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B.M.Barbashov, A.L.Koshkarov, V.V.Nesterenko

RELATIVISTIC STRING IN THE CONSTANT HOMOGENEOUS ELECTROMAGNETIC FIELD

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S U M M A R Y

The Lagrangian and Hamiltonian formalisms are constructed for the massless relativistic string in a constant homogeneous electromagnetic field in a special gauge dependent on the field. Solutions to the equations of motion and the mass spectrum of the string are found. The conserved quantities are the canonical (not dynamical) momenta of the string, therefore the squared mass of the string is defined as the square of its total canonical momentum. It turned out that even in the classical dynamics there are possible the states of the string with the negative squared mass (tachyons). The quantum description of the string in an electromagnetic field is considered and the problem on the relativistic invariance of quantum theory is discussed. The method based on the check of the Poincaré algebra turns out to be not valid in the presence of the external field as in this case the Lorentz rotation operators are time-dependent.

1. INTRODUCTION

The study of dynamics of the massless relativistic string has shown that the quantization of this object gives the spectrum of states of the string coincident with that of hadron masses in dual resonance models.[1]. To construct dual amplitudes, it is necessary to consider the interacting strings[2]. In view of this, exactly solvable examples of the interacting relativistic string are instructive. Such solutions have been found to the equations of motion of the string in the field of a plane electromagnetic wave[3], and, as is shown below, these can be obtained for a constant homogeneous electromagnetic field. The solution of the general Cauchy problem for the free relativistic string has been found in ref.[4].

II. Equations of Motion of the Relativistic String in Electromagnetic Field and the Choice of Gauge

The action of the string in an electromagnetic field is defined in the following way

$$S = \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \, (\mathcal{L}_0 + \mathcal{L}_{int}), \qquad (1)$$

where

$$\mathcal{L}_0 = -[(\dot{x}x')^2 - \dot{x}^2 x'^2]^{\frac{1}{2}}, \qquad \mathcal{L}_{int} = g\, x'_\mu \dot{x}_\nu F^{\mu\nu}(x),$$

$$x_\mu = x_\mu(\sigma,\tau), \quad \dot{x}_\mu = \frac{\partial x_\mu(\sigma,\tau)}{\partial\tau}, \quad x'_\nu = \frac{\partial x_\nu(\sigma,\tau)}{\partial\tau}.$$

This type of interaction, as in the case of the free string, is covariant under arbitrary change of parameters σ and τ

$$\tilde{\sigma} = f_1(\sigma,\tau), \quad \tilde{\tau} = f_2(\sigma,\tau).$$

This statement becomes clear when the term in eq. (1) for the interaction is transformed in the following way

$$S_{int} = g\int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma\, x'_\mu \dot{x}_\nu F^{\mu\nu}(x) =$$

$$= g\int_{\tau_1}^{\tau_2} d\tau \frac{dx^\nu}{d\tau} A_\nu(x)\Big|_{\sigma=\sigma_2(\tau)} - g\int_{\tau_1}^{\tau_2} d\tau \frac{dx^\nu}{d\tau} A_\nu(x)\Big|_{\sigma=\sigma_1(\tau)}, \tag{2}$$

where

$$\frac{dx^\nu}{d\tau} = \dot{x}^\nu(\sigma,\tau) + x'^\nu(\sigma,\tau)\dot{\sigma}(\tau).$$

Evidently, the term S_{int} describes the interaction with electromagnetic field of two point-like charges at edges of the string. These charges are equal in magnitude and opposite in sign $g_1 = -g_2 = g$.

The variation of action (1) results in the equations of motion

$$\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}_0}{\partial\dot{x}_\mu}\right) + \frac{\partial}{\partial\sigma}\left(\frac{\partial\mathcal{L}_0}{\partial x'_\mu}\right) \tag{3}$$

and in the boundary conditions

$$\frac{\partial\mathcal{L}_0}{\partial x'^\nu} - \frac{\partial\mathcal{L}_0}{\partial\dot{x}^\nu}\dot{\sigma} + g F_{\nu\mu}(x)(\dot{x}^\mu + x'^\mu\dot{\sigma}) = 0, \tag{4}$$

$$\sigma = \sigma_i(\tau), \quad (i = 1,2),$$

where $\sigma_i(\tau)$ is the value of the parameter σ at the string ends.

The variation of the action S over functions $\sigma_i(\tau)$ produces the condition

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} = 0, \quad \sigma = \sigma_1(\tau), \quad \sigma = \sigma_2(\tau). \tag{5}$$

Therefore the interaction with electromagnetic field does not change the equations of motion of the string and changes only the boundary conditions. This is a direct consequence of equality (2).

As in the free case, eqs.(3) are dependent and obey two identities[4]. Projections of these eqs. onto $\dot{x}^\mu$ and x'^μ are zero. Thus eqs.(3) do not determine completely all variables $x_\mu(\sigma,\tau)$ and on searched solutions one may impose the two subsidiary conditions (the orthonormal gauge conditions)

$$\dot{x}x' = 0, \quad \dot{x}^2 + x'^2 = 0. \tag{6}$$

These equations can be treated as conditions on the choice of parameters σ and τ on the world sheet of the string.

It is easy to show that from eqs.(4) and (6) eq. (5) follows. Therefore in what follows we will not consider the condition (5) as an independent one. With eq. (6) taken into account the equations of motion (3) and boundary conditions (4) become linear

$$x''_\mu(\sigma,\tau) - \ddot{x}_\mu(\sigma,\tau) = 0, \tag{7}$$

$$x'_\nu + g F_{\nu\mu}\dot{x}^\mu + (\dot{x}_\nu + g F_{\nu\mu} x'^\mu)\dot{\sigma} = 0, \quad \sigma = \sigma_i(\tau), \; i = 1,2. \tag{8}$$

As was mentioned above, the variation of action (5) over $\sigma_i(\tau)$ did not produce new equations for determining the functions $\sigma_i(\tau)$, $(i = 1,2)$ describing in coordinates σ,τ on the world sheet of the string the motion of its ends. Therefore for simplicity the functions $\sigma_i(\tau)$ can be taken such that $\dot{\sigma}_i(\tau) = 0$ and

$$\sigma_1(\tau) = 0, \quad \sigma_2(\tau) = l.$$

Now the problem of the string motion in the electromagnetic field is formulated as follows: It is required to find a solution to equations of motion (7) which obeys the subsidiary conditions (6) and boundary conditions

$$x'_\nu + g F_{\nu\mu}\dot{x}^\mu = 0, \quad \sigma = 0, \quad \sigma = l. \tag{9}$$

This problem is the covariant Lagrangian formulation of equations of the relativistic string in the electromagnetic field.

Another approach to the problem under consideration is possible (so-called non-covariant formalism). As in the free case, on searched solutions one can impose, in addition, the gauge conditions, for the following reasons. The equations of motion (7), boundary conditions (8) and subsidiary conditions (6) do not fix completely the choice of variables σ and τ. These equations stay covariant in passing to new variables $\tilde\sigma$ and $\tilde\tau$ such that

$$\tilde\sigma - \tilde\tau = \tilde\alpha(\sigma - \tau) = \tilde\alpha(\alpha), \quad \tilde\sigma + \tilde\tau = \tilde\beta(\sigma + \tau) = \tilde\beta(\beta) \tag{10}$$

with arbitrary functions $\tilde\alpha(\alpha)$ and $\tilde\beta(\beta)$. This allows to impose the gauge conditions on the searched functions $x_\mu(\sigma,\tau)$, which finally fix the choice of parameters σ and τ. The gauge conditions should be taken so that they do not contradict the requirement $\dot\sigma_i(\tau) = 0$, $i = 1,2$. Let us project the boundary conditions (8) on a constant vector n

$$n^\nu x'_\nu + g n^\nu F_{\nu\mu}\dot{x}^\mu + (n^\nu \dot{x}_\nu + g n^\nu F_{\nu\mu} x'^\mu)\dot\sigma = 0, \quad \sigma = \sigma_i(\tau), \ i = 1,2. \tag{11}$$

The choice of gauge (that of σ, τ) is fixed by the requirements

$$n^\nu x'_\nu + g n^\nu F_{\nu\mu}\dot{x}^\mu = 0, \tag{12}$$

$$n^\nu \dot{x}_\nu + g n^\nu F_{\nu\mu} x'^\mu = \mathcal{P}, \tag{13}$$

where $\mathcal{P}$ is an arbitrary nonzero constant. As will be shown below, this gauge means that the projection of the string canonical momentum of each its point on the vector n is the constant equal to $\mathcal{P}$. Now from (11), (12) and (13) it follows that $\dot\sigma_i(\tau) = 0$, $(i = 1,2)$, i.e., the gauge (12), (13) fixes uniquely the functions $\sigma_i(\tau)$.

In Appendix A it is shown that the gauge (12), (13) for the constant homogeneous electromagnetic field corresponds to the transition to new parameters $\tilde\sigma$ and $\tilde\tau$ by formulae (10) with functions $\tilde\alpha(\alpha)$ and $\tilde\beta(\beta)$ with already defined forms.

The gauge conditions (12), (13) and subsidiary conditions (6) allow one to express two components of the vector $x_\mu(\sigma,\tau)$ in terms of other independent components which in the following will be denoted by $x_\perp(\sigma,\tau)$.

As a result, the initial, essentially nonlinear (because of conditions (6)) problem reduces to finding of solutions to the linear equations of motion for independent components of the vector $x_\mu(\sigma,\tau)$ obeying the linear boundary conditions (9). In what follows we will use just this formalism though it results in the loss of the explicit relativistic covariance of the theory.

Now we formulate the Hamiltonian formalism for describing the string motion in the electromagnetic field. Let us introduce the canonical momenta $\pi_\mu = -\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$. Taking account of (6) one can easily obtain the following expression for π^μ:

$$\pi^\mu = \dot{x}^\mu + F^{\mu\lambda} x'_\lambda. \tag{14}$$

The canonical momentum π^μ is an analog of the generalized momentum of a charged particle moving in the electromagnetic field. The total canonical momentum of the string is determined as follows

$$\Pi^{\mu} = \int_0^{l} d\sigma\, \pi^{\mu}(\sigma,\tau) . \tag{15}$$

Using eqs. (14), (7) and (9) we show that the momentum Π^{μ} is conserved if $F_{\mu\nu}=const.$:

$$\frac{d\Pi^{\mu}}{d\tau} = \int_0^{l} (\ddot{x}^{\mu} + F^{\mu\lambda}\dot{x}'_{\lambda})\, d\sigma = \int_0^{l} (x''^{\mu} + F^{\mu\lambda}\dot{x}'_{\lambda})\, d\sigma =$$

$$= (x'^{\mu} + F^{\mu\lambda}\dot{x}_{\lambda}) \Big|_{\sigma=0}^{\sigma=l} = 0 .$$

Then the rest mass of the string is naturally defined as the square of the conserved total canonical momentum

$$M^2 = \Pi_{\mu}\Pi^{\mu} = \Pi^2 . \tag{16}$$

Like for the free string, the phase space of the considered system is bounded by the constraints between $x_{\mu}(\sigma,\tau)$ and $\pi_{\mu}(\sigma,\tau)$:

$$\pi x' = 0, \qquad (\pi^{\mu} - g F^{\mu\lambda} x'_{\lambda})^2 + x'^2 = 0 . \tag{17}$$

This is a consequence of the singularity of the total Lagrangian (1):

$$\det \left\| \frac{\partial^2 \mathcal{L}}{\partial \dot{x}_{\mu}\, \partial \dot{x}_{\nu}} \right\| = 0 .$$

The Hamiltonian of the system, as in the free case, appears to equal identically zero

$$\mathcal{H} = -\pi \dot{x} - \mathcal{L} = 0 .$$

In addition to the primary constraints (17), on the canonical variables also the gauge conditions[5] can be imposed. To get the direct correspondence with the Lagrangian formalism these conditions may be taken in the form

$$n x'_{\nu} + g n^{\nu} F_{\nu\mu} (\pi^{\mu} - g F^{\mu\sigma} x'_{\sigma}) = 0, \tag{18}$$

$$n\pi = \mathcal{P} . \tag{19}$$

In the Hamiltonian formalism we also will distinguish independent ("transverse") canonical variables $x_{\perp}(\sigma,\tau)$, $\pi_{\perp}(\sigma,\tau)$ and dependent ones $x_{\parallel}$, $\pi_{\parallel}$.

III. Solution to Equations of Motion in the Non-Covariant Formalism

As is known[6], any electromagnetic field constant in space and time can be reduced by the appropriate Lorentz transformation to the following four cases:

1. $\vec{E} \neq 0$, $\vec{H} = 0$, ($E^2 - H^2 > 0$, $\vec{E}\vec{H} = 0$);
2. $\vec{E} = 0$, $\vec{H} \neq 0$, ($E^2 - H^2 < 0$, $\vec{E}\vec{H} = 0$);
3. $\vec{E}$ and $\vec{H}$ differ from zero and are parallel to each other (the invariant $E^2 - H^2$ takes arbitrary values, $\vec{E}\vec{H} \neq 0$);
4. Electric and magnetic fields in all reference frames are equal in magnitude and perpendicular to each other ($E^2 - H^2 = 0$, $\vec{E}\vec{H} = 0$).

In all the four cases solutions to the equations of motion and boundary conditions can be found as the Fourier series and the squared mass operator can be determined for the string. And it appears that the electric field changes the distance between equidistant levels of this operator and displaces the square of mass of the ground state to the negative region. The magnetic field gives only the negative contribution to the squared mass of the ground state. It is important that even in the classical solutions there appear the states with imaginary mass (tachyons). Therefore for the string in the constant electromagnetic field the situation with tachyon states is complicated. From solutions it is seen that there exist limiting values of the electric field at which the solutions change their behaviour. These are: $E = \pm E_{cr}$, where

$$E_{cr} = \frac{1}{2\pi\hbar c\alpha' g}.$$

If one puts that charges at the string ends are equal in magnitude to the electron charge, and $\alpha' \approx 0.9\ GeV^{-2}$ (as is made in comparing the free relativistic string to the dual models), then $E_{cr} \sim 10^{+22}\ volt/cm$. For comparison note that this value is by a factor of 10^{12} larger than the intensity of the electric field acting in the hydrogen atom on electron. For values of the external fields $E_{ext} \ll E_{cr}$ all solutions for the string in the electromagnetic field turn into the free solutions.

Consider the solution to the first case. Let the vector $\vec{E}$ be directed along the x-axis then

$$gF_{01} = -gF_{10} = E \tag{20}$$

and the vector n be taken as the vector with components $n^0 = n^1 = 1,\ n^\alpha = 0,\ \alpha = 2,3$. The gauge conditions (10) and (11) take the form

$$t' - x' = (\dot t - \dot x)E, \tag{21}$$

$$\dot t - \dot x = (t' - x')E + \mathcal{P}. \tag{22}$$

We put $E^2 \neq E_{cr}$ since the opposite case requires a special consideration carried out in Appendix B.

In the gauge (21), (22) subsidiary conditions can be solved with respect to t and x by expressing them through independent variables $x_\perp = (0, 0, y, z)$:

$$t' = -\frac{E}{2\mathcal{P}}(\dot x_\perp^2 + x_\perp'^2) + \frac{1}{\mathcal{P}}\dot x_\perp x_\perp' + \frac{E\mathcal{P}}{2(1-E^2)} = f_1(\sigma,\tau),$$

$$\dot t = \frac{1}{2\mathcal{P}}(\dot x_\perp^2 + x_\perp'^2) - \frac{E}{\mathcal{P}}\dot x_\perp x_\perp' + \frac{\mathcal{P}}{2(1-E^2)} = f_2(\sigma,\tau), \tag{23}$$

$$x' = -\frac{E}{2\mathcal{P}}(\dot x_\perp^2 + x_\perp'^2) + \frac{1}{\mathcal{P}}\dot x_\perp x_\perp' - \frac{\mathcal{P}E}{2(1-E^2)} = \varphi_1(\sigma,\tau),$$

$$\dot x = \frac{1}{2\mathcal{P}}(\dot x_\perp^2 + x_\perp'^2) - \frac{E}{\mathcal{P}}\dot x_\perp x_\perp' - \frac{\mathcal{P}}{2(1-E^2)} = \varphi_2(\sigma,\tau).$$

Then $t(\sigma,\tau)$ and $x(\sigma,\tau)$ are obtained via integration:

$$t(\sigma,\tau) = \int_{\sigma_0}^{\sigma} f_1(\sigma',\tau)\,d\sigma' + \int_{\tau_0}^{\tau} f_2(\sigma,\tau')\,d\tau' - \int_{\tau_0}^{\tau} d\tau' \int_{\sigma_0}^{\sigma} d\sigma' \dot f_1(\sigma',\tau') + t_0, \tag{24}$$

$$x(\sigma,\tau) = \int_{\sigma_0}^{\sigma} \varphi_1(\sigma',\tau)\,d\sigma' + \int_{\tau_0}^{\tau} \varphi_2(\sigma,\tau')\,d\tau' - \int_{\tau_0}^{\tau} d\tau' \int_{\sigma_0}^{\sigma} d\sigma' \dot\varphi_1(\sigma',\tau') + x_0,$$

where t_0 and x_0 are constants.

The boundary conditions (9) in the case under consideration are written for each component of the vector x_μ in the following way

$$\left.\begin{aligned} t' + E\dot x &= 0, \\ E\dot t + x' &= 0, \\ x_\perp' &= 0. \end{aligned}\right\}\ \sigma = 0, l. \tag{25, 26}$$

The solution $x_\mu(\sigma,\tau)$ is obtained by the following scheme: first we find independent components $x_\perp(\sigma,\tau)$ which obey the d' Alembert equation and boundary conditions (26), using the initial data $\dot x_\perp(\sigma, 0)$ and $x_\perp(\sigma, 0)$ which are given arbitrary. Then by formulae (23) and (24) we get $t(\sigma,\tau)$ and $x(\sigma,\tau)$.

The initial data for components $t(\sigma,\tau)$ and $x(\sigma,\tau)$ are defined uniquely by formulae (23), (24) through the initial data $x_\perp(\sigma,0)$ and $\dot{x}_\perp(\sigma,0)$.

The independent variables $x_\perp(\sigma,\tau)$ can be represented as the Fourier series

$$x_\perp(\sigma,\tau)=\frac{x_{\perp 0}}{\ell}+P_\perp\tau+\sum_{n=1}^{\infty}\frac{1}{\sqrt{n\pi}}\left(a_{n\perp}^{+}e^{i\frac{n\pi}{\ell}\tau}+a_{n\perp}e^{-i\frac{n\pi}{\ell}\tau}\right)\cos\left(\frac{n\pi}{\ell}\sigma\right). \quad (27)$$

Now let us find Π^2, where $\Pi_\mu=\int_0^\ell d\sigma\,\pi_\mu(\sigma,\tau)$. This quantity, as was mentioned above, will be defined as the squared mass of the string. According to (14) the vector π_μ in the constant electric field (20) has the following components:

$$\begin{aligned}&\pi_x=\dot{x}+Et', \quad &&\pi_t=\dot{t}+Ex',\\ &\pi_y=\dot{y}, &&\pi_z=\dot{z}.\end{aligned} \quad (28)$$

Using expansions (27) and (28) we get

$$M^2=-E^2\ell^2P_\perp^2+(1-E^2)\sum_{n=1}^{\infty}n\pi\left(a_{n\perp}^{+}a_{n\perp}+a_{n\perp}a_{n\perp}^{+}\right). \quad (29)$$

The squared mass of the ground state is negative and equals

$$M_0^2=-E^2\ell^2P_\perp^2.$$

For $|E|<1$ the distance between equidistant levels of the operator M^2 is smaller by a factor of $(1-E^2)$ than that for the free string*). For $|E|>1$ the squared mass of the string as a whole is always negative.

The Hamiltonian function which produces correct equations of motion for independent canonical variables $x_\perp(\sigma,\tau)$ and $\pi_\perp(\sigma,\tau)$ may be taken in the form

*) To pass from dimensionless quantities E and H to the usual intensities of the electromagnetic field it is necessary to multiply E and H by $2\pi\hbar c\alpha' g$ in all formulae.

$$H=\frac{\mathcal{P}}{1-E^2}\Pi_t=\frac{1}{2}\int_0^\ell(\pi_\perp^2+x_\perp'^2)\,d\sigma+\frac{\mathcal{P}^2\ell}{1-E^2}.$$

Indeed, from the variational principle in the Hamiltonian formalism/7/

$$\delta S=\delta\int_{\tau_1}^{\tau_2}d\tau\int_0^\ell d\sigma\,(\pi_\perp\dot{x}_\perp-\mathcal{H})=0$$

we obtain the Hamilton equations

$$\begin{aligned}\dot{x}_\perp&=\frac{\partial\mathcal{H}}{\partial\pi_\perp}=\pi_\perp,\\ \dot{\pi}_\perp&=\frac{\partial}{\partial\sigma}\left(\frac{\partial\mathcal{H}}{\partial x_\perp'}\right)=x_\perp''\end{aligned} \quad (30)$$

and the boundary conditions

$$\frac{\partial\mathcal{H}}{\partial x_\perp'}=x_\perp'=0, \quad \sigma=0,\ell.$$

Thus, the Hamiltonian formalism coincides with the Lagrangian derivation of the equations of motion.

Next, let us examine the string motion in the constant magnetic field. We suppose that the field is directed along the x-axis, then

$$gF_{23}=gF^{23}=-H.$$

The vector n again is taken with components $n^0=n^1=1$, $n^\alpha=0$, $\alpha=2,3$.

The gauge conditions (12), (13) in this case are written in the form

$$t'-x'=0, \qquad \dot{t}-\dot{x}=\mathcal{P} \quad (31)$$

or

$$t-x=\mathcal{P}\tau.$$

The boundary conditions (9) are

$$\left.\begin{aligned}&t'=0, \quad x'=0\\ &y'+H\dot{z}=0, \quad z'-H\dot{y}=0.\end{aligned}\right\}\ \sigma=0,\ell. \quad (32)$$

As independent variables we take $y(\sigma,\tau)$ and $z(\sigma,\tau)$. These components of the vector $x_\mu(\sigma,\tau)$ obey more complicated boundary conditions than t and x, however, in this way one can solve the subsidiary conditions (6) allowing for gauge (31) without roots:

$$t' = \frac{1}{\mathcal{P}}\,\dot{x}_\perp x'_\perp\,, \qquad \dot{t} = \frac{1}{2\mathcal{P}}(\dot{x}_\perp^2 + x'^2_\perp) + \frac{\mathcal{P}}{2}, \tag{33}$$

$$x' = \frac{1}{\mathcal{P}}\,\dot{x}_\perp x'_\perp\,, \qquad \dot{x} = \frac{1}{2\mathcal{P}}(\dot{x}_\perp^2 + x'^2_\perp) - \frac{\mathcal{P}}{2}.$$

The boundary conditions (32) for independent components are fulfilled if y and z are represented as the Fourier series

$$y(\sigma,\tau) = \left[\frac{y_0}{l} + P_y\tau - HP_z\left(\sigma - \frac{l}{2}\right)\right]\frac{1}{\sqrt{1+H^2}} +$$

$$+ \frac{1}{\sqrt{1+H^2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n\pi}}\left(a_n^+ e^{i\frac{n\pi}{l}\tau} + a_n e^{-i\frac{n\pi}{l}\tau}\right)\cos\left(\frac{n\pi}{l}\sigma\right) -$$

$$- \frac{iH}{\sqrt{1+H^2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n\pi}}\left(b_n^+ e^{i\frac{n\pi}{l}\tau} - b_n e^{-i\frac{n\pi}{l}\tau}\right)\sin\left(\frac{n\pi}{l}\sigma\right), \tag{34}$$

$$z(\sigma,\tau) = \left[\frac{z_0}{l} + P_z\tau + HP_y\left(\sigma - \frac{l}{2}\right)\right]\frac{1}{\sqrt{1+H^2}} +$$

$$+ \frac{1}{\sqrt{1+H^2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n\pi}}\left(b_n^+ e^{i\frac{n\pi}{l}\tau} + b_n e^{-i\frac{n\pi}{l}\tau}\right)\cos\left(\frac{n\pi}{l}\sigma\right) +$$

$$+ i\frac{H}{\sqrt{1+H^2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n\pi}}\left(a_n^+ e^{i\frac{n\pi}{l}\tau} - a_n e^{-i\frac{n\pi}{l}\tau}\right)\sin\left(\frac{n\pi}{l}\sigma\right). \tag{35}$$

Thus, eqs. (33), (34), (35) define completely the string motion in the constant magnetic field.

The canonical momentum π_μ has the following components:

$$\pi_t = \dot{t}\,, \qquad \pi_x = \dot{x}\,,$$

$$\pi_y = \dot{y} + Hz'\,, \qquad \pi_z = \dot{z} - Hy'\,.$$

From expansions (34) and (35) it follows that

$$\pi_y = P_y\sqrt{1+H^2} + i\sqrt{1+H^2}\sum_{n>0}\frac{\sqrt{n\pi}}{l}\left(a_n^+ e^{i\frac{n\pi}{l}\tau} - a_n e^{-i\frac{n\pi}{l}\tau}\right)\cos\left(\frac{n\pi}{l}\sigma\right),$$

$$\pi_z = P_z\sqrt{1+H^2} + i\sqrt{1+H^2}\sum_{n>0}\frac{\sqrt{n\pi}}{l}\left(b_n^+ e^{i\frac{n\pi}{l}\tau} - b_n e^{-i\frac{n\pi}{l}\tau}\right)\cos\left(\frac{n\pi}{l}\sigma\right).$$

In this case the mass of the whole string equals

$$M^2 = \Pi^2 = -H^2 l^2 P_\perp^2 + \sum_{n=1}^{\infty} n\pi\left(a_n^+ a_n + a_n a_n^+ + b_n^+ b_n + b_n b_n^+\right). \tag{36}$$

The Hamiltonian may be taken as follows

$$H = \mathcal{P}\Pi_t = \frac{1}{2}\int_0^l\left[(\pi_y - Hz')^2 + (\pi_z + Hy')^2 + y'^2 + z'^2\right]d\sigma + \frac{\mathcal{P}^2 l}{2}.$$

Consider the third case when the electric E and magnetic field H are nonzero and parallel to each other. We orientate them along the x-axis

$$gF_{01} = -gF_{10} = E\,, \qquad gF_{32} = -gF_{23} = H\,.$$

The vector n, as before, is with components:

$$n^\mu = (1, 1, 0, 0)\,.$$

The gauge conditions have the same form as for the electric field only:

$$t' - x' = (\dot{t} - \dot{x})E,$$

$$\dot{t} - \dot{x} = (t' - x')E + \mathcal{P}.$$

The boundary conditions (9) for components x_μ are

$$\left.\begin{aligned} t' + E\dot{x} = 0, \\ x' + E\dot{t} = 0, \\ y' + H\dot{z} = 0, \\ z' - H\dot{y} = 0. \end{aligned}\right\} \sigma = 0, l. \qquad (37)$$

The independent variables are taken to be y and z. The boundary conditions for them (37) are the same as for the magnetic field only (32). Therefore for y and z one may use the expansions (34) and (35). The dependent components t and x are defined by formulae (23) and (24).

The vector π_μ has the form

$$\pi_t = \dot{t} + E x', \qquad \pi_y = \dot{y} + H z',$$
$$\pi_x = \dot{x} + E t', \qquad \pi_z = \dot{z} - H y'.$$

The square of mass of the string again is nonpositive definite

$$M^2 = -(E^2 + H^2) l^2 P_\perp^2 + (1-E^2)\sum_{n=1}^{\infty} n\pi(a_n a_n^+ + a_n^+ a_n + b_n b_n^+ + b_n^+ b_n). \qquad (38)$$

The Hamiltonian function which produces correct equations of motion and boundary conditions for the independent canonical variables may be taken in the form

$$H = \frac{\mathcal{P}}{1-E^2}\Pi_t = \frac{1}{2}\int_0^l d\sigma\,[(\pi_y - Hz')^2 + (\pi_z + Hy')^2 + y'^2 + z'^2] + \frac{\mathcal{P}^2 l}{1-E^2}.$$

The case $|E| = 1$ is exceptional again. However, it can be shown that solutions in this case are the same as for the string in the electric field E at $|E| = 1$ (see App. B).

And finally, consider the fourth case (E and H are equal and perpendicular to each other). The field E is directed along the x-axis, H along the y-axis:

$$g F_{01} = -g F_{10} = E, \quad g F_{13} = -g F_{31} = H$$

and $E = H = F$.

The vector n is chosen as $n^\mu = (1, 0, 0, 1)$. Then the gauge conditions take the form

$$t' - z' = 0, \qquad \dot{t} - \dot{z} = \mathcal{P}.$$

The boundary conditions are as follows

$$\left.\begin{aligned} t' + E\dot{x} = 0, \\ z' + H\dot{x} = 0, \\ y' = 0, \\ x' + \mathcal{P}F = 0. \end{aligned}\right\} \sigma = 0, l.$$

Now the independent variables are $x(\sigma,\tau)$ and $y(\sigma,\tau)$, the dependent ones t and z. Obviously, the components t and z are expressed through x and y by the formulae (analogous to (33)):

$$t' = \frac{1}{\mathcal{P}}\dot{x}_\perp x'_\perp, \qquad \dot{t} = \frac{1}{2\mathcal{P}}(\dot{x}_\perp^2 + x_\perp'^2) + \frac{\mathcal{P}}{2},$$

$$z' = \frac{1}{\mathcal{P}}\dot{x}_\perp x'_\perp, \qquad \dot{z} = \frac{1}{2\mathcal{P}}(\dot{x}_\perp^2 + x_\perp'^2) - \frac{\mathcal{P}}{2},$$

where $x_\perp$ is the two-dimensional vector with components $x_\perp = (0, x(\sigma,\tau), y(\sigma,\tau), 0)$.

Solutions to the equations of motion and boundary conditions for the independent variables x and y are the following Fourier expansions

$$y(\sigma,\tau) = \frac{y_0}{l} + P_y\tau + \sum_{n=1}^{\infty}\frac{1}{\sqrt{n\pi}}\left(a_n^+ e^{i\frac{n\pi}{l}\tau} + a_n e^{-i\frac{n\pi}{l}\tau}\right)\cos\left(\frac{n\pi}{l}\sigma\right),$$

$$x(\sigma,\tau)=\frac{x_0}{\ell}+P_x\tau+\sum_{n=1}^{\infty}\frac{1}{\sqrt{n\pi}}\left(g_n^+ e^{i\frac{n\pi}{\ell}\tau}+g_n e^{-i\frac{n\pi}{\ell}\tau}\right)\cos\left(\frac{n\pi}{\ell}\sigma\right)-\mathcal{P}F\sigma .$$

The canonical momenta of the string are

$$\pi_t=\dot t+E x',$$
$$\pi_x=\dot x,$$
$$\pi_y=\dot y,\qquad \pi_z=(\dot z+H x')$$

and the square of mass is

$$M^2=\Pi^2=-F^2\mathcal{P}^2\ell^2+\sum_{n=1}^{\infty}n\pi(a_n^+a_n+a_na_n^+ +$$
$$+g_n^+g_n+g_ng_n^+). \qquad (39)$$

The Hamiltonian is given by the formula

$$H=\mathcal{P}\Pi_t=\int_0^{\ell}d\sigma\,\frac{\pi_\perp^2+x_\perp'^2}{2}+\frac{\mathcal{P}^2\ell}{2}$$

where

$$\pi_\perp=(\pi_x,\pi_y),\qquad x'_\perp=(x',y').$$

IV. Transition to Quantum Theory

With the explicit solutions of the equations of motion of the string in the constant homogeneous electromagnetic field, one can construct also the quantum theory. As an example, consider the case with the parallel electric and magnetic fields. The problem in the other cases is solved analogously. It is convenient to write the subsidiary conditions (6) in terms of the Fourier components of x_μ. To this end, we introduce the variables $x_\pm=t\pm x$. According to (37) the boundary conditions for x_+ have the form

$$x'_+ + E\dot x_+=0,\quad \sigma=0,\ell.$$

These conditions and equations of motion hold automatically if x_+ has the following Fourier expansion

$$x_+(\sigma,\tau)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{1}{n}\left[(1-E)\alpha_n e^{in(\sigma+\tau)}+(1+E)\alpha_n^+e^{in(\sigma-\tau)}\right]+$$
$$+x_{+0}+P_+(\tau-E\sigma). \qquad (40)$$

Inserting (34), (35) and (40) into (23) we get

$$\alpha_n=-\frac{i}{\mathcal{P}\pi}L_n,\quad L_n=\frac{1}{2}\sum_{m=-\infty}^{+\infty}(A_{n-m}A_m+B_{n-m}B_m),\ n=0,1,2,\ldots, \qquad (41)$$

where

$$A_m=-\frac{m}{\sqrt{|m|}}a_{-m},\ a_{-m}=a_m^+,\ B_m=\frac{-m}{\sqrt{|m|}}b_{-m},\ b_{-m}=b_m^+,\ m\neq 0,$$
$$A_0=iP_y,\ B_0=iP_z,\ \alpha_0=-(i/2)P_+ .$$

Thus, the dependent Fourier components α_n are expressed via the independent ones a_n, a_n^+, b_n, b_n^+ just in the same way as for the free string[1]. Therefore the transition to the quantum description of the string in the constant homogeneous electromagnetic field may be achieved following the quantum theory of the free string.

The expansions of the canonical variables (34) and (35) are considered to be operator expressions with the usual commutation relations:

$$[y_0,P_y]=[z_0,P_z]=i,$$
$$[a_n,a_m^+]=[b_n,b_m^+]=\delta_{n,m},$$
$$[A_k,A_m]=[B_k,B_m]=k\,\delta_{k+m,0}.$$

Other commutators are put to equal zero.

In the transition to the operator form in (41) the problem arises, as is known, about the order of operators, and only in L_0. Therefore in the quantum case L_n is written, as usually[1], in the normal form (with adding of constant $-\alpha_0$ to L_0):

$$L_n=\frac{1}{2}\sum_{m=-\infty}^{+\infty}(:A_{n-m}A_m: + :B_{n-m}B_m:)-\alpha_0\delta_{n,0}\,. \qquad (42)$$

The commutator of L_n is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D-2}{12}m(m^2-1)\delta_{m,-n}\,,$$

where D is the dimension of space-time.

In accordance with (38) and (42) the operator of the square of mass has the form

$$M^2=-(E^2+H^2)\ell^2P^2-\alpha_0+2(1-E^2)\sum_{n=1}^{\infty}n\pi\,(:a_n^+a_n: + :b_n^+b_n:)\,.$$

For other configurations of the electromagnetic field the operator M^2 is obtained from the classical formulae (29), (36), (39) by changing the usual product of the Fourier amplitudes to the normal product of operators minus the constant α_0 .

For the free string α_0 and D were determined from the requirement of the relativistic invariance of quantum theory[1] which is fulfilled only at $\alpha_0=1$ and $D=26$.

The check of the relativistic invariance of quantum theory of the string in the constant homogeneous electromagnetic field meets difficulties. On the basis of Lagrangian (1) one cannot construct the conserved generatórs of the Lorentz rotations. To find these it is necessary to consider also the quantized electromagnetic field that extremely complicates the problem. Therefore the question on the relativistic invariance of quantum theory in the considered case remains open.

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Appendix A

Let us show that it is always possible to pass to new parameter $\tilde\alpha(\alpha)$ and $\tilde\beta(\beta)$ such that conditions (12), (13) hold. Functions $x_\mu(\sigma,\tau)$ obey the d' Alembert equation (7) therefore these can be represented in the form

$$x^\mu(\alpha,\beta)=\psi_-^\mu(\alpha)+\psi_+^\mu(\beta)\,. \qquad (A.1)$$

Inserting (A.1) into (12) we get

$$n^\nu\psi'_{+\nu}(\beta)+gn_\nu F^{\nu\mu}\psi'_{+\mu}(\beta)+n^\nu\psi'_{-\nu}(\alpha)-gn_\nu F^{\nu\mu}\psi'_{-\mu}(\alpha)=0, \qquad (A.2)$$

where the prime means the differentiation with respect to the function argument.

Next, we pass to new parameters $\tilde\alpha(\alpha)$ and $\tilde\beta(\beta)$ by the formulae

$$\frac{d\alpha}{d\tilde\alpha}=-C\,[n^\nu\psi'_{-\nu}(\alpha)-gn_\nu F^{\nu\mu}\psi'_{-\mu}(\alpha)]^{-1},$$
$$\frac{d\beta}{d\tilde\beta}=C\,[n^\nu\psi'_{+\nu}(\beta)+gn_\nu F^{\nu\mu}\psi'_{+\mu}(\beta)]^{-1}, \qquad (A.3)$$

where C is an arbitrary constant. Also we introduce the new functions

$$\tilde\psi_+(\tilde\beta)=\psi_+(\beta(\tilde\beta))\,,\qquad \tilde\psi_-(\tilde\alpha)=\psi_-(\alpha(\tilde\alpha))\,.$$

Now it is easy to show that eq. (A.2) for the new functions $\tilde\psi_-(\tilde\alpha)$, $\tilde\psi_+(\tilde\beta)$ and new variables $\tilde\alpha,\tilde\beta$ are satisfied identically. Really, we have

$$\tilde\psi_-'(\tilde\alpha)=\frac{d}{d\tilde\alpha}\tilde\psi_-(\tilde\alpha)=\frac{d}{d\tilde\alpha}\psi_-(\alpha(\tilde\alpha))=\frac{d}{d\alpha}\psi_-(\alpha)\cdot\frac{d\alpha}{d\tilde\alpha}=\psi_-'(\alpha)\frac{d\alpha}{d\tilde\alpha}\,,\ \tilde\psi_+'(\tilde\beta)=\psi_+'(\beta)\frac{d\beta}{d\tilde\beta},$$

therefore, according to (A.3), we get

$$n\tilde\psi_+'(\tilde\beta)+gnF\tilde\psi_+'(\tilde\beta)+n\tilde\psi_-'(\tilde\alpha)-gnF\tilde\psi_-'(\tilde\alpha)=C-C=0.$$

Here for simplicity we omit the Lorentz indices over which the summation is made.

Analogously, we find that in the new variables:

$$n\dot{x} + g n F x' = 2C.$$

Appendix B

Consider the motion of the string in the constant homogeneous electric field when the field intensity $E = \pm 1$ in dimensionless units. From the gauge conditions (21), (22) it follows that the constant $\mathcal{P}$ equals zero, and instead of two equations fixing the gauge one has only one equation

$$t' - x' = \pm (\dot{t} - \dot{x}). \tag{B.1}$$

Since t and x obey the wave equation, the gauge condition (B.1) may be represented as follows

$$t - x = U(\sigma + \tau), \tag{B.2}$$

where U is an arbitrary function, and $E = +1$, for definiteness.

Inserting (B.2) into the subsidiary conditions (6) and boundary conditions (25), (26) gives the following result. The independent components of the vector x_r here can be only constants

$$x_\perp(\sigma,\tau) = (y(\sigma,\tau),\, z(\sigma,\tau)) = const$$

and dependent components t and x are defined by the formulae

$$x(\sigma,\tau) = \frac{1}{2}\left[U(\sigma+\tau) + \psi(\sigma-\tau)\right],$$

$$t(\sigma,\tau) = -\frac{1}{2}\left[U(\sigma+\tau) - \psi(\sigma-\tau)\right], \tag{B.3}$$

where U is an arbitrary function entering into the gauge

condition (B.2), ψ a new arbitrary function. The boundary conditions (25), (26) give no rise to the requirement of periodicity on the function U and ψ. Thus, in the case under consideration the string is moving only along the direction of the electric field $\vec{E}$ without periodicity. Its canonical momenta (28) are zero therefore the mass of the whole string is zero, too.

The simplest example of solutions to eqs. (B.3) is the case when $U(\sigma+\tau) = \sigma + \tau$, $\psi(\sigma-\tau) = \tau - \sigma$. Then

$$t = \tau, \quad x = -\sigma, \quad 0 \le \sigma \le \ell.$$

This solution describes the string which is at rest on the x-axis. The force of the electric field acting on the string charges just compensates the internal tension of the string and does not allow it to oscillate that would occur in the case of the free string/7/. It is important that this compensation occurs at arbitrary length of the string ℓ that is a result of the nonlinear character of the considered system. The length of a usual elastic body (e.g., rod or spring with charges at the ends) would depend linearly on the intensity of the electric field E.

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