middle part of Fig. 3). These two objects are completely equivalent as to their interaction with external EMF. The magnetic field inside TS is given by $H_\phi = g/\rho$, where $\rho$ is the cylindrical radius. If the major radius $d_T$ of $TS$ is much larger than its minor radius $R$, we may put $H_\phi = H_T = g/d_T$, $g = d_T H_T$. Finally, for $\mathcal{E}_{TL}$, we get

$$\mathcal{E}_{TL} = -\frac{\pi^2 I_L H_T d_L^2 d_T^2 R^2}{c^5 R_{TL}^2}\vec{R}_{TL}(\vec{n}_L\times\vec{n}_T)D_L^{(2)}\omega^3(\sin\omega t_r - \frac{1}{kr}\cos\omega t_r).$$

Its maximal absolute value is

$$|\mathcal{E}_{TL}| = \pi^2 I_L H_T d_L^2 d_T^2 R^2\omega^3/c^5 R_{TL}.$$

This expression should be multiplied by the number $N_L$ of the turns in a circular loop. The typical value of magnetic field inside the ferromagnetic sample is about 1000 gauss. Let $N_L = 1000$, $I_L = 1$ ampere , the dimensions of a current loop and TS are of the order of few centimeters and the distance between sources about 10 cm. In order the motion of TS can be observed, the frequency should be of the order few hertzs (otherwise, positive and negative values of $\mathcal{E}_{TL}$ compensate each other for the finite observation time). For these parameters, $\mathcal{E}_{TL} \sim 10^{-32}$ ergs and the corresponding force $F_{TL} \sim \mathcal{E}_{TL}/R_{TL} \sim 10^{-33}$ dynes. Such a small force could be hardly observed experimentally for the realistic cosine or sine current dependences. Under the influence of a force from a current loop, TS begins to move. The EMF strengths are non-zero outside TS, when it moves uniformly in medium ([47,48]), or when it is accelerated (both in medium and vacuum [13]). The moving TS will act on a current loop which, in its turn, begins to move. But these, next order effects, are beyond the present consideration.

## 7.3 Back to the Lorentz and Feld-Tai lemmas

### 7.3.1 Lorentz lemma

Proceeding in the same way as for interaction energies, we get for the integrals $\mathcal{E}_{12}$ and $\mathcal{E}_{21}$ entering into the formulation of the Lorentz lemma

$$\mathcal{E}_{12} = -\int \dot{\rho}_1(\vec{r}_1,t)\rho_2(\vec{r}_2,\tau)\delta(\tau - t + R_{12}/c)\frac{1}{R_{12}}dV_1 dV_2 d\tau +$$

$$+\frac{1}{c^2}\int \vec{j}_1(\vec{r}_1,t)\vec{j}_2(\vec{r}_2,\tau)\dot{\delta}(\tau - t + R_{12}/c)\frac{1}{R_{12}}dV_1 dV_2 d\tau, \tag{7.17}$$

$$\mathcal{E}_{21} = -\int \dot{\rho}_2(\vec{r}_2,t)\rho_1(\vec{r}_1,\tau)\delta(\tau - t + R_{12}/c)\frac{1}{R_{12}}dV_1 dV_2 d\tau +$$

$$+\frac{1}{c^2}\int \vec{j}_2(\vec{r}_2,t)\vec{j}_1(\vec{r}_1,\tau)\dot{\delta}(\tau - t + R_{12}/c)\frac{1}{R_{12}}dV_1 dV_2 d\tau, \tag{7.18}$$

where the dot above $\rho$ means derivative w.r.t. $t$ and the dot above the $\delta$ function means derivative w.r.t. its argument. Again we see that, in general, $\mathcal{E}_{12}(t) \neq \mathcal{E}_{21}(t)$. Let now the time dependences of $\rho$ and $\vec{j}$ be separated in the same way as in (7.5). Then,

$$\mathcal{E}_{12} = -\int \dot{\rho}_1(t)\rho_1(\vec{r}_1)\rho_2(\vec{r}_2)\rho_2(\tau)\frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)dV_1 dV_2 d\tau +$$

$$+\frac{1}{c^2}\int j_1(t)\vec{j}_1(\vec{r}_1)\vec{j}_2(\vec{r}_2)j_2(\tau)\frac{1}{R_{12}}\dot{\delta}(\tau - t + R_{12}/c)dV_1 dV_2 d\tau, \tag{7.19}$$

$$\mathcal{E}_{21} = -\int \dot{\rho}_2(t)\rho_2(\vec{r}_2)\rho_1(\vec{r}_1)\rho_1(\tau)\frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)dV_1 dV_2 d\tau +$$

$$+\frac{1}{c^2}\int j_2(t)\vec{j}_2(\vec{r}_2)\vec{j}_1(\vec{r}_1)j_1(\tau)]\frac{1}{R_{12}}\dot{\delta}(\tau - t + R_{12}/c)dV_1 dV_2 d\tau, \tag{7.20}$$

Similarly to the interaction energies, we see that $\mathcal{E}_{12}(t) = \mathcal{E}_{21}(t)$ for arbitrary time dependences $\rho_1$ and $j_1$ coinciding with $\rho_2$ and $j_2$, i.e., when the conditions (7.5) are fulfilled.



### 7.3.2 Feld-Tai lemma

Direct evaluation of integrals entering into the Feld-Tai lemma gives

$$\mathcal{H}_{12} = \epsilon_{ijk}\int j_{1i}(\vec{r}_1,t)\frac{\partial A_{2k}}{\partial x_{1j}}dV_1 = \epsilon_{ijk}\int j_{1i}(\vec{r}_1,t)j_{2k}(\vec{r}_2,\tau)\frac{\partial}{\partial x_{1j}}\frac{1}{R_{12}}\delta(\tau - t + \frac{R_{12}}{c})dV_1 dV_2 d\tau,$$

$$\mathcal{H}_{21} = \epsilon_{ijk}\int j_{2i}(\vec{r}_2,t)\frac{\partial A_{1k}}{\partial x_{2j}}dV_2 = \epsilon_{ijk}\int j_{2i}(\vec{r}_2,t)j_{1k}(\vec{r}_1,\tau)\frac{\partial}{\partial x_{2j}}\frac{1}{R_{12}}\delta(\tau - t + \frac{R_{12}}{c})dV_1 dV_2 d\tau. \tag{7.21}$$

Here $\epsilon_{ijk}$ is the unit antisymmetrical tensor of the third rank. When the time dependences in current densities are separated ($\vec{j}(r,t) = j(t)\vec{j}(r)$), these equations are reduced to

$$\mathcal{H}_{12} = \epsilon_{ijk}j_1(t)\int j_2(\tau)j_{1i}(\vec{r}_1)j_{2k}(\vec{r}_2)\frac{\partial}{\partial x_{1j}}\frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)dV_1 dV_2 d\tau,$$

$$\mathcal{H}_{21} = \epsilon_{ijk}j_2(t)\int j_1(\tau)j_{2i}(\vec{r}_2)j_{1k}(\vec{r}_1)\frac{\partial}{\partial x_{2j}}\frac{1}{R_{12}}\delta(\tau - t + R_{12}/c)dV_1 dV_2 d\tau. \tag{7.22}$$

Obviously.

$$\mathcal{H}_{12} = \mathcal{H}_{21}$$

when the arbitrary time dependences of sources 1 and 2 coincide ($j_1(t) = j_2(t)$).

### 7.3.3 The physical meaning of the Lorentz and Feld-Tai lemma for the interacting current sources

We conclude: the Lorentz and Feld-Tai lemmas are fulfilled when the following two conditions are satisfied:

i) Time dependences are separated from space variables in the charge-current densities. This means that the time dependence should be the same for all space points of a particular source.

ii) The separated time dependence is the same for sources 1 and 2.

The physical meaning of the Lorentz lemma is as follows. The time- dependent magnetic flux penetrating a particular turn of winding, creates an electric field directed along this turn. Being summed, they give potential difference between ends of the winding if it is not closed and induce the current in the winding if it is closed. This voltage (or current) can be measured. To obtain voltage, we omit in $\mathcal{E}_{12}$ the time dependent current force $I_1$ (not the current density $\vec{j}_1$). Thus obtained $\mathcal{E}_{12}(t)$ gives time-dependent voltage induced in the winding 1 by the time-dependent current flowing in the winding 2. Similarly, if we omit in $\mathcal{E}_{21}$ the time dependent current force $I_2$, then $\mathcal{E}_{21}(t)$ gives time-dependent voltage induced in the winding 2 by the time-dependent current flowing in the winding 1. Thus obtained $\mathcal{E}_{12}$ and $\mathcal{E}_{21}$ coincide if $I_1 = I_2$. We observe that in the first case 1 is a receiver and 2 is a transmitter. In the second case, the situation is opposite. This means that an induced voltage is invariant under the replacement of the detector and transmitter.

We illustrate this using point-like TS and current loop as an example. Turning to (3.16) and (3.46), we observe that $f_T$ and $f_L$ in $\mathcal{E}_{TL}$ may be presented as

$$f_T = \frac{\pi N I_T d_T R^2}{2}\tilde{f}_T, \quad f_L = \pi I_L d_L^2 \tilde{f}_L$$

where $I_T$ and $I_L$ are the current forces in TS and current loop, resp.; $\tilde{f}_T$ and $\tilde{f}_L$ are their time dependences. Omitting the factor $I_T\tilde{f}_T$, we get for the voltage induced in TS

$$V_{TL} = -\frac{\pi^2 N d_T R^2 d_L^2}{2c^5 R_{TL}^2} D^{(3)}(I_L\tilde{f}_L).$$

In the same way, omitting the factor $I_L\tilde{f}_L$, we get for the voltage induced in a current loop

$$V_{LT} = -\frac{\pi^2 N d_T R^2 d_L^2}{2c^5 R_{TL}^2} D^{(3)}(I_T\tilde{f}_T).$$

We see that, indeed, $V_{TL} = V_{LT}$ if $I_T\tilde{f}_T = I_L\tilde{f}_L$, i.e., when the time-dependent currents flowing in a current loop and toroidal solenoid are the same.
The physical meaning of the Feld-Tai lemma for interacting current sources is not clear to us. A time-dependent electric field penetrating a particular turn of winding, creates the magnetic field directed along this turn. If the free magnetic charges existed, then integrals entering into the Feld-Tai lemma (after omitting the corresponding factors as in the Lorentz lemma) would gave the magnetic voltage between the ends of the winding (if it is not closed). Their equality would gave the symmetry between the transmitter and receiver. Since the monopoles up to now were not found, this interpretation of the Feld-Tai lemma has no relation to reality. However, Lakhtakia [31] and Monzon [30], seem to have found numerous application of the Feld-Tai lemma.

### 7.3.4 Another viewpoint on the Lorentz and Feld-Tai lemmas

In the Fourier representation ($\vec{E}(t) = \int \vec{E}(\omega)\exp(i\omega t)d\omega$, etc.) the *curl* parts of Maxwell equations look like

$$curl\vec{E} = -ik\vec{H}, \quad curl\vec{H} = ik\vec{E} + \frac{4\pi}{c}\vec{j}, \quad k = \omega/c.$$

Then, the Lorentz and Feld-Tai lemmas are satisfied trivially. For example, the proof of the Lorentz lemma without appeal to the Maxwell equations takes three lines

$$\mathcal{E}_{12} = \int \vec{j}_1(\vec{r}_1)\vec{E}_{12}(\vec{r}_2)dV_1 = \int \vec{j}_1(\vec{r}_1)[-\vec{\nabla}\Phi_{12}(\vec{r}_1) - ik\vec{A}_{12}(\vec{r}_1)]dV_1 =$$

$$= -i\omega\int[\rho_1(\vec{r}_1)\Phi_{12}(\vec{r}_1) + \frac{1}{c}\vec{j}_1(\vec{r}_1)\vec{A}_{12}(\vec{r}_1)]dV_1 =$$

$$= -i\omega\int[\rho_1(\vec{r}_1)\rho_2(\vec{r}_2) + \frac{1}{c^2}\vec{j}_1(\vec{r}_1)\vec{j}_2(\vec{r}_2)]\frac{\exp(-ikR_{12})}{R_{12}}dV_1dV_2 = \mathcal{E}_{21}.$$

Therefore, the Lorentz and Feld-Tai lemmas may be viewed as the integral relations between the Fourier transforms of the current densities and field strengths. This, in its turn, may be used to derive new identities. For example, multiplying $\mathcal{E}_{12}$ by $\exp(i\omega t)$ and integrating over $\omega$, one gets

$$\int \vec{j}_1(\vec{r}_1,\omega)\vec{E}_{12}(\vec{r}_1,\omega)\exp(i\omega t)dV_1d\omega =$$



$$= \frac{1}{4\pi^2}\int \vec{j}_1(\vec{r}_1,t')\vec{E}_{12}(\vec{r}_1,t'')\exp[i\omega(t-t'-t'')]dV_1d\omega dt'dt'' =$$

$$= \frac{1}{2\pi}\int \vec{j}_1(\vec{r}_1,t')\vec{E}_{12}(\vec{r}_1,t'')\delta(t-t'-t'')dV_1dt'dt'' = \frac{1}{2\pi}\int \vec{j}_1(\vec{r}_1,t-t')\vec{E}_{12}(\vec{r}_1,t')dV_1dt'. \quad (7.23)$$

Performing the same operation with $\mathcal{E}_{21}$ and equalizing the result to (7.23), one arrives at

$$\int \vec{j}_1(\vec{r}_1,t-t')\vec{E}_{12}(\vec{r}_1,t')dV_1dt' = \int \vec{j}_2(\vec{r}_2,t-t')\vec{E}_{21}(\vec{r}_2,t')dV_2dt'. \quad (7.24)$$

This equation was obtained by Feld ([49]). We make one step further, excluding electric strengths. Then, the LHS of (7.24) is reduced to

$$\frac{1}{2\pi}\frac{\partial}{\partial t}\int[\rho_1(\vec{r}_1,t-t')\rho_2(\vec{r}_2,t'-\frac{R_{12}}{c}) + \frac{1}{c^2}\vec{j}_1(\vec{r}_1,t-t')\vec{j}_2(\vec{r}_2,t'-\frac{R_{12}}{c})]\frac{1}{R_{12}}dt'dV_1dV_2.$$

Therefore, the following equation should be satisfied:

$$\int[\rho_1(\vec{r}_1,t-t')\rho_2(\vec{r}_2,t'-\frac{R_{12}}{c}) + \frac{1}{c^2}\vec{j}_1(\vec{r}_1,t-t')\vec{j}_2(\vec{r}_2,t'-\frac{R_{12}}{c})]\frac{1}{R_{12}}dt'dV_1dV_2 =$$

$$= \int[\rho_2(\vec{r}_2,t-t')\rho_1(\vec{r}_1,t'-\frac{R_{12}}{c}) + \frac{1}{c^2}\vec{j}_2(\vec{r}_2,t-t')\vec{j}_1(\vec{r}_1,t'-\frac{R_{12}}{c})]\frac{1}{R_{12}}dt'dV_1dV_2. \quad (7.25)$$

Performing the same operation for the integrals entering into the Feld-Tai lemma, one gets

$$\int \exp(i\omega t)\vec{j}_1(\vec{r}_1,\omega)\vec{H}_{12}(\vec{r}_1,\omega)d\omega dV_1 = \frac{1}{2\pi}\int \vec{j}_1(\vec{r}_1,t-t')\vec{H}_{12}(\vec{r}_1,t')dt'dV_1 =$$

$$= -\frac{1}{2\pi c}\int \frac{1}{R_{12}}curl\vec{j}_1(\vec{r}_1,t-t')\vec{j}_2(\vec{r}_2,t'-R_{12}/c)dV_1dV_2dt'.$$

Therefore, the following equalities should be fulfilled

$$\int \vec{j}_1(\vec{r}_1,t-t')\vec{H}_{12}(\vec{r}_1,t')dt'dV_1 = \int \vec{j}_2(\vec{r}_2,t-t')\vec{H}_{21}(\vec{r}_2,t')dt'dV_2,$$

$$\int \frac{1}{R_{12}}curl\vec{j}_1(\vec{r}_1,t-t')\vec{j}_2(\vec{r}_2,t'-R_{12}/c)dV_1dV_2dt' =$$

$$= \int \frac{1}{R_{12}}curl\vec{j}_2(\vec{r}_2,t-t')\vec{j}_1(\vec{r}_1,t'-R_{12}/c)dV_1dV_2dt'. \quad (7.26)$$

It is important that Eqs.(7.24)-(7.26), contrary to the equations defining the Lorentz and Feld-Tai lemmas, are satisfied for any charge-current densities. No assumption on the separation of space and time dependences as well the equality of time dependences for two interacting sources is needed.
As the author is not the specialist in the applied aspects of reciprocity-like theorems, he cannot appreciate the meaning of the results obtained. On the other hand, there are outstanding experts in this field (A.Lakhtakia, J.C.R. Monzon and others). It would be nice to hear their opinion on the treated questions.

## 8 Discussion and Conclusion

Now we analyze assumption on the separability time and spatial variables in charge-current densities. Take at first the single circular current loop. Since there are no other turns, there is no resistive or capacity connections between them. Therefore, the current is the same along the whole wire (due to the continuity equation $div\vec{j} = 0$) and the time dependence is clearly separated from the space variables. On the other hand, consider the winding with many overlapping turns, e.g. the toroidal solenoid. If the turns are close to each other, there is a finite capacitance between them. For high frequencies, the leakage currents appear between particular turns and the current will be changed along the wire. This does not have any relation to the violation of the continuity equation $div\vec{j} = 0$. It will be fulfilled due to the presence of other $\vec{j}$ components, having the direction different from that of wire. Since the current density changes along the wire, the time dependence is not separated now. This should lead to the violation of reciprocity theorem. We conclude: the violation of the reciprocity is possible for high frequencies and large number of overlapping coils.

We briefly enumerate the main results obtained:

1. We evaluated the electromagnetic field of the toroidal solenoid with a periodic current in its winding. Various particular cases are considered and conditions for their validity are given.

2. We applied the reciprocity theorem ( Lorentz and Feld-Tai lemmas) to the electromagnetic fields of time dependent electric dipole, current loop, toroidal solenoid and higher order electromagnetic field sources. It is shown that the proportionality of time derivatives of EMF strengths to the EMF strengths themselves is not a necessary condition for the fulfillment of the reciprocity theorem.

3. The alternative proof of the reciprocity theorem is given. It is shown that the reciprocity theorem works for more general time dependences than it was suggested up to now. The conditions for its validity are reduced to the two following ones:
i) The time dependence should be separated from the spatial one in the charge-current densities of interacting sources;
ii) The time dependences of these sources should be the same.
These conditions are essentially the same as ones needed for the equality of action and reaction between two interacting electromagnetic sources.

## Acknowledgements.

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## Appendix

We begin with the well-known relation

$$\cos\nu\theta J_\nu(k\sqrt{d^2+R^2-2dR\cos\psi} = \sum_{-\infty}^{\infty} J_m(kR)J_{m+\nu}(kd)\cos m\psi, \quad R<d, \qquad (A.1)$$

where $\tan\theta = R\sin\psi/(d-R\cos\psi)$. For $R << d$, the angle $\theta$ may be put to zero. Then,

$$J_\nu(k\sqrt{d^2+R^2-2dR\cos\psi} \approx \sum_{-\infty}^{\infty} J_m(kR)J_{m+\nu}(kd)\cos m\psi, \quad R << d. \qquad (A.2)$$

We cannot put $R = 0$ in the r.h.s. of this equation, since for high frequencies, $kR$ may be large. Further ,

$$j_{2n+1}(ky) = \sqrt{\frac{\pi}{2ky}}J_{2n+3/2}(ky) \approx \sqrt{\frac{\pi}{2kd}}J_{2n+3/2}(ky).$$

Here we changed $y$ by $d$ outside the Bessel function. This is possible since $R << d$. Therefore,

$$j_{2n+1}(ky) \approx \sum_{-\infty}^{\infty}(-1)^m J_m(kR)J_{2n+1+m}(kd)\cos m\psi, \quad R << d. \qquad (A.3)$$

and

$$\int_0^{2\pi} j_{2n+1}(ky)\sin^2\psi d\psi = \pi\{J_0(kR)j_{2n+1}(kd) - \frac{1}{2}J_2(kR)[j_{2n+3}(kd) + j_{2n-1}(kd)]\}. \qquad (A.4)$$

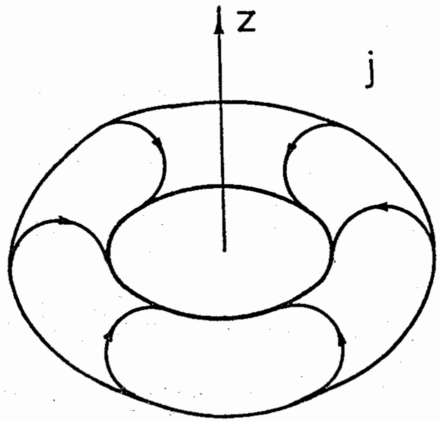


Fig.1: The poloidal current flowing on the torus surface.

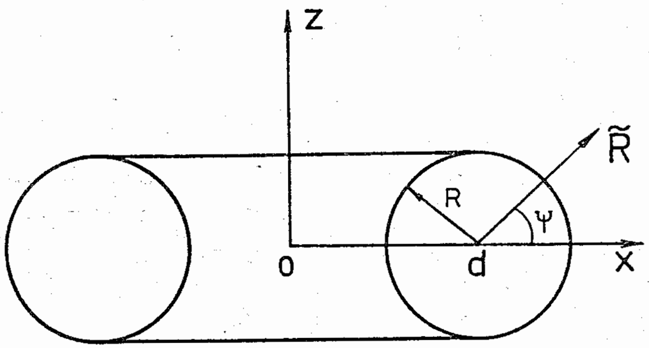


Fig.2: The coordinates $\tilde{R}, \psi$ parametrizing the torus.



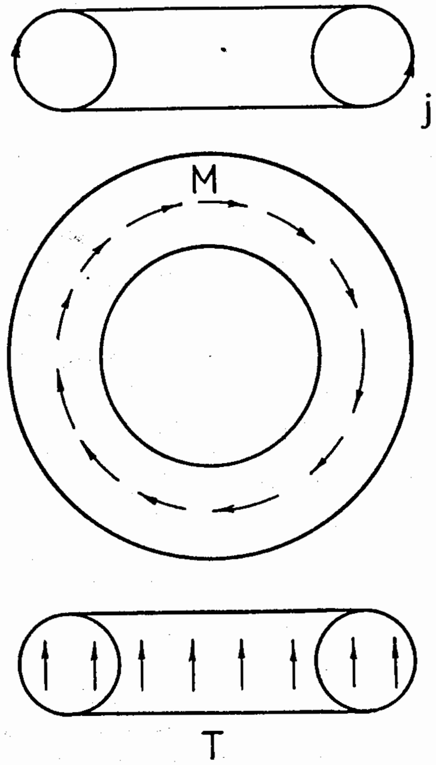


Fig.3: The poloidal current $\vec{j}$ flowing on the torus surface is equivalent to the magnetization $\vec{M}$ confined to the interior of the torus and to the toroidization $\vec{T}$ directed along the torus axis.

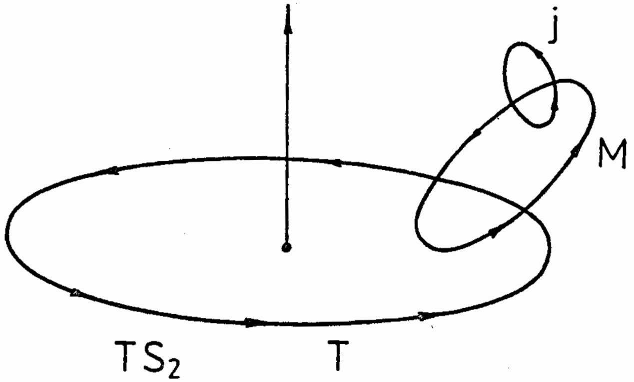


Fig.4: Torodal source of the second order is obtained if, instead of each particular turn of a usual toroidal solenoid, a new infinitely thin toroidal solenoid $ts$ is substituted with the current $\vec{j}$ in its winding. It generates the magnetization $\vec{M}$ covering the surface of the original toroidal solenoid and directed along its meridians. The complete magnetization from all $ts$ generates the closed tube of toroidal moments $T$ filling the interior of the original toroidal solenoid and generating in its turn the second order toroidal moment shown by the vertical arrow.



# References


[1] Peshkin M. and Tonomura A., 1989, The Aharonov-Bohm Effect (Berlin, Springer).

[2] Afanasiev G.N., 1988, J.Phys.A, 21, 2095; Afanasiev G.N., 1989, Phys. Lett. A, 142, 222; Afanasiev G.N. and Shilov V.M., 1993, J.Phys.A, 26, 743.

[3] Zeldovich Ya.B., 1957, JETP, 33. 1531.

[4] Dubovik V.M. and Tugushev V.V., 1990, Phys, Rep., 187, 145.

[5] Vaidman L., 1990, Amer. J. Phys., 58, 278.

[6] Luboshitz V.L. and Smorodinsky Ya.A, 1978, JETP, 75, 40.

[7] Afanasiev G.N., 1987, J. Comput. Phys., 69, 196 .

[8] Afanasiev G.N., 1993, J.Phys. A, 26, 731.

[9] Afanasiev G.N., Nelhiebel M. and Stepanovsky Yu.P., 1996, Physica Scripta, 54, 417.

[10] Afanasiev G.N., 1993, Physica Scripta, 48, 385.

[11] Page C.H., 1971, Amer.J.Phys., 39,1039; Amer.J.Phys., 39,1206.

[12] Afanasiev G.N., 1990, J.Phys. A, 23, 5755.

[13] Afanasiev G.N. and Dubovik V.M., 1992, J.Phys. A, 25, 4869.

[14] Afanasiev G.N., 1994, J.Phys. A, 27, 2143.

[15] Nevessky N.E., 1993, Electricity, No 12, 49 (In Russian).

[16] Afanasiev G.N. and Stepanovsky Yu. P., 1995, J.Phys. A, 28, 4565.

[17] Afanasiev G.N, 1999, Topological Effects in Quantum Mechanics (Dordrecht: Kluwer Acad. Publ.).

[18] Rayleigh J.W.S., 1945. Theory of Sound, vol. 1 (New York, Dover).

[19] Rayleigh J.W.S., 1900, Phil.Mag., 49, 324.

[20] Helmholtz H., 1886, Crelles J.. 100. 213.

[21] Lorentz H.A., 1895-1896, Amsterdammer Akademie der Wetenschappen, 176, No 4.

[22] Kami Y., 1992, IEICE Trans. Commun., E75-B, 115.

[23] Kim M.J., 1988,Applied Optics,27,2645.

[24] Bouche D. and Mittea R., 1993, Radio Science, 28, 527.

[25] Qin L.C. and Goodman P., 1989, Ultramicroscopy, 27,115.

[26] Alpert Ya.L., Ginzburg V.L., Feinberg E.L., 1953, Propagation of Radio Waves (Moscow: GITTL), in Russian.

[27] Rumsey V.H., 1954, Phys. Rev., 94, 1483.

[28] Feld Ya.N., 1992, Sov.Phys.Doklady, 37, 235.

[29] Tai C.T., 1992, IEEE Trans. Antennas Propagation, 40, 675.

[30] Monzon J.C., 1996, IEEE Trans. Microwave Theory Tech., 44, 10.

[31] Lakhtakia A., 1994, Beltrami fields in chiral media (Singapore, World Scientific).

[32] Rose M.E., 1955, Multipole fields (New York, Wiley).

[33] Blatt J.M. and Weisskopf, 1952, Theoretical Nuclear Physics (New York: Wiley).

[34] Jackson J.D., 1975, Classical Electrodynamics (New York: Wiley).

[35] Watson G.N., 1958, A treatise on the Theory of Bessel Functions (Cambridge: Cambr. Univ. Press).

[36] Ginzburg V.L., 1985, Izn. Vysch. Uchebn. Zaved., ser. Radiofizika, 28, 1211.

[37] Lorentz H.A., 1895, Versuch einer Theorie der electrischen und optischen Ersheinungen in bewegter Korpern (Leiden: E.J.Brill).

[38] Cullwick E.G.,1957, Electromagnetism and Relativity (London: Longmans), Chapter 17.

[39] French A.P., 1968, Special Relativity (New York: W.W. Norton and Co),p.224.

[40] Jefimenko O.D., 1992, Amer. J. Phys., 61, 218.

[41] Cornille P., 1995, Can. J.Phys., 73, 619.

[42] Rambaut M., 1991, Phys.Lett. A, 154, 210.

[43] Cornille P., 1989, J.Phys.A, 22, 4075.

[44] Ternan J.G.,1985, J.Appl.Phys.,57, 1743.

[45] Graneau P., Thompson D.S. and Morrill S.L., 1990, Phys. Lett.A, 145, 396.

[46] Peoglos V., 1988, J.Phys.D, 21, 1055.

[47] Ginzburg V.L. and Tsytovich V.N., 1985, Sov. Phys. JETP, 61, 48.

[48] Afanasiev G.N. and Stepanovsky Yu.P, 2000, Physica Scripta, 61, 704.

[49] Feld Ya.N., 1991, Doklady Akad. Nauk SSSR, 318, 325.

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