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E4 - 11984

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# THE SCHWINGER VARIATIONAL PRINCIPLE IN THE QUANTUM MECHANICAL THREE-BODY PROBLEM

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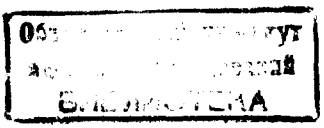

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Подкопаев А.П., Вжещонко Е., Зубарев А.Л. E4 - 11984

Вариационный принцип Швингера в квантово-механической задаче трех тел

В работе исследуется применение вариационного принципа Швингера (ВПШ) к задачам атомной ( $e^{+}H$ -рассеяние), мезоатомной ( $p(d\mu)$ -рассеяние) и ядерной физики (рассеяние пионов на дейтронах). Изучается сходимость вариационно-итерационного метода Швингера (ВИМШ). Показано, что из сходимости ВИМШ еще не следует сходимость именно к точному решению. Найдены примеры, когда реализуется такая патологическая ситуация. Метод сильной связи каналов переформулирован на основе ВПШ, что позволило провести эффективное суммирование по всем закрытым каналам. Полученные уравнения используются для решения задач $e^{+}H$ и $\pi d$ упругого рассеяния. Модифицированная связь 1s и 2p -состояний для задачи $e^{+}H$ -рассеяния дает результаты, близкие к точным. Рассматривается задача $\pi d$ -рассеяния с двухчастичными взаимодействиями в виде прямоугольных ям. Исследуется зависимость от длин $\pi N$ -рассеяния и эффективных радиусов. Показано, что вклад закрытых каналов в длину $\pi^{-}d$ -рассеяния составляет 30%.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1978

Podkopayev A.P., Wrzecionko J., Zubarev A.L. E4 - 11984

The Schwinger Variational Principle in the Quantum Mechanical Three-Body Problem

The Schwinger Variational Principle is applied to problems of atomic ($e^{+}H$ scattering) and mesoatomic ($p(d\mu)$ scattering) physics. The convergence of the Variational Iteration Method of Schwinger is investigated. The method of Strong Coupling of channels is reformulated on the basis of the Schwinger Variational Principle. The obtained equations are applied in the calculations of the low energy scattering parameters of the following processes: $e^{+}H \to e^{+}H$; $\pi d \to \pi d$.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1978



## 1. INTRODUCTION

Varioational calculations have two essential points: the choice of the trial function and the way of improving it systematically. Clearly, the best of all variational principles is that one which is least-dependent on the form of the trial function. As compared to the variational principles of Kohn, Hulten and Ritz, just the Schwinger variational principle (S.V.P.), possesses this property[1].

This paper is devoted to the systematic improvement of the trial function in the framework of the Schwinger Variational Principle. The developed methods are applied to the atomic problems (positron-hydrogen and $p(d\mu)$ -meson-deuterium scattering) and to the nuclear ones (pion-deuteron scattering).

In section 1 the problem of the convergence of the Schwinger variational iteration principle is investigated. It has been shown that in some cases there occurs the pathological convergence. It means that the iterational procedure is convergent, but not to the exact solution. Such a situation has been observed in positron-hydrogen ($e^{+}H$) and proton-deuteron $\mu$-mesoatom ($p(d\mu)$ scattering. In section 2 the method of the strong coupling of channels is reformulated on the bases of the (S.V.P.). For the wave function the approximate one-dimensional integral equations have been derived. These equations have been applied to calculate the low energy parameters for $e^{+}H$, $p(d\mu)$ and $\pi d$ scattering.

## 2. VARIATIONAL ITERATION METHOD

The Schwinger variational-iteration method (S.V.I.M.) is based on the integral equation for the wave function and in contrast to

other variational methods puts no constraints on the asymptotics of the trial function. In the two-body problem the SVIM for the scattering phase-shift can be formulated as follows[2]

$$|\psi_\ell^{(N+1)}> = \bar{G}_\ell^{(N)} V |\psi_\ell^{(N)}>,$$

$$\bar{G}_\ell^{(N)}(r,r') = G_\ell(r,r') - j_\ell(kr) j_\ell(kr') / (k \operatorname{tg}\delta_\ell^{(N)}),$$

$$G_\ell(r,r') = \begin{cases} k^{-1} j_\ell(kr) n_\ell(kr') & r \le r', \\ k^{-1} n_\ell(kr) j_\ell(kr') & r \ge r', \end{cases} \qquad (1)$$

$$\operatorname{tg}\delta_\ell^{(N)} = -k^{-1} (<j_\ell | V | \psi_\ell^{(N)}>)^2 / <\psi_\ell^{(N)} | (V - V G_\ell V) | \psi_\ell^{(N)}>,$$

where $\psi_\ell^{(N)}$ is the trial wave function in N-iteration, V is the potential operator, $j_\ell$, $n_\ell$ are the Riccati-Bessel functions, k - the wave vector of a scattering particle, $\delta_\ell^{(N)}$ the scattering phase shift in iteration N.

In paper[3] the iteration-separable method (I.S.M.) has been proposed. In this method the exact potential operator V is replaced by the first-rank separable operator

$$V_s^{(0)} = V |\phi_\ell^{(0)}> <\phi_\ell^{(0)}| V / <\phi_\ell^{(0)}| V |\phi_\ell^{(0)}>, \qquad (2)$$

where the initial trial function $\phi_\ell^{(0)}$ is taken to be an approximate solution of the equation

$$|\psi_\ell> = |j_\ell> + G_\ell V |\psi_\ell>. \qquad (3)$$

Equation (3) with the potential (2) is solved explicitly thus producing the first approximation of the wave function $\phi_\ell^{(1)}$, which is then used as a separable one in (2), etc. Inserting into (3) the separable potential in N-approximation

$$V_s^{(N)} = V |\phi_\ell^{(N)}> <\phi_\ell^{(N)}| V / <\phi_\ell^{(N)}| V |\phi_\ell^{(N)}> \qquad (4)$$

we obtain

$$|\phi_\ell^{(N+1)}> = |j_\ell> + G_\ell V |\phi_\ell^{(N)}> C^{(N)}(k). \qquad (5)$$



and

$$C^{(N)}(k) = <\phi_\ell^{(N)}| V |\phi_\ell^{(N+1)}> / <\phi_\ell^{(N)}| V |\phi_\ell^{(N)}>. \qquad (6)$$

Inserting (5) into (6) we express $C^{(N)}(k)$ in terms of the function of N iteration only:

$$C^{(N)}(k) = <j_\ell | V | \phi_\ell^{(N)}> / <\phi_\ell^{(N)}| (V - V G_\ell V) | \phi_\ell^{(N)}>. \qquad (7)$$

The ISM gives N-iteration of the scattering phase also in terms of the separable potential (4) as follows:

$$\operatorname{tg}\delta_\ell^{(N)} = -k^{-1} <j_\ell | V | \phi_\ell^{(N)}> C^{(N)}(k). \qquad (8)$$

The ISM (5) and (7) was constructed by assuming the fulfilment of the condition

$$<\phi_\ell^{(N)} | V | \phi_\ell^{(N)}> \neq 0 \qquad (9)$$

at each iteration step.

All the potentials of fixed sign fulfil this condition.

Now let us show that if the relation

$$<j_\ell | V | (\psi_\ell - \phi_\ell^{(N)})> = \epsilon, \qquad (10)$$

with $\epsilon \ll 1$ holds after N-iteration, then

$$A = <j_\ell | V | (\psi_\ell - C^{(N)} \phi_\ell^{(N)})> = O[\epsilon^2] \qquad (11)$$

Introducing the function $|\chi>$

$$|(\psi_\ell - \phi_\ell^{(N)})> = \epsilon |\chi>,$$

inserting (7) into (11), and making some transformations, we obtain

$$A = \epsilon^2 <\chi | (W - W_s^{(N)}) | \chi>,$$

$$W = V - V G_\ell V, \quad W_s^{(N)} = W |\phi_\ell^{(N)}> <\phi_\ell^{(N)}| W / <\phi_\ell^{(N)}| W |\phi_\ell^{(N)}>.$$

thus proving the relation (11).

Though the procedure of constructing the ISM and SVIM seems to be of different nature, both the methods provide identical expressions for the scattering phases that can be easily seen from exp. (1), (8) and (7). Hence, from the convergence of the ISM (5.7) the convergence of the SVIM follows. Though the scattering phases at each step of iteration coincide for both the methods, the wave functions $|\phi_\ell^{(N)}>$ and $|\psi_\ell^{(N)}>$, except for $|\phi_\ell^{(0)}>=|\psi_\ell^{(0)}>$, differ by the coefficient

$$|\psi_\ell^{(N)}> = [C^{(0)}C^{(1)}....C^{(N-1)}]^{-1}|\phi_\ell^{(N)}>$$

It can be shown (by analogy with the proof of (11)), that

$$|\mathrm{tg}\,\delta_\ell - \mathrm{tg}\,\delta_\ell^{(N+1)}| = O[|C^{(N)}-1|^2], \qquad (12)$$

if $|C^{(N)}-1|<<1$.

This relation provides an estimate for the results at each iteration step.

One may verify that the series $C^{(N)}(k)$ for the convergent ISM (5,7) should tend to unity:

$$\lim_{N\to\infty} C^{(N)}(k)=1. \qquad (13)$$

The expression (13), in turn, is a necessary condition of the convergence of ISM (5,7). If (13) holds, the integral equation for the function

$$|\phi_\ell> = \lim_{N\to\infty} |\phi^{(N)}>$$

and $|\psi_\ell>$ (3) coincide. Thus, the ISM produces not only the convergent scattering phases, as the SVIM, but also the convergent wave functions, so that at each iteration step the scattering phases are defined by the asymptotics of the wave function.

If the condition (9) is not fulfilled and for $N\to\infty$ the limit

$$\lim_{N\to\infty} <\phi_\ell^{(N)}|V|\phi_\ell^{(N)}> = 0, \qquad (14)$$

and the relation

$$<\psi_\ell|V|\psi_\ell> \neq 0,$$

for the exact wave function (3) take place, then the relation (13) cannot be proved, and the ISM may be divergent.



Therefore, the convergence of the SVIM under the condition (14) does not imply its convergence to the exact scattering phase. However, in the case (14), the condition (13) can, in principle, be fulfilled, and the SVIM will converge to the exact solution (provided $<\psi_\ell|V|\psi_\ell>=0$ ).

Many problems in the atomic and nuclear physics are considered on the basis of the method of strong coupling of channels. In this case we are dealing with the many-channel system of the Lippman-Schwinger equations [4]:

$$|F_\alpha> = |\phi_0>\delta_{\alpha,0} + G_\alpha V_{\alpha\alpha'}|F_{\alpha'}>, \qquad (15)$$

where $\phi_0$ is the wave function in the initial channel, $G_\alpha$ is the Green function in channel $\alpha$.

Repeated indices mean throughout the summation. The ISM can be simply generalized to the multi-channel problems both for elastic and inelastic scattering. For elastic processes the exact potential operator, $V_{\alpha\alpha'}$, in (15) is replaced by the separable potential of first rank

$$V^s_{\alpha\alpha'} = V_{\alpha\beta}|\chi_\beta><\chi_\gamma|V_{\gamma\alpha'}/<\chi_\delta|V_{\delta\rho}|\chi_\rho>, \qquad (16)$$

and the problem is solved explicitly. The ISM for multi-channel scattering is constructed by analogy with formulae (5,7,8).

As an illustration of the ISM for multi-channel problem, we have considered the elastic scattering of positrons on the hydrogen atom and of protons on the deuteron mesoatom:

$$e^+ + H \to e^+ + H$$

$$p + d\mu \to p + d\mu$$

We shall consider the scattering with zero total angular momentum of the system. Then, in this case the ISM is constructed as follows

$$|\chi^{(N+1)}> = |j_0(k_n r)>\delta_{n,1}\delta_{\ell,0} + G_{n\ell}V_{n\ell,n'\ell'}|\chi^{(N)}_{n'\ell'}>C^{(N)}(k_1), \qquad (17)$$

$$C^{(N)}(k_1) = \langle j_0 | V_{10,n_1\ell_1} | \chi^{(N)}_{n_1\ell_1} \rangle / \langle \chi^{(N)}_{n_2\ell_2} | (V_{n_2\ell_2,n_3\ell_3} -$$

$$- V_{n_2\ell_2,n_4\ell_4} G_{n_4\ell_4} V_{n_4\ell_4,n_3\ell_3}) | \chi^{(N)}_{n_3\ell_3} \rangle .$$

$$\mathrm{tg}\,\delta_0^{(N)} = -k_1^{-1} \langle j_0 | V_{10,n\ell} | \chi^{(N)}_{n\ell} \rangle C^{(N)}(k_1), \qquad (18)$$

where $n, \ell$ are quantum numbers specifying the state of the hydrogen atom ( $n$ is the principal quantum number, $\ell$ - orbital momentum), $k_n$ is the wave vector of a scattered particle in channel $n$ .

The SVIM gives the expression for the scattering phase which coincides with formula (18). The necessary condition for the convergence of ISM (17) in the multi-channel problem is the same as (13). The ISM cannot be constructed if there holds the condition of the type (14)

$$\lim_{N \to \infty} \langle \chi^{(N)}_{n\ell} | V_{n\ell,n'\ell'} | \chi^{(N)}_{n'\ell'} \rangle = 0 . \qquad (19)$$

In this case one may observe a pathological convergence of the SVIM. To check this, we calculated the scattering phases and scattering lengths for mentioned processes ($e^+H$ , $p(d\mu)$ ), taking into account different number of channels. The results of this calculation are listed in *Table 1*.

To obtain the convergent values for scattering lengths and phase shifts, with an accuracy better than 1%, from 3 to 15 iterations have been performed. It turns out that for $e^+H$ scattering the iteration process converges rapidly (3 to 4 iterations is enough), and for $p(d\mu)$ scattering the convergence is slow (sometimes more than 10 iterations are required). The results of the numerical calculations show that the sequence $C^{(N)}(k)$ does not always tend to unity (see 2,3,4,9,10 rows in *table 1*), though the convergence is achieved. In these cases the fulfilment of the condition (19) has been verified. It turns out that it is satisfied with the accuracy of calculation of the scattering phase shift or length. So in these cases the pathological convergence of the VIMS is observed: the calculated phase shift is not exact. In other cases (see 1,5,6,7,8 rows in *Table 1*) the sequence



*Table 1*

| E (ev) | Included channels | Number of channels | C | $\mathrm{tg}\,\delta_0$ or scattering length | Process |
|---|---|---|---|---|---|
| 0.000 | $\ell = 0$; $n = 1,2$ | 2 | 1.011 | 0.562(0.564) | $e^+H$ |
| 0.000 | $\ell = 0,1$; $(\ell+1) \le n \le 2$ | 3 | 0.373 | 0.171 | |
| 0.544 | $\ell = 0,1$; $(\ell+1) \le n \le 2$ | 3 | 0.641 | -0.0533(-0.0426) | |
| 0.896 | $\ell = 0,1$; $(\ell+1) \le n \le 6$ | 11 | 0.545 | -0.0584 | |
| 2.176 | $\ell = 0,1$; $(\ell+1) \le n \le 2$ | 3 | 0.908 | -0.1476(-0.1472) | |
| 4.896 | $\ell = 0,1$; $(\ell+1) \le n \le 2$ | 3 | 0.978 | -0.2462(-0.2460) | |
| 4.896 | $\ell = 0,1$; $(\ell+1) \le n \le 3$ | 5 | 0.965 | -0.231 | |
| 4.896 | $\ell = 0,1$; $(\ell+1) \le n \le 6$ | 11 | 0.943 | -0.218 | |
| 0.000 | $\ell = 0,1$; $(\ell+1) \le n \le 2$ | 3 | 0.145 | 0.339 | $p(d\mu)$ |
| 10.000 | $\ell = 0,1$; $(\ell+1) \le n \le 2$ | 3 | 0.238 | -0.0726 | |

In brackets there are given the values obtained by the numerical calculation of the equations of the channel strong coupling.

$C^{(N)}(k)$ tends to unity. At the same time, the condition (19) is not satisfied. So, in these cases we hope that calculated effects (sc. lengths or phase shifts) converge to the exact one.

As zero approximations, we took

$$|\chi^{(0)}_{n\ell}> = |j_0> \delta_{n,1}\,\delta_{\ell,0} \tag{20}$$

and the set of wave functions of the hydrogen atom for discrete spectrum

$$|\chi^{(0)}_{n\ell}> = |\psi_{n\ell}> \quad . \tag{21}$$

For the case when the pathological convergence of the SVIM takes place, the sequences $C^{(N)}$ and $tg\delta_0^{(N)}$ tend to the values independent of the choice of the initial trial function (20) or (21). This indicates that both the initial approximations here are far from the exact wave function.

Note that for the choice of $|\chi^{(0)}_{n\ell}>$ as close as possible to the exact function, the ISM (17) will provide our iteration function $|\chi^{(N)}_{n\ell}>$ to converge just to $|\chi^{(0)}_{n\ell}>$.

For a small number of closed channels (1s–2s) the initial approximation is rather close to the exact wave function, therefore the condition (13) is fulfilled, and the SVIM converges to the exact solution. With increasing number of closed channels the approximation (20) spoils so that the SVIM provides only the pathological convergence for $C^{(N)}$.

Thus, for the scattering when many channels are included, the convergence of the SVIM itself does not mean that the exact solution is achieved, if the necessary condition of the convergence of ISM (13) is not fulfilled. Such a situation is realised in the $e^+H$, $p(d\mu)$ scattering, if many channels are included.

## 3. *REFORMULATION OF THE METHOD OF STRONG COUPLING OF CHANNELS ON THE BASIS OF THE SCHWINGER VARIATIONAL PRINCIPLE*

Equations of the method of strong coupling of channels (EMSC) can be derived in two ways. First, one can look for a solution of the equation



$$(\hat{H}_0 + \hat{h} + V)|\psi> = E\,|\psi> \tag{22}$$

($\hat{H}_0$ is the kinetic energy operator of the incident particle, $h$ is the target Hamiltonian, $V$ is the potential of interaction of the projectile with all target particles) as an expansion over the target eigenfunctions $|n>$

$$|\psi> = \sum_n F_n\,|n>; \quad \hat{h}\,|n> = E_n\,|n>. \tag{23}$$

Inserting (23) into (22) and using the orthonormalization of $|n>$, we obtain the EMSC. Second, the EMSC can be found by taking the trial function in the Kohn-Hulten variational principle in form (23) and then varying with respect to $F_n$. Equations for $F_n$ can be deduced by using the SVP as well.

Let us rewrite eq. (22) in the integral form

$$|\psi^{(\pm)}> = |\phi> + \frac{1}{E-\hat{H}_0-h\pm i\epsilon}\,V\,|\psi^{(\pm)}>. \tag{24}$$

For eq. (24) the Schwinger Variational functional has the form

$$T_{ba} = -\frac{1}{2\pi}\left[<\phi_b|V|\psi_a^{(+)}> + <\psi_b^{(-)}|V|\phi_a> - \right.$$
$$\left. - <\psi_b^{(-)}|(V - V\,\frac{1}{E-\hat{H}_0-\hat{h}+i\epsilon}\,V)|\psi_e^{(+)}>\right] \tag{25}$$

Substituting in (25) the expansion (23) with $N$ terms for the trial functions $|\psi^{(-)}>$ and $|\psi^{(+)}>$, and varying with respect to $F_n$ we get [6]:

$$\sum_{m=1}^{N} V_{nm}\,|F_m> = V_{n1}\,|\vec{k}_1> + \sum_{m=1}^{N} W_{nm}\,|F_m>,$$
$$V_{nm} = <n|V|m>; \quad W_{nm} = <n|V\,\frac{1}{E-\hat{H}_0-\hat{h}+i\epsilon}\,V|m>. \tag{26}$$

For illustration, we put $N=1$. Then

$$V_{11}|F_1> = V_{11}\,|\vec{k}_1> + W_{11}\,|F_1>,$$

$$W_{11} = \langle 1 | V \frac{1}{E - \hat{H}_0 - \hat{h} + i\epsilon} V | 1 \rangle =$$

$$\sum_{\nu=1}^{\infty} V_{1\nu} \frac{1}{E - E_\nu - \hat{H}_0 + i\epsilon} V_{\nu 1} . \qquad (27)$$

Here the summation over $\nu$ includes the integration over continuous spectrum. Thus, though (25) is a single equation, its kernel takes into account the virtual transitions into continuous spectrum.

Consider in more detail to what extent (26) is applicable. Since the Schwinger variational principle is equivalent to the change of the interaction potential by the operator of finite rank [11] we should analyse a possibility of making such on approximation for V. For simplicity, let us consider the typical three-body problem, elastic scattering of particle 1 on the bound state of 2nd and 3rd particles:

$$V(\vec{r}_{23}, \vec{\rho}) = V_{12}(|\vec{\rho} - \frac{m_3}{m_2 + m_3} \vec{r}_{23}|) + V_{31}(|\vec{\rho} + \frac{m_2}{m_2 + m_3} \vec{r}_{23}|). \qquad (28)$$

However, it is known that $V(\vec{r}_{23}, \vec{\rho})$ is not fully continuous operator and, hence, it cannot, in general, be approximated by a finite-dimensional operator. We will show that in some cases it can be so approximated. For reactions without rearrangement the wave function $\psi(\vec{r}_{23}, \vec{\rho})$ with respect to the target variables $(\vec{r}_{23})$ differs from zero only in the limited region $(r_{23} \leq R)$, i.e., it is a quadratically integrable function of these variables. Therefore $V(\vec{r}_{23}, \vec{\rho})$ can be replaced by the operator $V(\vec{r}_{23}, \vec{\rho})\theta(R - r_{23})$ which can be approximated by a finite-rank operator. Consequently, the range of applicability of (26) is the region of energies at which the channels with rearrangement are closed.

To calculate the kernels (26), (27), one should sum over intermediate states. To this end, we apply to an approximation of ref.[7] which consists in the finite-dimensional approximation of $\hat{h}$. Indeed, if $\psi(\vec{r}_{23}, \vec{\rho})$ versus target variables $(\vec{r}_{23})$ is considered quadratically integrable, $\hat{h}$ on such functions is a fully continuous operator, therefore the spectral decomposition

$$\hat{h} = \sum_n E_n | n \rangle \langle n |$$



can be truncated with a finite number of terms

$$\hat{h}^{(N)} = \sum_{n=1}^{N} E_n | n \rangle \langle n | , \qquad (29)$$

and then summed up. The kernel $W_{11}$ with $\hat{h}^{(1)}$ is given by the formula

$$W_{11}^{(1)} = \langle 1 | V \frac{1}{E - \hat{H}_0 - \hat{h}^{(1)} + i\epsilon} V | 1 \rangle =$$

$$= \langle 1 | V | 1 \rangle \left( \frac{1}{E - \hat{H}_0 - E_1} - \frac{1}{E - \hat{H}_0} \right) \langle 1 | V | 1 \rangle +$$

$$+ \langle 1 | V \frac{1}{E - \hat{H}_0} V | 1 \rangle . \qquad (30)$$

Analogous summation can be made for $W^{(N)}$ as well.

The method presented above has been applied to calculate the positron-hydrogen scattering length, $a$, and $s$-wave phase shift, $\delta_0$, for different energies $E$ of incoming positrons. In the expansion (29) the $(1s,-2p)$ channels have been taken into account. The results of this calculation are given in *Table 2*. For comparison in this table we display the results by Schwartz[8] and those obtained, in the framework of the usual method of strong coupling of channels (with $1s-2s-2p$ states taken into account)[9].

From comparison with the Schwartz data it is seen that the method we proposed, though being rather simple, takes the polarization of H atom into account accurately enough, whereas the results of the method of strong coupling of channels are not acceptable.

The pion-deuteron scattering is the simplest example of the problem of pion-nuclear scattering, and it is natural to believe that the correct solution of the problem of pion scattering on deuterons will provide further information on pion-nucleon forces. It is also interesting to analyse a possibility for the quantitative description of properties of the $\pi d$ system on the basis of two-particle $\pi N$-potentials. Two-particle parameters (scattering lengths, etc.) are known experimentally with great uncertainty, therefore it is important to establish to what extent the $\pi d$-system properties depen on $\pi N$ parameters.

*Table 2*

| a in atomic units. or $\delta_0$ in radians. | E (ev) 0 | 0.544 | 1.224 | 2.176 | 4.896 | 6.664 | 8.704 |
|---|---|---|---|---|---|---|---|
| Our results | -1.6 | +0.062 | +0.012 | -0.035 | -0.180 | - | -0.275 |
| Schwartz's calculations | -2.1 | +0.188 | +0.168 | +0.120 | +0.007 | -0.054 | - |
| S.C.C.E. | - | -0.043 | -0.093 | -0.147 | -0.246 | - | -0.322 |

*Table 3*

| $r_0^{1/2}$[f] | $r_0^{3/2}$[f] | Static approxim. [f] | Our results [f] | Results of ref. /11/ [f] |
|---|---|---|---|---|
| 68.21 | -97.86 | 0.0509 | 0.0497 | 0.0510 |
| 18.55 | -22.94 | 0.0646 | 0.0590 | 0.0614 |
| 2.725 | -1.882 | 0.1165 | 0.0890 | 0.0729 |
| 1.220 | -0.437 | 0.1646 | 0.1203 | 0.0753 |



For the first time, the problem of elastic $\pi$d scattering was considered by Brueckner in the model of fixed centers[10]. In papers[11] this problem was solved on the basis of the Faddeev equations, and nucleon-nucleon and pion-nucleon potentials were taken in separable forms.

We shall analyse the pion-deuteron scattering on the basis of eq. (27). The operator $\hat{h}$ is approximated by (29) with N=1. All the pair interactions between particles are described by the central local potentials of the type of square well

$$V_i = \begin{cases} V_i^\circ & r \leq R_i \\ 0 & r > R_i \end{cases}$$

The parameters of the nucleon-nucleon potential were fitted to reproduce the deuteron binding energy $\epsilon_d$ = 2.25 *MeV* and triplet NN scattering length of $a^t$ = 5.378 f, while those for the pion-nucleon potential - the pion-nucleon scattering lengths $a^{1/2}$ and $a^{3/2}$ and effective radii of $\pi$N-interaction, $r_0^{1/2}$ and $r_0^{3/2}$.

*Table 3* shows the dependence of $\pi$d-scattering lengths $A_{\pi d}$ on effective radii of $\pi$N interaction. For comparison we present also the results of ref.[11] and calculations in the static approximation, i.e., in the standard method of strong coupling of channels with one equation. The calculation was performed with the experimental set of $\pi$N scattering lengths $a_{\pi N}^{1/2} = -0.257$ f ; $a_{\pi N}^{3/2} = 0.154$ f. From *Table 3* one may observe the very weak dependence of $A_{\pi d}$ on effective $\pi$N-radii. From comparison of our results with the static approximation, here the static approximation is understood as an approximation which corresponds to the replacement of the system of eqs. (15) by one integral equation with only one open channel, it follows that in the region of physical effective $\pi$N-radii (3rd line) the contribution of closed channels exceeds 30%*. We think this fact is very important. As to some difference of our results from those of ref.[11], it can be explained by the dependence on

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* Values of effective radii in the 1st and 2nd line have nothing in common with the real $\pi$N-situation, and the calculation with them has been made to demonstrate the weak sensitivity.

*Table 4*

| | $a_{\pi N}^{1/2}$ [fm] | $a_{\pi N}^{3/2}$ [fm] | $A_{\pi d}$ [fm] | Results of ref./18/ [fm] |
|---|---|---|---|---|
| /13/ | -0.242 | 0.124 | 0.054 | 0.050 |
| /12/ | -0.257 | 0.154 | 0.095 | 0.088 |
| /14/ | -0.262 | 0.145 | 0.081 | 0.072 |
| /15/ | -0.244 | 0.126 | 0.057 | 0.052 |
| /16/ | -0.257 | 0.126 | 0.052 | 0.044 |
| /17/ | -0.240 | 0.130 | 0.064 | - |

the potential shape (in ref.[11] the calculations have been done with the use of the separable Yamaguchi-type potential).

*Table 4* gives the dependence of $A_{\pi d}$ on different experimental sets of $\pi N$ scattering lengths. The strong dependence on $\pi N$-scattering lengths is observed here, however experimentally, $A_{\pi d}$ is known with large error $A_{\pi d} = (0.074 \, {}^{+0.031}_{-0.024}) f$ [19], and, hence, it is difficult to choose a certain set of $a_{\pi N}^{1/2}$, $a_{\pi N}^{3/2}$. The only thing is clear that the experimental data on low-energy $\pi d$-scattering can be interpreted within the potential model neglecting the contributions from inelastic processes of type $\pi^- d \to 2N$. In ref.[20] the absorptive part of the scattering length has been estimated to equal 0.007 f. The results in *Table 4* indicate independently that the absorptive part of $\pi d$ scattering length is small.

Thus, the reformulation of the method of strong coupling of channels on the basis of the Schwinger variational principle, which permits the summation over all states of discrete and continuous spectra with definite angular momenta in closed channels, provides good results even with the system of several ($N = 1,2$) coupled equations (26). An approach of this type to the



multiparticle problems may turn out to be effective if the closed channels are essential in the considered scattering process like in the problems of low-energy $e^+H$ and $\pi d$ elastic scattering we have studied.

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*Received by Publishing Department on October 27 1978.*