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Новые аналитические решения в R-матричной теории рассеяния

Среди точно решаемых задач квантовой механики широко известны случаи движения частицы в кулоновском, осцилляторном, прямоугольном и нескольких других потенциалах. Класс аналитических решений уравнения Шредингера существенно расширяется с помощью баргмановских потенциалов. В данной работе сочетание формализма R-матричной теории с техникой баргмановских потенциалов дополняет множество точно решаемых квантовых моделей для прямой и обратной задач рассеяния, отвечающих целому семейству потенциалов конечного радиуса действия.

Полученные результаты могут быть использованы также для приближенного восстановления взаимодействий по данным рассеяния.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Plekhanov E.B., Suzko A.A., Zakhariev B.N. **E4 - 12913**

New Analytical Solutions in R-Matrix Scattering Theory

The combination of R-matrix formalism with the technique of Bargmann potentials gives the class of exactly solvable quantum models for direct and inverse scattering problems with finite range interaction.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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INTRODUCTION

The well-known analytical solutions of Schrödinger equation (i.e., solutions of the direct problem) have been found by choosing the most simple forms of potentials V ($V = \mathrm{const}$; r; r^2; r^{-1}; r^{-2}, etc.). If the scattering data S correspond exactly to the given potential, then the connection $S \to V$ is simultaneously a solution both for the direct $(V \to S)$ and the inverse $(S \to V)$ scattering problems. It is natural therefore that the class of above-mentioned potentials can be enlarged provided the simplest forms of S are chosen, for which exact solutions of the inverse scattering problem equation

$$K(x, y) + Q(x, y) + \int K(x, z) Q(z, y)\, dz = 0 \tag{1}$$

are known. The integration limits in (1) depend on specific conditions of the formulation of the inverse problem; the kernel Q is determined by S, and the function K, a solution of (1), defines the potential V and the corresponding wave function ψ ($\overset{\circ}{\psi}$ corresponds to $V \equiv 0$):

$$V(x) = -2 \frac{d^2}{dx^2} K(x, x); \tag{2}$$

$$\Psi(x) = \overset{\circ}{\Psi} + \int K(x, y) \overset{\circ}{\Psi}(y)\, dy. \tag{3}$$

For degenerated kernels

$$Q(x, y) = \sum_n^N \phi_n(x) \cdot f_n(y) \tag{4}$$

eq. (1) reduces to a system of algebraic equations*, whose

* The dependence on y-variable can be removed from the integral in (1) and instead of the integral equation (1) for K we get linear algebraic equations for functions $F_n(x) = \int K(x,z) \phi_n(z) dz$.

solution gives a simple expression for the potential

$$V(x) = -2\frac{d^2}{dx^2}\ln \operatorname{Det} M(x), \tag{5}$$

where M is a matrix of coefficients of this system.

The kernels Q of the form (4) are obtained if, for example, the scattering S-function has a form of the ratio of two polynomials in k [1-3]:

$$S(k) = \prod_n^N \frac{k+i\alpha_n}{k+i\beta_n}\cdot\frac{k-i\beta_n}{k-i\alpha_n}. \tag{6}$$

The corresponding potentials are called the Bargmann potentials.

V.V.Malyarov and his collaborators[4] have applied this formalism to the case when S has the form (6) but with a variable $\lambda = \ell + ½$, where ℓ is the orbital momentum.

Up to now, however, this technique was not used in the R-matrix scattering theory. Equations of the inverse problem were written in this case in [5]*.

In this paper analytical solutions for the direct and inverse problem in the case of finite-range interaction $V(r \geq a) = 0$ (when the R-matrix formalism is valid) are derived.

THE EXACT SOLUTIONS IN R-MATRIX SCATTERING THEORY

The energy dependence of R-matrix

$$R(E) = \sum_\lambda \frac{\gamma_\lambda^2}{E_\lambda - E} \tag{7}$$

is determined by an infinite number of constants: E_λ - positions of resonances, and their reduced widths, γ_λ^2. In terms of these parameters the kernel of the integral equation (1) is written in the form [5]:

$$Q(r,r') = \sum_\lambda^\infty \gamma_\lambda^2 \overset{\circ}{\phi}(E_\lambda, r)\overset{\circ}{\phi}(E_\lambda, r') - \sum_\lambda^\infty \overset{\circ}{\gamma}{}_\lambda^2 \overset{\circ}{\phi}(\overset{\circ}{E}_\lambda, r)\overset{\circ}{\phi}(\overset{\circ}{E}_\lambda, r'), \tag{8}$$

where $\overset{\circ}{\phi}$ are known solutions of the Schrödinger equation

* In paper [6] devoted to the inverse problem, the discrete parametrization of scattering data by means of the R-matrix theory is also used, but the usual integral Gelfand-Levitan-Marchenko equation is considered.

$(\overset{\circ}{\phi}(a)=0;\ \overset{\circ}{\phi}{}'(a)=1)$ with the reference potentials $\overset{\circ}{V}$ (which is usually taken to be zero), to which R-matrix parameters $\overset{\circ}{E}_\lambda, \overset{\circ}{\gamma}_\lambda$ correspond.

It is obvious from (8) that if only the finite number of parameters E_λ, γ_λ, is different from corresponding values for potential $\overset{\circ}{V}$, the kernel Q becomes degenerate ("separable")*.

The most simple form of Q is obtained when only for a single value $\lambda = \lambda_1$, $\gamma_{\lambda_1} \neq \overset{\circ}{\gamma}_{\lambda_1}$, $(\delta\gamma_{\lambda_1}^2 = \gamma_{\lambda_1} - \overset{\circ}{\gamma}_{\lambda_1}$ and all the $E_\lambda = \overset{\circ}{E}_\lambda$):

$$Q(r,r') = \delta\gamma_{\lambda_1}^2 \overset{\circ}{\phi}(\overset{\circ}{E}_{\lambda_1}, r)\overset{\circ}{\phi}(\overset{\circ}{E}_{\lambda_1}, r'). \tag{9}$$

Substituting (9) into (1), multiplying both sides of the equation by $\overset{\circ}{\phi}(\overset{\circ}{E}_{\lambda_1}, r')$ and integrating it over r', we get an algebraic equation for the function

$F_{\lambda_1}(r) = \int_r^a \overset{\circ}{\phi}(\overset{\circ}{E}_{\lambda_1}, r'')K(r,r'')dr''$.

The solution of this equation together with (1) gives:

$$K(r,r') = -\frac{\delta\gamma_{\lambda_1}^2 \overset{\circ}{\phi}(\overset{\circ}{E}_{\lambda_1}, r)\overset{\circ}{\phi}(\overset{\circ}{E}_{\lambda_1}, r')}{1+\delta\gamma_{\lambda_1}^2 \int_{r'}^a \overset{\circ}{\phi}{}^2(\overset{\circ}{E}_{\lambda_1}, r'')dr''}. \tag{10}$$

With this function K we obtain the potential and wave function ϕ:

$$V(r) = \overset{\circ}{V}(r) + \frac{4\delta\gamma_{\lambda_1}^2 \overset{\circ}{\phi}(\overset{\circ}{E}_{\lambda_1}, r)\overset{\circ}{\phi}{}'(\overset{\circ}{E}_{\lambda_1}, r)}{1+\delta\gamma_{\lambda_1}^2 \int_r^a \overset{\circ}{\phi}{}^2(\overset{\circ}{E}_{\lambda_1}, r')dr'} + \frac{2(\delta\gamma_{\lambda_1}^2)^2 \overset{\circ}{\phi}{}^4(\overset{\circ}{E}_{\lambda_1}, r)}{(1+\delta\gamma_{\lambda_1}^2 \int_r^a \overset{\circ}{\phi}{}^2(\overset{\circ}{E}_{\lambda_1}, r')dr')^2}, \tag{11}$$

$$\phi(E,r) = \overset{\circ}{\phi}(E,r) - \frac{\delta\gamma_{\lambda_1}^2 \overset{\circ}{\phi}(\overset{\circ}{E}_{\lambda_1}, r)\int_r^a \overset{\circ}{\phi}(\overset{\circ}{E}_{\lambda_1}, r')\overset{\circ}{\phi}(E,r')dr'}{1+\delta\gamma_{\lambda_1}^2 \int_r^a \overset{\circ}{\phi}\ (\overset{\circ}{E}_{\lambda_1}, r'')dr''}. \tag{12}$$

The integrals in (11), (12) for simple $\overset{\circ}{V}$ (particularly for $\overset{\circ}{V} \equiv P$) can be calculated exactly.

* The separable kernels give local potentials but not separable ones.

It is noteworthy that the simplest case (9)-(12) in the R-matrix theory corresponds to S-function of the type (6) with the infinite number of factors.

In the general case, when several parameters $E_\lambda, \gamma_\lambda$ are different from $\overset{\circ}{E}_\lambda, \overset{\circ}{\gamma}_\lambda$ there remains only a finite number of terms in (8) after cancellation of those with $E_\lambda = \overset{\circ}{E}_\lambda$, $\gamma_\lambda = \overset{\circ}{\gamma}_\lambda$, and for V we get expression (5) with matrix M which is calculated within the R-matrix scattering theory.

MULTICHANNEL EQUATION

Bargmann potentials in the case of several coupled channels were obtained by Cox /7/. Here it will be done in the R-matrix approach. The relevant derivations will be carried out by following the same scheme as in the single-channel examples above.

It is convenient to treat the multichannel problems as a matrix generalization of single-channel ones. The system of the Schrödinger equations is then written in the form ($\hbar = 1$, $m = ½$):

$$-\hat{\phi}''(\hat{K}, r) + \hat{V}(r)\hat{\phi}(\hat{K}, r) = K^2\hat{\phi}(K, r), \tag{13}$$

where $\hat{\phi}$, $\hat{V}$, $\hat{K}^2 = \hat{E}$ are $n\times n$-matrices (it is assumed that n is the number of channels), $K_{ij} = k_i\delta_{ij}$; $k_1^2 = k_i^2 + \Delta_i^2$; $\Delta_i \geq 0$ is the threshold energy of an i-th channel, and n columns of the solution matrix $\hat{\phi}$ correspond to n linearly independent boundary conditions (which we choose so that the matrix $\overset{\circ}{\phi}$, corresponding to $V(x)\equiv 0$, be diagonal). The inverse-problem equations will be the same as (1), with $n\times n$ matrices $\hat{K}$ and $\hat{Q}$. The kernel $\hat{Q}$ in the multichannel R-matrix theory has the form:

$$\hat{Q}(r, r') = \sum_\lambda \hat{\overset{\circ}{\Phi}}(E_\Lambda, r)\hat{\Gamma}_\lambda \hat{\Gamma}_\lambda^T \hat{\overset{\circ}{\Phi}}(E_\lambda, r') - \sum_\lambda \hat{\overset{\circ}{\Phi}}(\overset{\circ}{E}_\lambda, r)\hat{\overset{\circ}{\Gamma}}_\lambda \hat{\overset{\circ}{\Gamma}}{}_\lambda^T \hat{\overset{\circ}{\Phi}}(\overset{\circ}{E}_\lambda, r'), \tag{14}$$

where $\hat{\Gamma}$ is a column-vector of partial reduced widths amplitudes $\gamma_{a\lambda}$ in channels a ; and $\hat{\Gamma}^T$ the corresponding row (T means the matrix transposition).

Supposing that only a finite number of resonance parameters $\hat{E}_\lambda$, $\hat{\Gamma}_\lambda$ are different from $\hat{\overset{\circ}{E}}_\lambda$, $\hat{\overset{\circ}{\Gamma}}_\lambda$ (for $\hat{\overset{\circ}{V}} \equiv 0$) so that only N terms remain in (14), $\hat{Q}$ can be written in the form /7/

$$\hat{Q}(r, r') = \hat{\mathcal{E}}^T(r)\hat{A}\hat{\mathcal{E}}(r'), \tag{15}$$

where $\hat{\mathcal{E}}$ is an $Nn\times n$-matrix (a column with N matrices $n\times n$) with elements expressed by $\hat{\overset{\circ}{\Phi}}(E_\lambda, r)$, $\hat{\overset{\circ}{\Phi}}(\overset{\circ}{E}_\lambda, r)$; $\hat{A}$ is an $Nn\times Nn$-matrix with elements $A_{pp'} = \delta_{pp'}a_p$ and a_p is an $n\times n$-matrix with elements determined by amplitudes of partial reduced widths $\gamma_{a\lambda}$, $\overset{\circ}{\gamma}_{a\lambda}$.

Substituting (15) into (1), multiplying the resulting equation from the right by $\hat{\mathcal{E}}(E, r')$, and integrating over r' (from r up to "a") we get the algebraic equation for $n\times Nn$-matrix $\hat{F}(r) = \int_r^a \hat{K}(r, r'')\hat{\mathcal{E}}^T(r'')dr''$:

$$\hat{F}(r) + \hat{\mathcal{E}}^T(r)\hat{A}\int_r^a \hat{\mathcal{E}}(r')\hat{\mathcal{E}}^T(r')dr' + \hat{F}(r)\hat{A}\int_r^a \hat{\mathcal{E}}(r')\hat{\mathcal{E}}^T(r')dr' = 0, \tag{16}$$

which immediately gives $\hat{F}(r)$. Substituting $\hat{F}$ into (1) and taking into account eq. (15) we determine $\hat{K}$*:

$$\hat{K}(r, r') = -\hat{\mathcal{E}}^T(r)\hat{A}\frac{1}{1+\hat{A}\hat{I}}\hat{\mathcal{E}}(r'), \quad \text{where } \hat{I} = \int_r^a \hat{\mathcal{E}}(r')\hat{\mathcal{E}}^T(r')dr'. \tag{17}$$

The potential matrix $\hat{V}$ is obtained by differentiating $\hat{K}(r, r')$ using the rule of differentiation for the inverse matrix $\hat{M}^{-1}$: $(\hat{M}^{-1})' = \hat{M}^{-1}\hat{M}'\hat{M}^{-1}$.

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* In order to derive. eq. (17) the following permutation rule is used:

$$\frac{1}{(1+AI)^T} I = I\frac{1}{1+AI} \tag{18}$$

This can be easily verified: $A = AII^{-1}$; $I^{-1}(AI)^T = I^{-1}I^TA^T = A$, where we take into account that matrices I and A are symmetric. So $(1+AI)I^{-1} = I^{-1}(1+AI)^T$ and multiplying the last equation from the right and from the left by I we get: $I(1+AI) = (1+AI)^T I$. Multiplying finally this equation from the right by $(1+AI)^{-1}$ and from the left by $\{(1+AI)^T\}^{-1}$ we see that the above-mentioned permutation rule is correct.

APPROXIMATE SOLUTION OF INVERSE PROBLEM

It is convenient to use R-matrix Bargmann potentials for the approximate reconstruction of finite-range potentials $(V(r \geq a)=0)$, or if V has a known behaviour at $r \geq a$. It is more easy to find real parameters $E_\lambda, \gamma_\lambda$, corresponding to a given scattering function S(E), than the complex constants α_n, β_n in (6) /8/. The values E_λ are calculated as zeros of the denominator in the R-matrix, expressed in terms of S(E) /5/:

$$R(E) = \frac{I(a) - S(E)\cdot O(a)}{I'(a) - S(E)O'(a) - B/a\, I(a) + B/a\, S(E)O(a)}, \tag{19}$$

where I(a), O(a) are the incoming and outgoing waves at point $r = a$ (if $\overset{\circ}{V}(r \geq a)=0$, then $I(a) = e^{-ika}$, $O(a) = e^{ika}$). And γ_λ^2 are determined according to (7), (19):

$$\gamma_\lambda^2 = \lim_{E \to E_\lambda} \{R(e)(E - E_\lambda)\},$$

using the Lapital rule for calculating the limit of $\frac{0}{0}$ type.

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