ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

ДУБНА

E2 - 9948

D.I.Kazakov, V.N.Pervushin, S.V.Pushkin

INVARIANT RENORMALIZATION FOR THE FIELD THEORIES WITH NONLINEAR SYMMETRY

1976

E2 - 9948

D.I.Kazakov, V.N.Pervushin, S.V.Pushkin*

INVARIANT RENORMALIZATION FOR THE FIELD THEORIES WITH NONLINEAR SYMMETRY

Submitted to ТМФ

*Moscow State University, Physical Department

1. Introduction

In recent years there has appeared a number of papers on the background-field method in quantum field theory [1-6]. The application of the continuous dimensional regularization in this method has allowed 't Hooft to derive a simple algorithm to construct the one-loop counterterms for the various types of the Lagrangians in field theories [3]. This algorithm allows us not only to simplify the calculations as compared to the usual method of renormalization, that is clearly seen in the complex gauge and chiral theories and in quantum gravity, but also to construct explicitly invariant renormalization procedure. The counterterms will automatically possess the same symmetry as the initial Lagrangian. It should be noted, however, that in the case of Lagrangians being the nonlinear realization of symmetry group the background-field method can produce noninvariant counterterms, at least, off mass shell.

The present paper is devoted to the invariant formulation of the background-field method for the Lagrangians with nonlinear realization of symmetry. The main point is taking into account of geometry of the curved space of fields when separating variables into "background" and "quantum" fields. In conclusion an example of the one-loop approximation is considered.

2. The Method

An excellent description of the background-field method of renormalization can be found in the original paper by 't Hooft [3].

Therefore we remind only the main points of the method and describe the proposed modification for the case of nonlinear realizations.

In the initial classical Lagrangian $\mathcal{L}(A)$ the change of variables

$$A \to A + \varphi \tag{1}$$

is carried out, where the field A is called the "background" (or classical , or external) one and the filed φ is called the "quantum" (or internal) one. Then the generating functional for the loop diagrams is

$$F[A] = \frac{1}{N} \int \prod_x d\mu(\varphi) \exp i \int dx \left\{ \mathcal{L}(A+\varphi) - \mathcal{L}(A) - \varphi \frac{\delta \mathcal{L}(A)}{\delta A} \right\} \tag{2}$$

The counterterms in the Lagrangian are obtained by expanding $\mathcal{L}(A+\varphi)$ in the Taylor series in φ . To construct the counterterms of a given order we need only the finite number of the expansion terms. (To remove the divergences in the subgraphs the same expansion should be done in the counterterms of lower order which, for this purpose, should be known off mass shell).

If the Lagrangian possesses some king of linear symmetry, it is easy to show that the generating functional (2) is also invariant. The counterterms obtained automatically satisfy all the Ward identities.

However, in the case of nonlinear realizations, the replacement (1) breaks the initial group of symmetry. The generating functional (2) does no longer lead to the invariant counterterms off mass shell. Therefore we propose another method for the separation of the background fields. The idea is the following: (see ref. /7/).

Let the Lagrangian be invariant under some group G of the field transformation

$$G_{A'} = G_g G_A \,, \tag{3}$$

where G_g is the transformation of the group G and

$$\mathcal{L}(G_{A'}) = \mathcal{L}(G_A) \,. \tag{4}$$

The construction of the nonlinear realizations and, on the basis of them, of the invariants defining the structure of the phenomenological Lagrangian for an arbitrary group of symmetry can be carried out by a standard procedure for the homogeneous space G/H (G is the symmetry group under consideration, H is its maximum subgroup leaving the vacuum invariant) /8/. The parameters of the group are identified with the fields of particles and the transformation of the group (3) defines the nonlinear realization of the group on the coordinates of the particle space. The latter is the Riemann space of constant curvature. Then the natural way for separating the classical fields without violating the symmetry is to use the geometric properties of the curved Riemann space of fields, namely to mean the sum of vectors (1) as the addition of vectors in the curved space (addition of vectors in the quotient space G/H), i.e.,

$$G_A \to G_A G_\varphi \,. \tag{1'}$$

The transformation (1') has the simple geometrical interpretation. It is the shift of the origin on the sphere which corresponds to the transformation of quantum fields with the parameters being the classical ones.

For such an "addition" of fields the "sum" is an element of the same space and has the same transformation properties under the group G:

$$G_{A'} G_{\varphi'} = G_g G_A G_\varphi \tag{5}$$

From eqs. (3), (4) and (5) it follows that both the Lagrangian $\mathcal{L}(A(+)\varphi)$ and all its Taylor series are invariant under the group G. We show this for the first variation of the Lagrangian representing it in the form

$$\frac{\delta \mathcal{L}(A(+)\varphi)}{\delta \varphi}\bigg|_{\varphi=0} = \frac{\delta \mathcal{L}(A)}{\delta A} = \frac{\delta \mathcal{L}(G_A G_\varphi)}{\delta (G_A G_\varphi)} \frac{\delta (G_A G_\varphi)}{\delta G_\varphi} \frac{\delta G_\varphi}{\delta \varphi}\bigg|_{\varphi=0} = \frac{\partial \mathcal{L}(G_A)}{\partial G_A} G_A \frac{\partial G_\varphi}{\delta \varphi}\bigg|_{\varphi=0}$$

Then we have:

$$\frac{\delta \mathcal{L}(A')}{\delta A'} = \frac{\delta \mathcal{L}(G_{A'})}{\delta G_{A'}} G_{A'} \frac{\delta G_{\varphi'}}{\delta \varphi'}\bigg|_{\varphi'=0} = \frac{\partial \mathcal{L}(G_{A'})}{\partial G_A} \frac{\partial G_A}{\partial G_{A'}} G_{A'} \frac{\delta G_{\varphi'}}{\partial \varphi'}\bigg|_{\varphi'=0} =$$

$$= \frac{\partial \mathcal{L}(G_A)}{\partial G_A} G_g^{-1} G_g G_A \frac{\delta G_\varphi}{\partial \varphi}\bigg|_{\varphi=0} = \frac{\delta \mathcal{L}(A)}{\delta A} .$$

Here we use the formulas (3) and (4) and the fact that $\frac{\partial G_\varphi}{\partial \varphi}\Big|_{\varphi=0} = const$, and is not transformed.

Hence, taking into account the invariance of the integration measure $d\mu(\varphi)$, we obtain

$$F[A'] = F'[A] .$$

So the use of addition law (1') enables us to construct explicitly invariant background formalizm in the case of nonlinear realizations. We do not use here the equations of motions for the classical fields which is important for the construction of finite Green functions off mass shell.

3. The Realization

For the realization of the procedure of addition on the group we use the covariant formalizm in terms of Cartan forms/9/. Consider the Lagrangian for the classical Bose field A in the form:/10/

$$\mathcal{L}(A) = \frac{1}{4} Sp\{\partial_\mu G_{2A}^{-1} \partial_\mu G_{2A}\} = \frac{1}{2} Sp\{\omega_\mu(A)\,\omega_\mu(A)\}. \qquad (6)$$

Here $\omega_\mu(A)$ is a differential Cartan form of a semisimple continuous group G defined by the equation

$$G_A^{-1}\partial_\mu G_A = i[\omega_\mu(A) + \theta_\mu(A)] ,$$

$$\omega_\mu(A) = \omega_\mu^i(A) X_i , \quad \theta_\mu(A) = \theta_\mu^\alpha(A) Y_\alpha$$

and Y_α and X_i are the generators of the subgroups H and G/H, correspondingly, with the algebra

$$[Y_\alpha, Y_\beta] = i A_{\alpha\beta}^\gamma Y_\gamma \; ; [X_i, Y_\beta] = i B_{i\beta}^k X_k \; ; [X_i, X_k] = i C_{ik}^\alpha Y_\alpha .$$

Lagrangian (6) is, evidently, invariant under the transformations of the group G (3) (we restrict the consideration here to the global symmetries). The replacement of the fields (1') gives the following modification of the Lagrangian/7/

$$\mathcal{L}(A(+)\varphi) = \frac{1}{2} Sp\{\bar{\omega}_\mu(A,\varphi)\bar{\omega}_\mu(A,\varphi)\}, \qquad (7)$$

where $\bar{\omega}_\mu(A,\varphi)$ is the solution of the Cartan fundamental equations with the nonzero boundary conditions $\bar{\omega}_\mu(A,0) = \omega_\mu(A)$. For the case of continuous semisimple group, $\bar{\omega}_\mu^i$ can be represented as follows:/7/

$$\bar{\omega}_\mu^i(A,\varphi) = \sum_{n=0}^{\infty} (-1)^n (\mathcal{M}_\varphi^n)_l^i \left[\frac{\omega_\mu^l(A)}{(2n)!} + \frac{(D_\mu\varphi)^l}{(2n+1)!} \right] , \qquad (8)$$

where $(D_\mu\varphi)^i = \partial_\mu\varphi^i + \varphi^k B_{km}^i \theta_\mu^m(A)$; $(\mathcal{M}^0)_l^i = \delta_{il}$; $\mathcal{M}_l^i = -B_{km}^i \varphi^k C_{jl}^m \varphi^j$.

Using the representation (8) we can expand the Lagrangian (7) in quantum fields φ according to the background-field method. For instance, according to/3/ the generating Lagrangian for the one-loop approximation is convenient to be written in the form

$$\mathcal{L}(\varphi) = \frac{1}{2}(\partial_\mu\varphi^i + N_\mu^{ik}\varphi^k)^2 + \frac{1}{2}\varphi^i X_{ij}\varphi^j . \qquad (9)$$

Comparing this expression with (7), (8) we get the following expressions for the coefficient functions

$$X_{ij}(A) = \omega_\mu^k(A)\,\omega_\mu^l(A)\,B^k_{im}C^m_{jl}\,,$$
$$(D_\mu\varphi)^i = \partial_\mu\varphi^i + B^i_{km}\theta^m_\mu(A)\varphi^k = \partial_\mu\varphi^i + N^{ik}_\mu\varphi^k\,. \qquad (10)$$

Further for simplicity we shall suppose the structure constants B and C to be the same.

The obtained functions automatically possess the required properties of (anti)symmetry in indices /3/. To find the counterterms $\Delta\mathcal{L}_n$ in the n-loop approximation we, first, have to expand the Lagrangian over quantum fields up to φ^{2n} and then to carry out the expansion of $\Delta\mathcal{L}_{n-1}$, up to φ^{2n-2}, changing $\omega_\mu(A)$ by $\overline{\omega}_\mu(A,\varphi)$. The n-loop expansion of $\Delta\mathcal{L}_{n-1}$ reproduces the subtraction in the subgraphs.

In paper /3/ the counterterms are constructed due to the invariance properties of the Lagrangian (9) with respect to the special local transformations. This property is the general one for the various types of the Lagrangians. The presence of the symmetry group G in our theory reduces the class of the counterterms and enables us to use another algorithm of counterterms construction based on the invariants of the group G.

4. Algorithm for Counterterms Construction

The counterterms construction procedure is based on the initial symmetry group. According to /10/ a complete set of the invariants of the classical fields of any power in Cartan forms can be written in the form

$$Sp\left(D^{L_1}_{\dots}\omega,[\dots,[D^{L_{n-1}}_{\dots}\omega, D^{L_n}_{\dots}\omega]\dots]]\right), \qquad (11)$$

where L_i are the powers of the covariant differentials, points stand for the Lorentz indices over which summation is made. From the analysis of diagrams and (anti) symmetry properties of the coefficient functions it follows that in the n-loop approximation the power of the invariants is maximal and equal to $[D\omega]^{2k}[\omega]^{2(n+1-2k)}$, $k=0,1,\dots,n-1$. However, these invariants are not independent. This is due to the structure equation of the group, i.e.,

$$C_{\mu\nu} = \partial_\mu\theta_\nu - \partial_\nu\theta_\mu + i[\theta_\mu,\theta_\nu] = -i[\omega_\mu,\omega_\nu]. \qquad (12)$$

From (12) it is easy to get the following relations for covariant derivatives:

$$D^2_{[\mu,\nu]}\omega_\rho = [[\omega_\mu,\omega_\nu],\omega_\rho]\,,$$
$$Sp\,\omega_\mu D^2_{[\mu,\nu]}\omega_\nu = Sp(D_\mu\omega_\nu D_\nu\omega_\mu) - Sp(D_\mu\omega_\mu D_\nu\omega_\nu)\,. \qquad (13)$$

Using the relations like (13), we can construct the complete set of the linearly independent invariants of any power.

For instance, in the one-loop approximation (14a) and two-loop one (14b) we have

$$Sp\,\omega_\mu\omega_\nu\omega_\mu\omega_\nu\,,\quad Sp\,\omega_\mu\omega_\mu\omega_\nu\omega_\nu\,; \qquad (14a)$$

$$\begin{aligned}
&Sp\,\omega_\mu\omega_\mu\omega_\nu\omega_\nu\omega_\rho\omega_\rho\,,\quad Sp\,\omega_\mu\omega_\nu\omega_\mu\omega_\nu\omega_\rho\omega_\rho\,,\\
&Sp\,\omega_\mu\omega_\nu\omega_\rho\omega_\mu\omega_\nu\omega_\rho\,,\quad Sp\,\omega_\mu\omega_\nu\omega_\rho\omega_\mu\omega_\rho\omega_\nu\,,\\
&Sp\,\omega_\mu\omega_\nu\omega_\rho\omega_\nu\omega_\mu\omega_\rho\,,\quad Sp\,\omega_\mu\omega_\nu\omega_\nu\omega_\mu\omega_\rho\omega_\rho\,,\\
&Sp\,D_\mu\omega_\mu D_\nu\omega_\nu\omega_\rho\omega_\rho\,,\quad Sp\,D_\mu\omega_\mu D_\nu\omega_\rho\omega_\nu\omega_\rho\,,\\
&Sp\,D_\mu\omega_\nu D_\mu\omega_\nu\omega_\rho\omega_\rho\,,\quad Sp\,D_\mu\omega_\nu D_\mu\omega_\rho\omega_\nu\omega_\rho\,,\\
&Sp\,D_\mu\omega_\nu D_\mu\omega_\rho\omega_\rho\omega_\nu\,,\quad Sp\,D_\nu\omega_\rho D_\mu\omega_\mu\omega_\rho\omega_\nu\,.
\end{aligned} \qquad (14b)$$

It is natural to construct the counterterms in the form

$$\Delta \mathcal{L} = a_1 I_1 + a_2 I_2 + \cdots + a_N I_N ,$$

where $I_1, \dots, I_N$ is the complete set of linearly independent invariants (11) and $a_1, \dots, a_N$ are the functions of the regularization parameter determined by the contribution into invariants from different divergent diagrams generated by the Lagrangian (7), (8). Therefore it is convenient to write invariants in the form where they are directly reproduced by the Lagrangian. This problem can be solved as follows: Using the relations (12, 13) let us transform every invariant I_i until any other invariant I_j appears in the expansion in its explicit form. The result of the transformation looks as:

$$I_1 = T_1(\theta,\omega) + d_2^1 I_2 + d_3^1 I_3 + \dots + d_N^1 I_N ,$$

$$I_2 = T_2(\theta,\omega) + d_1^2 I_1 + d_3^2 I_3 + \dots + d_N^2 I_N ,$$

$$\vdots$$

$$I_N = T_N(\theta,\omega) + d_1^N I_1 + d_2^N I_2 + \dots + d_{N-1}^N I_{N-1} .$$

As follows from the structure of the invariants and equation (13) there are, at least, two equal (up to a factor) invariants in the set T_i. Let, for example, $T_k = T_\ell$. In this case we have to compose new linearly independent invariants

$$T_k' = T_k , \quad T_\ell' = I_k - I_\ell .$$

When all linearly independent invariants are separated in this way, we obtain the counterterms in the form

$$\Delta \mathcal{L} = b_1 T_1 + \dots + b_k T_k' + \dots + b_\ell T_\ell' + \dots + b_N T_N . \qquad (15)$$

To determine the coefficients b_i it is necessary to take, in each T_i (15), a combination of coefficient functions absent in all other invariants. (If it is impossible, one should repeat the described procedure). We call this combination of coefficient

functions A_i. To determine b_i, it is sufficient now to construct all divergent diagrams contributing to A_i and to calculate the corresponding n-loop integrals, using continuous dimensional regularization method. And also to perform the subtraction in the subgraphs, using the expansion of the counterterms of a lower order.

Let us demonstrate this procedure on a simple example.

5. Example. One-Loop Approximation

Consider as an example the one-loop approximation. The generating Lagrangian has the form (9), (10). It can be also rewritten as

$$\mathcal{L}(\varphi) = \frac{1}{2} Sp\, D_\mu \varphi D_\mu \varphi + \frac{1}{2} Sp\, \varphi X \varphi ,$$

where $X = \omega_\mu \omega_\mu$, $D_\mu \varphi = \partial_\mu \varphi I + i\theta_\mu \varphi$ and the matrices X_i and Y_i are chosen in the adjoint representation. The invariants, according to (14a), are

$$I_1 = Sp\, \omega_\mu \omega_\mu \omega_\nu \omega_\nu = Sp\, \omega_\mu \omega_\nu \omega_\mu \omega_\nu + i Sp\, \omega_\mu C_{\mu\nu} \omega_\nu ,$$

$$I_2 = Sp\, \omega_\mu \omega_\nu \omega_\mu \omega_\nu = Sp\, \omega_\mu \omega_\mu \omega_\nu \omega_\nu - i Sp\, \omega_\mu C_{\mu\nu} \omega_\nu ,$$

where eq. (12) was used. We take the new system of invariants as follows:

$$T_1' = i Sp\, \omega_\mu C_{\mu\nu} \omega_\nu = -\frac{1}{2} Sp\, C_{\mu\nu} C_{\nu\mu} ,$$

$$T_2' = I_1 + I_2 = Sp[\omega_\mu \omega_\mu \omega_\nu \omega_\nu + \omega_\mu \omega_\nu \omega_\mu \omega_\nu] .$$

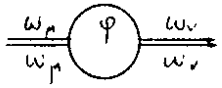
The counterterms are then

$$\Delta \mathcal{L}_1 = b_1 T_1' + b_2 T_2' . \qquad (16)$$

We choose the structures A_1 and A_2 in the form:

$$A_1 = Sp\, \omega_\mu \omega_\mu \omega_\nu \omega_\nu , \quad A_2 = Sp\, \theta_\mu \theta_\nu \theta_\mu \theta_\nu .$$

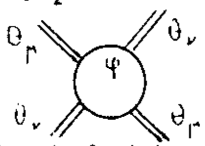
The contribution to A_1 comes from the diagram



$$\Longrightarrow \frac{1}{4\epsilon} A_1 , \qquad (17)$$

and to A_2 from the diagram

$$\left(\frac{1}{\epsilon} = \frac{1}{16\pi^2} \frac{1}{\frac{n-4}{2}} \right)$$



$$\Longrightarrow \frac{1}{12\epsilon} A_2 . \qquad (18)$$

Here the single internal lines denote the quantum fields φ and double external lines denote the forms of classical fields A contained in the Lagrangian. Substituting eq. (17) and (18) into (16) and taking into account (12), we have

$$\Delta \mathcal{L}_1 = \frac{1}{3\epsilon} \left[Sp\, \omega_\mu \omega_\mu \omega_\nu \omega_\nu + \frac{1}{2} Sp\, \omega_\mu \omega_\nu \omega_\mu \omega_\nu \right] .$$

6. Conclusion

In conclusion we mention the advantages of the developed formalizm.

1. The counterterms are explicitly invariant without taking into account of the equations of motion. This enables us to investigate the structure of the divergences of S-matrix and Green functions off mass shell.

2. The counterterms are constructed out of a small number of beforehand known invariants.

3. The coefficient determination procedure enables us to calculate the minimal possible number of diagrams.

In the nonlinear theories the question of their nonrenormalizability naturally arises. The increase of the power of invariants with the number of loops, apparently, indicates that pure-symmetry arguments do not lead to the closed form of the Lagrangian and to its renormalizability in the ordinary sense. However, the proposed formalizm is minimal and seems to be the

most convenient for investigation of this question in the theories with nonlinear realization of a symmetry group. At present, the described formalizm is used for the two-loop counterterms calculations.

The authors express their deep gratitude to D.V.Shirkov and M.K.Volkov for continued interest and also to A.A.Slavnov and A.A.Vladimirov for usefull discussions and stimulating remarks.

References

1. B.S.De Witt. Phys.Rev., 162, 1195, 1239 (1967).
2. J.Honerkamp. Nucl.Phys., B48, 269 (1972).
3. G.'t Hooft. Nucl.Phys., B62, 444 (1973).
4. S.Tamura. Lettere Nuovo Cim., 12, 266 (1975); 13, 639 (1975).
5. M.T.Grisaru, P. van Nieuwenhuizen, C.C.Wu. Phys.Rev., D12, 3203 (1975).
6. M.R.Brown, M.J.Duff. Phys.Rev., D11, 2124 (1975).
7. V.N.Pervushin. Teor.Mat.Fiz., 22, 291 (1975).
8. D.V.Volkov. "Particles and Nucleus", 4, 3 (1973).
9. E.Cartan. Lecons sur la geometrie des espaces de Riemann, Paris, 3785 (1946).
10. D.V.Volkov, V.D.Gershun, A.A.Zheltukhin, A.I.Pashnev. Teor.Mat.Fiz., 15, 245 (1973).

Received by Publishing Department
on July 7, 1976.