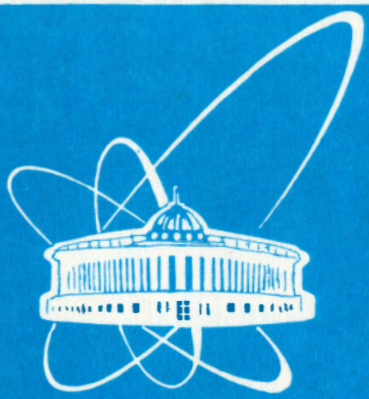

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D.V.Shirkov*

THE BOGOLIUBOV RENORMALIZATION GROUP IN THEORETICAL AND MATHEMATICAL PHYSICS

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*E-mail: shirkovd@thsun1.jinr.ru

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Ширков Д.В.
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Ренормгруппа Боголюбова в теоретической
и математической физике

Этот текст следует докладу на Симпозиуме в Рингберге, приуроченному к 70-летию Вольфхарта Циммермана. Историческое введение частично перекликается с текстом моего предыдущего мемуарного обзора, посвященного Боголюбову (см. Сообщение ОИЯИ E2-96-15). В то же время во вторую часть включены некоторые свежие результаты из КТП (раздел 2.1), а также суммированы (раздел 2.4) недавние впечатляющие достижения применения «квантовополевой ренормгруппы» в математической физике.

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова ОИЯИ.

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Shirkov D.V.
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The Bogoliubov Renormalization Group in Theoretical
and Mathematical Physics

This text follows the line of a talk on Ringberg symposium dedicated to Wolfhart Zimmermann 70th birthday. The historical overview (Part I) partially overlaps with corresponding text of my previous commemorative paper — see Ref. [61] in the list. At the same time second part includes some fresh results in QFT (Sect. 2.1) and summarize (Sect. 2.4) an impressive recent progress of the «QFT renormalization group» application in mathematical physics.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1 Early History of the RG in the Quantum Field Theory

1.1 The birth of Bogoliubov's renorm–group.

In the spring of 1955 a small conference on "Quantum Electrodynamics and Elementary Particle Theory" was organized in Moscow. It took place at the Lebedev Institute in the first half of April. Among the participants there were several foreigners, including Hu Ning and Gunnar Källén. Landau's survey lecture "Fundamental Problems in QFT", in which the issue of ultraviolet (UV) asymptotics in the QFT was discussed, constituted the central event of the conference. Not long before, the problem of short-distance behaviour in QED was advanced substantially in a series of articles [1] by Landau, Abrikosov, and Khalatnikov. They succeeded in constructing a closed approximation of the Schwinger–Dyson equations, which admitted an explicit solution in the massless limit and, in modern language, it resulted in the summation of the leading UV logarithms.

The most remarkable fact was that this solution turned out to be self–contradictory from the physical point of view because it contained a "ghost pole" in the renormalized amplitude of the photon propagator or, in terms of bare notions, the difficulty of "zero physical charge".

At that time our meetings with Nicolai Nicolaevich Bogoliubov (N.N. in what follows) were regular and intensive because we were tightly involved in the writing of final text[1] of our big book. N.N. was very interested in the results of Landau's group and proposed me to consider the general problem of evaluating their reliability by constructing, e.g., the second approximation (including *next-to-leading UV logs*) to the Schwinger–Dyson equations, to verify the stability of the UV asymptotics and the very existence of a ghost pole.

Shortly after the meeting at the Lebedev Institute, Alesha Abrikosov told me about Gell–Mann and Low's article[3] which had just appeared. The same physical problem was treated in this paper, but, as he put it, it was hard to understand and to combine it with the results obtained by the Landau group.

I looked through the article and presented N.N. with a brief report on the methods and results, which included some general assertions on the

[1] Just at that time the first draft of a central part of the book has been published [2] in the form of two extensive papers.

scaling properties of the electron charge distribution at short distances and rather cumbersome functional equations – see, below, Eq.(1).

N.N. immediate comment was that the Gell-Mann—Low approach is very important: it is closely related to the *la groupe de normalisation* discovered a couple of years earlier by Stueckelberg and Petermann [4] in the course of discussing the structure of the finite arbitrariness in the scattering matrix elements arising upon removal of the divergences. This group is an example of the continuous groups studied by Sophus Lie. This implied that functional group equations similar to those of paper [3] should take place not only in the UV limit but also in the general case as well.

Within the next few days I succeeded in recasting Dyson's finite transformations and obtaining the desired functional equations for the QED propagator amplitudes, which have group properties, as well as the group differential equations, that is, the S. Lie equations of the renormalization group (RG). Each of these resulting equations — see, below Eqs.(3) — contained a specific object, the product of the squared electron charge $\alpha = e^2$ and the transverse photon propagator amplitude $d(Q^2)$. We named this product, $e^2(Q^2) = e^2 d(Q^2)$, the *invariant charge*. From the physical point of view this function is an analogue of the so–called *effective charge* of an electron, first discussed by Dirac in 1933 [5], which describes the effect of the electron charge screening due to quantum vacuum polarization. Also, the term "renormalization group" was first introduced in our Doklady Akademii Nauk SSSR publication [6] in 1955 (and in the English language paper [7]).

At the above–mentioned Lebedev meeting Gunnar Källén presented a paper written with Pauli on the so–called "Lee model", the exact solution of which contained a *ghost pole* (which, in contrast to the physical one corresponding to a bound state, had negative residue) in the nucleon propagator. Källén–Pauli's analysis led to the conclusion that the Lee model is physically void.

In view of the argument on the presence of a similar pole in the QED photon propagator (which follows from the abovementioned solution of Landau's group as well as from an independent analysis by Fradkin [8]) obtained in Moscow, Källén's report resulted in a heated discussion on the possible inconsistency of QED. In the discussion Källén argued that no rigorous conclusion about the properties of sum of an infinite nonconvergent series can be drawn from the analysis of a finite number of terms.

Nevertheless, before long a publication by Landau and Pomeranchuk (see, e.g., the review paper[9]) appeared arguing that not only QED but also local QFT were self–contradictory.

Without going into details, remind that our analysis of this problem carried out [10] with the aid of the RG formalism just appeared led to the conclusion that such a claim cannot have the status of a *rigorous result, independent of perturbation theory.*

1.2 Renormalization and renormalization invariance.

As is known, the regular formalism for eliminating ultraviolet divergences in quantum field theory (QFT) was developed on the basis of covariant perturbation theory in the late 40s. This breakthrough is connected with the names of Tomonaga, Feynman, Schwinger and some others. In particular, Dyson and Abdus Salam carried out the general analysis of the structure of divergences in arbitrarily high orders of perturbation theory. Nevertheless, a number of subtle questions concerning so-called overlapping divergences remained unclear.

An important contribution in this direction based on a thorough analysis of the mathematical nature of UV divergences was made by Bogoliubov. This was achieved on the basis of a branch of mathematics which was new at that time, namely, the Sobolev–Schwartz *theory of distributions.* The point is that propagators in local QFT are distributions (similar to the Dirac delta–function) and their products appearing in the coefficients of the scattering matrix expansion require supplementary definition in the case when their arguments coincide and lie on the light cone. In view of this the UV divergences reflect the ambiguity in the definition of these products.

In the mid 50ies on the basis of this approach Bogoliubov and his disciples developed a technique of supplementing the definition of the products of singular Stueckelberg–Feynman propagators [2] and proved a theorem [11] on the finiteness and uniqueness (for renormalizable theories) of the scattering matrix in any order of perturbation theory. The prescription part of this theorem, namely, *Bogoliubov's R-operation*, still remains a practical means of obtaining finite and unique results in perturbative calculations in QFT.

The Bogoliubov algorithm works, essentially, as follows:

– To remove the UV divergences of one-loop diagrams, instead of introducing some regularization, for example, the momentum cutoff, and handling (quasi) infinite counterterms, it suffices to complete the definition of divergent Feynman integral by subtracting from it certain polynomial in

the external momenta which in the simplest case is reduced to the first few terms of the Taylor series of the integral.

– For multi-loop diagrams (including ones with overlapping divergencies) one should first subtract all divergent subdiagrams in a hierarchical order regulated by the R–operator.

The uniqueness of computational results is ensured by special conditions imposed on them. These conditions contain specific degrees of freedom (related to different renormalization schemes and momentum scales) that can be used to establish the relationships between the Lagrangian parameters (masses, coupling constants) and the corresponding physical quantities. The fact that physical predictions are independent of the arbitrariness in the renormalization conditions, that is, they are *renorm–invariant*, constitutes the conceptual foundation of the renormalization group.

An attractive feature of this approach is that it is free from any auxiliary nonphysical attributes such as bare masses, bare coupling constants, and regularization parameters which turn out to be unnecessary in computations employing Bogoliubov's approach. As a whole, this method can be regarded as *renormalization without regularization and counterterms.*

1.3 The discovery of the renormalization group.

The renormalization group was discovered by Stueckelberg and Petermann [4] in 1952-1953 as a group of infinitesimal transformations related to a finite arbitrariness arising in the elements of the scattering S-matrix upon elimination of the UV divergences. This arbitrariness can be fixed by means of certain parameters c_i:

> *"... we must expect that a group of infinitesimal operators* $\mathbf{P}_i = (\partial/\partial c_i)_{c=0}$, *exists, satisfying*
>
> $$\mathbf{P}_i S = h_i(m,e)\partial S(m,e,...)/\partial e\ ,$$
>
> *admitting thus a renormalization of* e*."*

These authors introduced the *normalization group* generated (as a Lie group) by the infinitesimal operators $\mathbf{P}_i$ connected with renormalization of the coupling constant e.

In the following year, on the basis of Dyson's transformations written in the regularized form, Gell-Mann and Low [3] derived functional equations for QED propagators in the UV limit. For example, for the renormalized

transverse part d of the photon propagator they obtained an equation of the form

$$d\left(\frac{k^2}{\lambda^2}, e_2^2\right) = \frac{d_C(k^2/m^2, e_2^2)}{d_C(\lambda^2/m^2, e_1^2)}\ ,\quad e_2^2 = e_1^2 d_C(\lambda^2/m^2, e_1^2)\ , \tag{1}$$

where λ is the cutoff momentum and e_2 is the physical electron charge. The appendix to this article contains the general solution (obtained by T.D. Lee) of this functional equation for the photon amplitude $d(x, e^2)$ written in two equivalent forms:

$$e^2 d\,(x, e^2) = F\left(xF^{-1}\left(e^2\right)\right)\ ,\quad \ln x = \int_{e^2}^{e^2 d} \frac{\mathrm{d}y}{\psi(y)}\ , \tag{2}$$

with

$$\psi(e^2) = \frac{\partial(e^2 d)}{\partial \ln x} \quad \text{at} \quad x = 1\ .$$

A qualitative analysis of the behaviour of the electromagnetic interaction at small distances was carried out with the aid of (2). Two possibilities, namely, infinite and finite charge renormalizations were pointed out:

> *Our conclusion is that the* **shape** *of the charge distribution surrounding a test charge in the vacuum does not, at small distances, depend on the coupling constant except through the scale factor. The behavior of the propagator functions for large momenta is related to the magnitude of the renormalization constants in the theory. Thus it is shown that the unrenormalized coupling constant* $e_0^2/4\pi\hbar c$, *which appears in perturbation theory as a power series in the renormalized coupling constant* $e_1^2/4\pi\hbar c$ *with divergent coefficients, many behave either in two ways:*
>
> *It may really be infinite as perturbation theory indicates;*
>
> *It may be a finite number independent of* $e_1^2/4\pi\hbar c$.

Note, that the latter possibility corresponds to the case when ψ vanishes at a finite point: $\psi(\alpha_\infty) = 0$. Here, α_∞ is known now as a fixed point of the renormalization group transformations.

The paper [3] paid no attention to the group character of the analysis and the results obtained there. The authors failed to establish a connection between their results and the standard perturbation theory and did not discuss the possibility that a ghost pole might exist.

The final step was taken by Bogoliubov and Shirkov [6, 12] – see also the survey [7] published in English in 1956. Using the group properties of finite Dyson transformations for the coupling constant and the fields, these authors derived functional group equations for the propagators and vertices in QED in the general case (that is, with the electron mass taken into account). For example, the equation for the transverse amplitude of the photon propagator and electron propagator amplitude were obtained in the form

$$d(x,y;e^2) = d(t,y;e^2)d\left(\frac{x}{t},\frac{y}{t};e^2d(t,y;e^2)\right),$$

$$s(x,y;e^2) = s(t,y;e^2)s\left(\frac{x}{t},\frac{y}{t};e^2d(t,y;e^2)\right) \qquad (3)$$

in which the dependence not only on momentum transfer $x = k^2/\mu^2$ (where μ is a certain normalizing scale factor), but also on the mass variable $y = m^2/\mu^2$ was taken into account.

As can be seen, the product e^2d of electron charge squared and photon propagator amplitude enters in both functional equations. This product is invariant with respect to Dyson transformation. We called this function – *invariant charge.*

In the modern notation, the first equation (which in the massless case $y = 0$ is equivalent to (1)) is an equation for the invariant charge (now widely known as an effective or running coupling) $\bar{\alpha} = \alpha d(x,y;\alpha = e^2)$:

$$\bar{\alpha}(x,y;\alpha) = \bar{\alpha}\left(\frac{x}{t},\frac{y}{t};\bar{\alpha}(t,y;\alpha)\right). \qquad (4)$$

Let us emphasize that, unlike in the Ref.[3] approach, in our case there are no simplifications due to the massless nature of the UV asymptotics. Here the homogeneity of the transfer momentum scale is violated explicitly by the mass m. Nevertheless, the symmetry (even though a bit more complex one) underlying the renormalization group, as before, can be stated as an *exact symmetry* of the solutions of the quantum field problem – see Eq. (10) below. This is what we mean when using the term *Bogoliubov's renormalization group* or *renorm-group* for short.

The differential group equations (DGEs) for $\bar{\alpha}$ and for the electron propagator:

$$\frac{\partial\bar{\alpha}(x,y;\alpha)}{\partial\ln x} = \beta\left(\frac{y}{x},\bar{\alpha}(x,y;\alpha)\right);$$

$$\frac{\partial s(x,y;\alpha)}{\partial\ln x} = \gamma\left(\frac{y}{x},\bar{\alpha}(x,y;\alpha)\right)s(x,y;\alpha), \qquad (5)$$

with

$$\beta(y,\alpha) = \frac{\partial\bar{\alpha}(\xi;y;\alpha)}{\partial\xi}, \quad \gamma(y,\alpha) = \frac{\partial s(\xi,y;\alpha)}{\partial\xi} \quad \text{at } \xi = 1 \qquad (6)$$

were first derived in [6] by differentiating the functional equations. In this way an explicit realization of the DGEs mentioned in the citation from [4] was obtained. These results established a conceptual link with the Stueckelberg–Petermann and Gell-Mann – Low approaches.

1.4 Creation of the RG method

Another important achievement of paper [6] consisted in formulating a simple algorithm for improving an approximate perturbative solution by combining it with the Lie differential equations (modern notation is used in this quotation from [6]):

> *Formulae (5) show that to obtain expressions for $\bar{\alpha}$ and s valid for all values of their arguments one has only to define $\bar{\alpha}(\xi,y,\alpha)$ and $s(\xi,y,\alpha)$ in the vicinity of $\xi = 1$. This can be done by means of the usual perturbation theory.*

In our adjacent publication [12] this algorithm was effectively used to analyse the UV and infrared (IR) asymptotic behaviour in QED. The one-loop and two-loop UV asymptotics

$$\bar{\alpha}^{(1)}_{RG}(x;\alpha) \equiv \bar{\alpha}^{(1)}_{RG}(x,0,\alpha) = \frac{\alpha}{1-\frac{\alpha}{3\pi}\cdot\ln x}, \qquad (7)$$

$$\bar{\alpha}^{(2)}_{RG}(x;\alpha) = \frac{\alpha}{1-\frac{\alpha}{3\pi}\ln x+\frac{3\alpha}{4\pi}\ln(1-\frac{\alpha}{3\pi}\ln x)} \qquad (8)$$

of the photon propagator as well as the IR asymptotics

$$s(x,y;\alpha) \approx (x/y-1)^{-3\alpha/2\pi} = (p^2/m^2-1)^{-3\alpha/2\pi}$$

of the electron propagator in transverse gauge were obtained. At that time these expressions had already been known only at the one–loop level. It should be noted that in the mid 50s the problem of the UV behaviour in local QFT was quite urgent. As it has been mentioned already a substantial progress in the analysis of QED at small distances was made by Landau and his collaborators [1]. However, Landau's approach did not provide a prescription for constructing subsequent approximations.

An answer to this question was found only within the new renorm–group method. The simplest UV asymptotics of QED propagators obtained in our paper [12], for example, expression (7), agreed precisely with the results of Landau's group.

Within the RG approach these results can be obtained in just a few lines of argumentation. To this end, the massless one-loop approximation

$$\bar{\alpha}^{(1)}_{PTh}(x;\alpha) = \alpha + \frac{\alpha^2}{3\pi}\ell + \dots \ , \qquad \ell = \ln x$$

of perturbation theory should be substituted into the right-hand side of the first equation in (6) to compute the generator $\beta(0,\alpha) = \psi(\alpha) = \alpha^2/3\pi$, followed by an elementary integration of the first of Eqs.(5).

Moreover, starting from the two-loop expression $\bar{\alpha}^{(2)}_{PTh}(x,;\alpha)$ containing the $\alpha^2\ell/4\pi^2$ term we arrive at the second renormalization group approximation (8) performing summation of the next-to-leading UV logs. Comparing solution (8) with (7) one can conclude that two-loop correction is extremely essential just in the vicinity of the ghost pole singularity at $x_1 = \exp(3\pi/\alpha)$. This demonstrates that the RG method is a regular procedure, within which it is quite easy to estimate the range of applicability of the results.

The second order renorm–group solution (8) for the invariant coupling first obtained in [12] contains the nontrivial log–of–log dependence which is now widely known of the two–loop approximation for the running coupling in quantum chromodynamics (QCD).

Quite soon [13] this approach was formulated for the case of QFT with two coupling constants g and h, namely, for a model of pion–nucleon interactions with self-interaction of pions. To the system of functional equations for two invariant couplings

$$\bar{g}^2(x,y;g^2,h) = \bar{g}^2\left(\frac{x}{t},\frac{y}{t},\bar{g}^2(t,y;g^2,h),\bar{h}(t,y;g^2,h)\right) ,$$
$$\bar{h}(x,y;g^2,h) = \bar{h}\left(\frac{x}{t},\frac{y}{t},\bar{g}^2(t,y;g^2,h),\bar{h}(t,y;g^2,h)\right)$$

there corresponds a coupled system of nonlinear differential equations. It was analysed [14] in one-loop apprixiamation to carry out the UV analysis of the renormalizable model of pion-nucleon interaction.

In Refs. [6, 12, 13] and [14] the RG was thus directly connected with practical computations of the UV and IR asymptotics. Since then this technique, known as the *renormalization group method* (RGM), has become the sole means of asymptotic analysis in local QFT.

1.5 Other early RG applications

Another important general theoretical application of the RG method was made in the summer of 1955 in connection with the (then topical) so-called ghost pole problem. This effect, first discovered in quantum electrodynamics [8, 15], was at first thought [15] to indicate a possible difficulty in QED, and then [9, 16] as a proof of the inconsistency of the whole local QFT.

However, the RG analysis of the problem carried out in [10] on the basis of massless solution (2) demonstrated that no conclusion obtained with the aid of finite–order computations within perturbation theory can be regarded as a complete proof. This corresponds precisely to the impression, one can get when comparing (7) and (8). In the mid 50s this result was very significant, for it restored the reputation of local QFT. Nevertheless, in the course of the following decade the applicability of QFT in elementary particle physics remained doubtful in the eyes of many theoreticians.

In the general case of arbitrary covariant gauge the renormalization group analysis in QED was carried out in [17]. Here, the point was that the charge renormalization is connected only with the transverse part of the photon propagator. Therefore, under nontransverse (for example, Feynman) gauge the Dyson transformation has a more complex form. This issue has been resolved by considering the treating the gauge parameter as another coupling constant.

Ovsyannikov [18] found the general solution to the functional RG equations taking mass into account:

$$\Phi(y,\alpha) = \Phi\left(y/x, \bar{\alpha}(x,y;\alpha)\right)$$

in terms of an arbitrary function Φ of two arguments, reversible in its second argument. To solve the equations, he used the differential group equations represented as linear partial differential equations of the form (which are now widely known as the Callan—Symanzik equations):

$$\left\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \beta(y,\alpha)\frac{\partial}{\partial\alpha}\right\}\bar{\alpha}(x,y,\alpha) = 0 \ .$$

The results of this "period of pioneers" were collected in the chapter "Renormalization group" in the monograph [19], the first edition of which appeared in 1957 (shortly after that translated into English and French [20]) and very quickly acquired the status of the "QFT folklore".

2 Further Bogoliubov's RG Development

2.1 Quantum field theory

The next decade and a half brought a calm period, during which there was practically no substantial progress in the renorm–group method.

1. New possibilities for applying the RG method were discovered when the technique of operator expansion at small distances (on the light cone) appeared [21]. The idea of this approach stems from the fact that the RG transform, regarded as a Dyson transformation of the renormalized vertex function, involves the simultaneous scaling of all its invariant arguments (normally, the squares of the momenta) of this function. The expansion on the light cone, so to say, "separates the arguments", as a result of which it becomes possible to study the physical UV asymptotic behaviour by means of the expansion coefficients (when some momenta are fixed on the mass shell). As an important example we can mention the evolution equations for moments of QCD structure functions [22].

2. In the early 70ies S. Weinberg [23] proposed the notion of the *running mass* of a fermion. If considered from the viewpoint of [17], this idea can be formulated as follows:

any parameter of the Lagrangian can be treated as a (generalized) coupling constant, and its effective counterpart should be included into the renorm-group formalism.

However, the results obtained in the framework of this approach turned out to be, practically, the same as before. For example, the most familiar expression for the fermion running mass

$$\bar{m}(x,\alpha) = m_\mu \left(\frac{\alpha}{\bar{\alpha}(x,\alpha)}\right)^\nu ,$$

in which the leading UV logarithms are summed, was known for the electron mass in QED (with $\nu = 9/4$) since the mid 50s (see [1], [12]).

3. The end of the calm period can be marked well enough by the year 1971, when the renormalization group method was applied in the quantum theory of non-Abelian gauge fields, in which the famous effect of *asymptotic freedom* has been discovered [24].

The one-loop renorm-group expression

$$\bar{\alpha}_s^{(1)}(x;\alpha_s) = \frac{\alpha_s}{1+\alpha_s\beta_1 \ln x} ,$$

for the QCD effective coupling $\bar{\alpha}_s$ exhibits a remarkable UV asymptotic behaviour thanks to β_1 being positive. This expression implies, in contrast to Eq.(7), that the effective QCD coupling decreases as x increases and tends to zero in the UV limit. This discovery, which has become technically possible only because of the RG method use, is the most important physical result obtained with the aid of the renorm–group approach in particle physics.

4. One more interesting application of the RG method in the multicoupling case, ascending back in 50ies [14], refers to special solutions, so-called separatrices in a phase space of several invariant couplings. These solutions relate effective couplings and represent a scale invariant trajectories, like, e.g., $g_i = g_i(g_1)$ in the phase space which are straight lines at the one-loop case.

Some of them, that are "attractive" (or stable) in the UV limit, are related to symmetries that reveal themselves in the high-energy domain. It has been conjectured that these trajectories may be connected to *hidden symmetries of a Lagrangian* and even could serve as a tool to find them. On this basis the method has been developed [25] for finding out these symmetries. It was shown that in the phase space of the invariant charges the internal symmetry corresponds to a singular solution that remain straight-line when taking into account the higher order corrections. Such solutions corresponding to supersymmetry have been found for some combinations of Yukawa and quartic interactions.

Generally, these singular solutions obey the relations

$$\frac{dg_i}{dt} = \frac{dg_i}{dg_1}\frac{dg_1}{dt}, \quad t = \ln x$$

which are known since Zimmermann's paper [26] as *the reduction equations*. In the 80ies they have been used [27] (see also review paper [28] and references therein) in the UV analyzis of asymptotically free models. Just for these cases the one-loop reduction relations are adequate to physics.

Quite recently some other application of this technique has been found in a supersymmetrical generalizations of Grand Unification scenario in the Standard Model. It has been shown [29, 30, 31] that it is possible to achieve complete UV finiteness of a theory if Yukawa couplings are related to the gauge ones in a way corresponding to these special solutions,that is to reduction relations.

5. A general method of approximate *solution of the massive RG equations* has been developed [32]. Analytic expressions of high level of accuracy

for an effective coupling and one-argument function have been obtained up to four- and three-loop order [33]. For example, the two-loop massive RG–solution for the invariant coupling

$$a_s(Q^2)^{(2)}_{\rm rg} = \frac{a_s}{1 + a_s A_1(Q^2, m^2) + a_s \frac{A_2(...)}{A_1(...)} \ln\left(1 + a_s A_1(...)\right)} \qquad (9)$$

at small α_s values corresponds to adequate perturbation expansion

$$a_s(Q^2)^{(2)}_{\rm pert} = a_s - a_s^2 A_1(Q^2, m^2) + + a_s^3 A_1^2 - a_s^3 A_2(Q^2, m^2) + \dots \ .$$

At the same time, it smoothly interpolates between two massless limits (with $A_\ell \simeq \beta_\ell \ln Q^2 + c_\ell$) at $Q^2 \ll m^2$ and $Q^2 \gg m^2$ described by equation analogous to Eq.(8). In the latter case it can be represented in the form usual for the QCD practice:

$$\bar{\alpha}_s^{-1}(Q^2/\Lambda^2)_{\rm rg,2} \to \beta_1 \left\{ \ln \frac{Q^2}{\Lambda^2} + b_1 \ln\left(\ln \frac{Q^2}{\Lambda^2}\right)\right\} \ ; \ \ b_1 = \frac{\beta_2}{\beta_1^2} \ .$$

The solution (9) demonstrate, in particular, that the threshold crossing generally changes the subtraction scheme [34].

Our investigation [32, 33, 35] was prompted by the problem of explicitly taking into account heavy quark masses in QCD. However, the results obtained are important from a more general point of view for a discussion of the scheme dependence problem in QFT. The method used could also be of interest for RG applications in other fields within the situation with disturbed homogeneity, such as, e.g., intermediate asymptotics in hydrodynamics, finite-size scaling in critical phenomena and the excluded volume problem in polymer theory.

In the paper [35] this method was used for the effective couplings evolution in Standard Model (SM). Here, new analytic solution of a coupled system of three mass-dependent two-loop RG evolution equations for three SM invariant gauge couplings has been obtained.

6. One more recent QFT development relevant to renorm-group is the *Analytic approach* to perturbative QCD (pQCD). It is based upon the procedure of *Invariant Analytization* [36] ascending to the end of 50ies.

The approach consists in a combining of two ideas: the RG summation of leading UV logs with analyticity in the Q^2 variable, imposed by spectral representation of the Källén–Lehmann type which implements general properties of local QFT including the Bogoliubov condition of microscopic

causality. This combination was first devised [38] to get rid of the ghost pole in QED about forty years ago.

Here, the pQCD invariant coupling $\bar{\alpha}_s(Q^2)$ is transformed into an "analytic coupling" $\alpha_{\rm an}(Q^2/\Lambda^2) \equiv \mathcal{A}(x)$, which, by constuction, is free of ghost singularities due to incorporating some nonperturbative structures.

This analytic coupling $\mathcal{A}(x)$ has no unphysical singularities in the complex Q^2-plane; its conventional perturbative expansion precisely coincides with the usual perturbation one for $\bar{\alpha}_s(Q^2)$; it has no extra parameters; it obeys an universal IR limiting value $\mathcal{A}(0) = 4\pi/\beta_0$ that is independent of the scale parameter Λ; it turns out to be remarkably stable [37] with respect to higher loop corrections and, in turn, to scheme dependence.

Meanwhile, the "analytized" perturbation expansion [39] for an observable F, in contrast with the usual case, may contain specific functions $\mathcal{A}_n(x)$, instead of powers $(\mathcal{A}(x))^n$. In other words, the pertubation series for $F(x)$, due to analyticity imperative, may change its form [40] turning into an asymptotic expansion à la Erdélyi over a nonpower set $\{\mathcal{A}_n(x)\}$.

2.2 Ways of the RG expanding

As is known, in the early 70ies Wilson [41] succeeded in transplanting the RG philosophy from relativistic QFT to a quite another branch of modern theoretical physics, namely, the theory of phase transitions in spin lattice systems. This new version of the RG was based on Kadanoff's idea[42] of joining in "blocks" of few neighbouring spins with appropriate change (renormalization) of the coupling constant.

To realize this idea, it is necessary to average spins in each block. This operation reducing the number of degrees of freedom and simplifying the system under consideration, preserves all its long-range properties under a suitable renormalization of the coupling constant. Along with this, the above procedure gives rise to a new theoretical model of the original physical system.

In order that the system obtained by averaging be similar to the original one, one must also discard those terms of a new effective Hamiltonian which happily turns out to be irrelevant in the description of infrared properties. As a result of this *Kadanoff–Wilson decimation*, we arrive at a new model system characterized by new values of the elementary scale (spacing between blocks) and coupling constant (of blocks interaction). By iterating this operation, one can construct a discrete ordered set of models. From the physical point of view the passage from one model to some other one is an

irreversible approximate procedure. Two passages of that sort applied in sequence should be equivalent to one, which gives rise to a group structure in the set of transitions between models. However, in this case the RG is an approximative and is realized as a semigroup.

This construction, obviously in no way connected with UV properties, was much clearer from the general physical point of view and could therefore be readily understood by many theoreticians. Because of this, in the seventies the RG concept and its algorithmic structure were successfully carried over to diverse branches of theoretical physics such as polymer physics [43], the theory of noncoherent transfer [44], and so on.

Apart from constructions analogous to that of Kadanoff–Wilson, in a number of cases the connection with the original quantum field renorm–group was established. This has been done with help of the functional integral representation. For example, the classic Kolmogorov–type turbulence problem was connected with the RG approach by the following steps [45]:

1. Define the generating functional for correlation functions.
2. Write for this functional the path integral representation.
3. By a change of functional integration variable establish an equivalence of the given classical statistical system with some QFT model.
4. Construct the Schwinger–Dyson equations for this equivalent QFT.
5. Use the Feynman diagram technique and perform a finite renormalization.
6. Write down the standard RG equations and use them to find fixed point and scaling behavior.

The physics of renormalization transformation in the turbulence problem is related to a change of UV cutoff in the wave-number variable.

Hence, in different branches of physics the RG evolved in two directions:

- The construction of a set of models for the physical problem at hand by direct analogy with the Kadanoff–Wilson approach (by averaging over certain degrees of freedom) — in polymer physics, noncoherent transfer theory, percolation theory, and others;
- The search for an exact RG symmetry directly or by proving its equivalence to some QFT: for example, in turbulence theory [45, 46] and turbulence in plasma [48].

What is the nature of the symmetry underlying the renormalization group?

a) In QFT the renorm–group symmetry is an exact symmetry of a solution described in terms of the notions of the equation(s) and some boundary condition(s).

b) In turbulence and some other branches of physics it is a symmetry of a solution of an equivalent QFT model.

c) In spin lattice theory, polymer theory, noncoherent transfer theory, percolation theory, and so on (in which the Kadanoff–Wilson blocking concept is used) the RG transformation involves transitions inside a set of auxiliary models (constucted especially for this purpose). To formulate RG, one should define an ordered set $\mathcal{M}$ of models M_i. The RG transformation connecting various models has the form $R(n)M_i = M_{ni}$. Here, the symmetry can be formulated only in the terms of whole set $\mathcal{M}$.

There is also a purely mathematical difference between the aforesaid RG realizations. In QFT the RG is a *continuous symmetry group.* On the contrary, in the theory of critical phenomena, polymers, and other cases (when an averaging operation is necessary) we have an *approximate discrete semigroup.* It must be pointed out that in dynamical chaos theory, in which RG ideas and terminology can sometimes be applied too, functional iterations do not constitute a group at all, in general. An entirely different terminology is sometimes adopted in the above–mentioned domains of theoretical physics. Terms like "the real–space RG", "the Wilson RG", "the Monte–Carlo RG", or "the chaos RG" are in use.

Nevertheless, the affirmative answer to the question *"Are there distinct renormalization groups?"* implies no more than what has just been said about the differences between cases a) and b) on the one hand and c) on the other.

For this reason, we shall use notation of the **Bogoliubov Renormalization Group** for the exact Lie group, as it emerged from the QFT original papers [4, 6, 7] (see also chapter "Renormalization Group" in the monograph [19, 20]) of mid-fifties. This is to make clear distinction between exact group and the Wilson construction for which the term *Renormalization Group* is widely used in the current literature.

2.3 Functional self–similarity.

An attempt to analyse the relationship between these formulations on a simple common basis was undertaken about fifteen years ago [49]. In this

paper (see also our surveys [50, 51, 52]) it was demonstrated that all the above–mentioned realizations of the RG could be considered in a unified manner by using only some common notions of mathematical physics.

In the general case it proves convenient to discuss the symmetry underlying the renorm–group with the aid of a continuous one–parameter transformation of two variables x and g

$$R_t : \{x \to x' = x/t,\ g \to g' = \bar{g}(t,g)\} \ . \tag{10}$$

Here, x is the basic variable subject to a scaling transformation, while g is a physical quantity undergoing a more complicated functional transformation. To form a group, the transform R_t must satisfy the composition law

$$R_t R_\tau = R_{t\tau} \ ,$$

which yields the functional equation for $\bar{g}$:

$$\bar{g}(x,g) = \bar{g}\left(x/t, \bar{g}(t,g)\right) \ . \tag{11}$$

This equation has the same form as the functional equation (4) for the effective coupling in QFT in the massless case, that is, at $y = 0$. It is therefore clear that the contents of RG equation can be reduced to the group composition law.

In physical problems the second argument g of the transformation usually is related to the boundary value of a solution of the problem under investigation. This means that the symmetry underlying the RG approach is a symmetry of a solution (not of equation) describing the physical system at hand, involving a transformation of the parameters entering the boundary conditions.

Therefore, in the simplest case the renorm–group can be defined as a continuous one–parameter group of transformations of a solution of a problem fixed by a boundary condition. The RG transformation affects the parameters of a boundary condition and corresponds to changing the way in which this condition is introduced for *one and the same solution.*

Special cases of such transformations have been known for a long time. If we assume that $F = \bar{g}$ is a factored function of its arguments, then from Eq.(11) it follows that $F(z,f) = fz^k$, with k being a number. In the particular case the group transform takes the form

$$P_t \ : \ \{\ z \to z' = z/t \ , \ \ f \to f' = ft^k \ \} \ ,$$

which is known in mathematical physics long since as a power *self-similarity transformation.* More general case R_t with functional transformation law (10) can be characterized [49] as a *functional self–similarity* (FSS) transformation.

2.4 Recent application in math. physics

We can now answer the question concerning *the physical meaning of the symmetry* that underlies FSS and the Bogoliubov renorm–group. As we have already mentioned, it is not a symmetry of the physical system or the equations of the problem at hand, but a *symmetry of the solution* considered as a function of the essential physical variables and suitable boundary conditions. A symmetry like that can be related, in particular, to the invariance of a physical quantity described by this solution with respect to the way in which the boundary conditions are imposed. The changing of this way constitutes a group operation in the sense that the group property can be considered as the transitivity property of such changes.

Homogeneity is an important feature of the physical systems under consideration. However, homogeneity can be violated in a discrete manner. Imagine that such a discrete inhomogeneity is connected with a certain value of x, say, $x = y$. In this case the RG transformation with canonical parameter t will have the form:

$$R_t \ : \ \{\ x' = x/t \ , \ \ y' = y/t \ , \ \ g' = \bar{g}(t,y;g)\ \} \ . \tag{12}$$

The group composition law yields precisely the functional equation (4).

The symmetry connected with FSS is a very simple and frequently encountered property of physical phenomena. It can easily be "discovered" in various problems of theoretical physics: in classical mechanics, transfer theory, classical hydrodynamics, and so on [51, 52, 53, 54].

Recently, some interesting attempts have been made to *use the RG concept in classical mathematical physics*, in particular, to study strong nonlinear regimes and to investigate asymptotic behavior of physical systems described by nonlinear partial differential equations (DEs).

About a decade ago the RG ideas were applied by late Veniamin Pustovalov with co-authors [56] to analyze a problem of generating of higher harmonics in plasma. This problem, after some simplification, was reduced to a couple of partial DEs with the boundary parameter – solution "characteristic" – explicitly included. It was proved that these DEs admit an exact

symmetry group, that takes into account transformations of this boundary parameter, which is related to the amplitude of the magnetic field at a critical density point. The solution symmetry obtained was then used to evaluate the efficiency of harmonics generation in cold and hot plasma. The advantageous use of the RG-approach in solving the above particular problem gave promise that it may work in other cases and this was illustrated in [57] by a series of examples for various boundary value problems.

Moreover, in Refs. [51, 57] the possibility of devising a regular method for finding a special class of symmetries of the equations in mathematical physics, namely, RG-type symmetries, was discussed. The latter are defined as solution symmetries with respect to transformations involving parameters that enter into the solutions through the equations as well as through the boundary conditions in addition to (or even rather than) the natural variables of the problem present in the equations.

As it is well known, the aim of modern group analysis [58, 59], which goes back to works of Sophus Lie [60], is to find symmetries of DEs. This approach does not include a similar problem of studying the symmetries of solutions of these equations. Beside the main direction of both the classical and modern analysis, there also remains the study of solution symmetries with respect to transformations involving not only the variables present in the equations, but also parameters entering into the solutions from boundary conditions.

From the aforesaid it is clear that the symmetries which attracted attention in the 50s in connection with the discovery of the RG in QFT were those involving the parameters of the system in the group transformations. It is natural to refer to these symmetries related to FSS as the *RG-type symmetries.*

It should be noted that the procedure of revealing the RG symmetry (RGS), or some group feature, similar to RG regularity, in any partial case (QFT, spin lattice, polymers, turbulence and so on) up to now is not a regular one. In practice, it needs some imagination and atypical manipulation "invented" for every particular case — see discussion in [61]. By this reason, the possibility to find a regular approach to constructing RGS is of principal interest.

Recently a possible scheme of this kind was presented in application to mathematical model of physical system that is described by DEs. The leading idea [54, 57, 62] in this case is based on the fact that solution symmetry for such system can be found in a regular manner by using the well-developed

methods of modern group analysis. The scheme that describes devising of RGS and its application is then formulated [63] as follows.

Firstly, a specific RG-manifold should be constructed. Secondly, some auxiliary symmetry, i.e., the most general symmetry group admitted by this manifold is to be found. Thirdly, this symmetry should be restricted on a particular solution to get the RGS. Fourthly, the RGS allows one to improve an approximate solution or, in some cases, to get an exact solution.

Depending on both a mathematical model and boundary conditions, the first step of this procedure can be realized in different ways. In some cases, the desired RG-manifold is obtained by including parameters, entering into a solution via equation(s) and boundary condition, in the list of independent variables. The extension of the space of variables involved in group transformations, e.g., by taking into account the dependence of coordinates of renorm–group operator upon differential and/or non-local variables (which leads to the Lie-Bäcklund and non-local transformation groups [59]) can also be used for constructing the RG-manifold. The use of the Ambartsumian's invariant embedding method [64] and of differential constraints sometimes allows reformulations of a boundary condition in a form of additional DE(s) and enables one to construct the RG-manifold as a combination of original equations and embedding equations (or differential constraints) which are compatible with these equations. At last, of particular interest is the perturbation method of constructing the RG-manifold which is based on the presence of a small parameter.

The second step, the calculating of a most general group $\mathcal{G}$ admitted by the RG-manifold, is a standard procedure in the group analysis and has been described in detail in many texts and monographs – see, for example, [58, 65, 66].

The symmetry group $\mathcal{G}$ thus constructed cannot as yet be referred to as a renorm–group. In order to obtain this, the next, third step should be done which consists in restricting $\mathcal{G}$ on a solution of a boundary value problem. This procedure utilizes the invariance condition and mathematically appears as a "combining" of different coordinates of group generators admitted by the RG-manifold.

The final step, i.e., constructing analytic expression for solution of boundary value problem on the basis of the RGS, usually presents no specific problems. A review of results, that were obtained on the basis of the formulated scheme can be found, for example, in [63, 67, 68].

Up to now the described regular method is feasible for systems that can

be described by DEs and is based on the formalism of modern group analysis. However, it seems also possible to extend our approach on physical systems that are not described just by differential equations. A chance of such extension is based on recent advances in group analysis of systems of integro-differential equations [69, 70] that allow transformations of both dynamical variables and functionals of a solution to be formulated [71]. More intriguing is the issue of a possibility of constructing a regular approach for more complicated systems, in particular to that ones having an infinite number of degrees of freedom. The formers can be represented in a compact form by functional (or path) integrals.

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