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GENERALIZED HAMILTONIAN DYNAMICS OF FRIEDMANN COSMOLOGY WITH SCALAR AND SPINOR MATTER SOURCE FIELDS

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Фридмановская Вселенная со скалярными и спинорными полями в обобщенной гамильтоновой динамике

Классическая и квантовая проблемы описания динамики Вселенной Фридмана–Робертсона–Уокера с безмассовыми скалярными и массивными спинорными полями обсуждаются в рамках обобщенной гамильтоновой динамики Дирака. Проведена гамильтонова редукция вырожденной теории в случае минимального и конформного взаимодействия гравитации и материи. Показано, что в обоих случаях для всех знаков кривизны $k = 0, \pm 1$ максимально симметричного пространства существует не зависящий от времени редуцированный локальный гамильтониан, который описывает динамику масштабного фактора. Анализируется роль конформного времениподобного вектора Киллинга в пространстве-времени Робертсона–Уокера в вопросе существования независимого от времени гамильтониана. Дается сравнение квантования Уиллера–ДеВитта расширенной системы с каноническим квантованием полученной редуцированной системы. Показано, что во втором подходе квантовые наблюдаемые, трактуемые как средние от дираковских наблюдаемых, корректно описывают классический предел теории.

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Khvedelidze A.M., Palii Yu.G. E2-99-72
Generalized Hamiltonian Dynamics of Friedmann Cosmology with Scalar and Spinor Matter Source Fields

The classical and quantum dynamics of the Friedmann–Robertson–Walker Universe with massless scalar and massive fermion matter field as a source is discussed in the framework of the Dirac generalized Hamiltonian formalism. The Hamiltonian reduction of this constrained system is realized for two cases of minimal and conformal coupling between gravity and matter. It is shown that in both cases for all values of curvature, $k = 0, \pm 1$, of maximally symmetric space there exists a time independent reduced local Hamiltonian which describes the dynamics of the cosmic scale factor. The relevance of conformal time-like Killing vector fields in FRW space-time to the existence of time independent Hamiltonian and the corresponding notion of conserved energy is discussed. The extendend quantization with the Wheeler–deWitt equation is compared with the canonical quantization of unconstrained system. It is shown that quantum observables treated as expectation values of the Dirac observables properly describe the original classical theory.

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I. INTRODUCTION

Cosmological models apart from the main task, to investigate the large scale structure of the Universe, are highly attractive objects with the standpoint of analysis of the conceptual problems in the theory of gravitation. By studying cosmological models instead of general spacetime we can to overcome the difficulties due to the infinite number of degrees of freedom and concentrate our attention to the problems arising solely from the time reparametrization invariance; such as the construction of observables. * In the present article we attempt a contribution to the discussion of some aspects of this problem by considering the simplest cosmological model, the Friedmann-Robertson-Walker (FRW) Universe filled in the scalar massless and massive spinor matter fields. The conventional Hamiltonian description of this model is based on the original Dirac [4] and the so-called Arnowitt-Deser-Misner (ADM) [5] formulation of general relativity. † The ADM method involves the choice of certain coordinate fixing conditions (gauge), solution of the constraints and construction of the observables such as energy, momentum and angular momentum, using the asymptotically flat boundary condition for gravitational field and assuming that three-dimensional space of constant time is open [7]. However, when the closed Universe is considered to build the ADM observables from initial data for canonical variables it is impossible. Since in this case there is no boundary of the space manifold and no asymptotic region can be used to construct the corresponding integrals of motion. This leads to the conclusion that for such cosmological models neither the natural notion of time evolution nor the corresponding energy definition is possible to find [8], [9]. To clear up this contradic-

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*The problem of observables consist in the determination of the invariant characteristics of gravitational field in terms of measurable quantities [1] and is closely related to that of time evolution [2], [3].

†For review of the cosmological models construction with applications of the ADM method see e.g. [6].

tion between the existence of widely used cosmological quantities and their absence in the corresponding field theoretical formulation the FRW cosmological model will be considered in the framework of the Dirac Generalized Hamiltonian formulation [10]- [12]. The key moment of the canonical treatment is the assumption that general relativity represents "already parametrized" theory due to the principle of general covariance, so that the problem of construction of observables can be solved automatically rewriting the theory in the equivalent "deparametrized" form. [‡] However, careful analysis of correctness of such deparametrized program carried out by Hajicek [14] shows that even for simple mechanical system with one quadratic Hamiltonian constraint there are topological obstructions to its implementation analogous of the well-known "Gribov ambiguity" in gauge theories. A direct way to clarify the topological structure of such a theory lies in the finding of integral curves of the dynamical equations and the investigation their global properties. Within this motivation the present note is devoted to the realization of local deparametrization of integrable cosmological FRW models considering it as a preparation for the study the global features of reduction procedure. [§] We will follow the method of Hamiltonian reduction to construct the observables and the corresponding dynamical equations which is well elaborated for gauge theories. This approach is based on an appropriate choice of canonical coordinates on phase space and deals without explicit introduction of any

[‡]To make agreement between the four-dimensional covariance and the possibility to extract from the canonical coordinates hidden variables appropriate for deparametrization theory is difficult task. To solve this problem Kuchar suggested to perform the "second parametrization" of general relativity by extending its phase space by the additional embedding variables [13].

[§]Apart from topological obstruction arising due to the projection onto the constraint shell it is necessary also to investigate the problems connected with the topological structure of spaces of constant curvature. The well elaborated classification of three-dimensional spacelike manifolds [15] allows to estimate the influence of topological properties on physical quantities. An interesting study of the role played by this global properties is under present consideration (see [16] and references therein).

gauge-fixing functions (see e.g. [17] and references therein).

The general plan of present article is as follows. In Section II we state the FRW cosmological model with real massless scalar and massive fermion fields as sources with different type of coupling to gravity. In Section III some generic features of the Hamiltonian reduction and the construction of observables in reparametrization invariant mechanical models is disscused. The aim of Section III is to explain the method to obtain the unconstrained system from reparametrized invariant one by considering the simplest example of free relativistic particle motion. Section IV is devoted to the construction of the unconstrained systems equivalent to FRW cosmology when the homogeneous matter is presented in different forms: as massless scalar field interacting with gravity minimally and conformaly, massive spinor field. Finally, in Section V we discuss the correspondence principle fulfilment for observables in quantum theories based either on Wheeler-deWitt equation or on the canonical quantization scheme of the unconstrained classical system. In Appendices we state some notations and technical detailes of derivations in order to simplify the reading of the main text.

II. MODEL WITH SPATIAL HOMOGENEITY AND ISOTROPY

By definition, the FRW spacetime is a four-dimensional pseudo-Riemannian manifold on which a six-dimensional Lie group G_6 acts as group of isometries. The group of isometries G_6 has a three-dimensional isotropy subgroup and three-dimensional subgroup which acts simply transitive on the one parameter ("time t") family of spacelike hypersurfaces Σ_t. The large group of isometries restricts both the dependence and the number of independent components of the metric tensor and leads to the so-called maximally symmetric three-dimensional space. After the choice of standard coordinates [18] one has the FRW metric

$$ds^2 = -N^2(t)\, dt \otimes dt + a^2(t)\, \gamma_{ab}\, dx^a \otimes dx^b, \qquad (1)$$

where γ_{ab} is the time independent metric of three-dimensional space

$$\gamma_{ab}\, dx^a \otimes dx^b = \frac{dr^2}{1 - \frac{kr^2}{r_o^2}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{2}$$

of constant curvature ${}^{(3)}R(\gamma_{ij}) = -6k/r_o^2, \quad k = 0, \pm 1$. The lapse function $N(t)$ and the cosmic scale factor $a(t)$ describe the remaining gravitational degrees of freedom whose classical behaviour is determined by varying the standard Hilbert action. However, constructed in this way the minisuperspace model is out of interest. Simple counting of the physical degrees of freedom shows that this vacuum FRW model is empty on the classical level; only unphysical degrees of freedom propagate. Thus in order to have some nontrivial observables it is necessary to introduce the source matter fields.

1. Lagrangian for scalar field with minimal coupling to gravity

The introduction of a massless scalar field as a source of gravity results in the simplest cosmological model which has direct correspondence to the classical Friedmann model. ** For a massless scalar field, the two most interesting couplings to gravity extensively considered are the so-called minimal coupling and the conformal one. ††

The Hilbert action for gravity minimally coupled to massless scalar field

$$W = \int d^4x\sqrt{-g}\left[-\frac{{}^4R}{2\kappa} + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\right] \tag{3}$$

**Below we will point out the correspondence the conventional Friedmann cosmology based on the Einstein equations supplemented by certain matter equation of state.

††It is well known that essentially all typs of couplings of free scalar field to the scalar curvature and its kinetic term can be reduced to minimal coupling form using rescaling of the metric and scalar field redefinition [19]. In 1974 based on this type of transformations Bekenstein [20] proposed the method of construction solution for particular case of conformally coupled Einstein-scalar equation from solution of the minimally coupled ones (see also [21]). The detailed investigation of this type solutions for FRW geometry with spatial homogeneous scalar fields can be found in [22].

reduces to the following

$$W = V_{(3)} \int dt \left[-\frac{3}{\kappa}\left(\frac{\dot a^2}{N_c} - \frac{ka^2}{r_o^2}N_c\right) + \frac{a^2}{2N_c}\dot\Phi^2 + \frac{3}{\kappa}\frac{d}{dt}\left(\frac{a\dot a}{N_c}\right)\right], \tag{4}$$

assuming the spatial homogeneity of the scalar field and FRW metric (1). Here $\kappa = 8\pi G$ and new variable $N_c = N/a$ has been introduced. Integration over the spatial hyperplane leads to the appearence of the factor $V_{(3)}$ – "volume" of the three-dimensional space of constant curvature. ‡‡

2. Lagrangian for a scalar field with conformal coupling to gravity

The conformally coupled scalar field is described by action

$$W[g,\Phi] = \int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa}{}^{(4)}R + \frac{1}{12}{}^{(4)}R\Phi^2 + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\right]. \tag{5}$$

Choosing the metric (1) this leads to the action for the FRW Universe filled in by massless homogeneous scalar field $\varphi(t) := a(t)\Phi(t)$

$$W[a, N_c, \varphi] = \int dt \left[-\frac{3}{\kappa}\left(\frac{\dot a^2}{N_c} - \frac{ka^2}{r_o^2}N_c\right) + \frac{1}{2}\left(\frac{\dot\varphi^2}{N_c} - \frac{k\varphi^2}{r_o^2}N_c\right) + \frac{3}{\kappa}\frac{d}{dt}\left(\frac{\dot a a}{N_c}\right)\right], \tag{6}$$

3. FRW Lagrangian with spinor matter fields

The action for a massive spinor field interacting with gravity is given by

‡‡In all formulaes this factor will be ommited, in order to simplify the numerical factors.

$$W = \int d^4x\sqrt{-g}\left[-\frac{^{(4)}R(g)}{2\kappa}\right. \tag{7}$$
$$\left. + \frac{i}{2}\left(\bar{\Psi}\gamma^{\mu}(x)\nabla_{\mu}\Psi - \nabla_{\mu}\bar{\Psi}\gamma^{\mu}(x)\Psi\right) - m\bar{\Psi}\Psi\right],$$

where the spinor field $\Psi(x)$ ($\bar{\Psi}$ Dirac conjugate spinor field) components are treated classically as a collection of Grassmann variables $\Psi_i\Psi_j + \Psi_j\bar{\Psi}_i = 0$ and ∇_{μ} is the covariant derivative (For detailed notation see Appendix). Assuming the homogeneity of the fermion fields and after the redefinition $\psi(t) := a^{3/2}(t)\Psi(t)$ Eqs.(7) reduces to the action of the finite dimensional system

$$W = \int dt\left[-\frac{3}{\kappa}\left(\frac{\dot{a}^2}{N_c} - \frac{ka^2}{r_o^2}N_c\right) + \frac{i}{2}(\bar{\psi}\gamma^o\dot{\psi} - \dot{\bar{\psi}}\gamma^o\psi)\right. \tag{8}$$
$$\left. -aN_c\mathcal{H}_D + \frac{3}{\kappa}\frac{d}{dt}\left(\frac{a\dot{a}}{N_c}\right)\right],$$

whith

$$\mathcal{H}_D = m\bar{\psi}\psi. \tag{9}$$

III. REDUCTION AND OBSERVABLES IN REPARAMETRIZATION INVARIANT MECHANICAL MODELS

It is the purpose of this part to discuss the construction of observables for a system with reparametrization invariance. For our aims we shall state the ideas using a mechanical system, i.e. a system with a finite number of degrees of freedom and restrict ourselves to the case of Abelian constraints.

Let us consider a system with $2n$ - dimensional phase space Γ whose dynamics is constrained to a certain submanifold Γ_c describing by the functionally independent set of m abelian constraints

$$\varphi_\alpha(p,q) = 0, \qquad \{\varphi_\alpha(p,q), \varphi_\beta(p,q)\} = 0\,. \tag{10}$$

Due to the presence of constraints the Hamiltonian dynamics is described by the Poincare-Cartan form

$$\Theta = \sum_{i=1}^{n} p_i dq_i - H_E(p,q)dt\,, \tag{11}$$

with the extended Hamiltonian $H_E(p,q)$ differing from the canonical Hamiltonian $H_C(p,q)$ by a linear combination of constraints with arbitrary multipliers $u_\alpha(t)$

$$H_E(p,q) = H_C(p,q) + u_\alpha(t)\varphi_\alpha(p,q)\,. \tag{12}$$

For the case of first class constraints the functions $u_\alpha(t)$ can't be fixed without using some additional requirements. This observation reflects the existence of the local (gauge) symmetry and the presence of coordinates in the theory whose dynamics is governed in an arbitrary way. However, according to the principle of gauge invariance, these coordinates do not affect physical quantities and thus can be treated as ignorable (gauge degrees of freedom). The question is how to identify these coordinates. If theory contains only Abelian constraints one can find these ignorable coordinates as follows. It is always possible [23] - [24] to define a canonical transformation to a new set of canonical coordinates

$$\begin{aligned} q_i &\mapsto Q_i = Q_i(q,p)\,, \\ p_i &\mapsto P_i = P_i(q,p)\,, \end{aligned} \tag{13}$$

so that m of the new momenta $(\overline{P}_1, \ldots, \overline{P}_m)$ become equal to the Abelian constraints

$$\overline{P}_\alpha = \varphi_\alpha(q,p)\,. \tag{14}$$

In the new coordinates $(\overline{Q}, \overline{P})$ and (Q^*, P^*) we have the following canonical equations

$$\begin{aligned} \dot{Q}^* &= \{Q^*, H_{Ph}\}, & \dot{\overline{P}} &= 0, \\ \dot{P}^* &= \{P^*, H_{Ph}\}, & \dot{\overline{Q}} &= u(t), \end{aligned} \tag{15}$$

with the physical Hamiltonian

$$H_{Ph}(P^*, Q^*) := H_C(P,Q)\Big|_{\overline{P}_\alpha = 0}. \tag{16}$$

The physical Hamiltonian H_{Ph} depends only on the $(n-m)$ pairs of new gauge-invariant canonical coordinates (Q^*, P^*). Moreover the form of the canonical system (15) expresses the explicit separation of the phase space into physical and unphysical sectors. Arbitrary functions $u(t)$ enter only into the part the equation for the ignorable coordinates $\overline{Q}_\alpha$, conjugated to the momenta $\overline{P}_\alpha$. §§

Trying to apply this program to any model with reparametrization invariance we as a rule reveal that the physical Hamiltonian defined by (16) is zero and thus we have got the dynamics of unconstrained system in the Maupertuis form

$$\Theta_{Ph} = \sum_{i=1}^{n-m} P_i^* dQ_i^* - dV, \tag{17}$$

where dV is a total differential. The problem is now how to deal with the zero Hamiltonian. This situation in some sence opposite to the case known from the Hamilton-Jacobi method of integration of equations of motion. The main idea of this method is to implement on the system with Hamiltonian $H(t,p,q)$ the canonical transformation with generating function $S(t,q,p)$, which is the solution for the equation

$$\frac{\partial S}{\partial t} + H\left(t, q, \frac{\partial S}{\partial q}\right) = 0\,. \tag{18}$$

As a result the new Hamiltonian is zero and the equation of motion in the new coordinates have the simplest form

$$\dot{Q} = 0, \qquad \dot{P} = 0\,. \tag{19}$$

§§This paper deals with Abelian constraints only, but a few remarks on the general non-Abelian case may be in order. A straightforward generalization to this situation is unattainable; identification of momenta with constraints is forbidden due to the non-Abelian character of constraints. However, one can replace the non-Abelian constraints by an equivalent set of constraints forming an Abelian algebra and after this implement the above mentioned Levi-Civita transformation. For proofs of this Abelianization statement see e.g. [11] – [12] and the description of iterative Abelianization conversion in [17].

After reduction we have got just a system in these coordinates and the problem is to reconstruct the nonzero Hamiltonian in any other coordinates for the obtained uncontsrained system. Two remarks to the picture described above may be in order. There is no difference between the local behaviour in systems obtained via the reduction of reparametrization invariant theories. The specific properties, which make a difference of systems are hidden in the total differential in the Poincare-Cartan form. Before passing to the construction of the reduced phase space for FRW Universe it seems worth to set forth our approach to the same problem of a free relativistic particle.

A. Digress: Reduced dynamics of free relativistic particle

For the presentation of our procedure to construct the reduced dynamical system from the degenerate system with reparametrization invariance let us start with the simplest case of free motion of a particle in Minkowski space-time writing its action in the form close to the cosmological Friedmann models (4), (6), (8)

$$W[x,e] := \frac{1}{2}\int_{T_1}^{T_2} d\tau \left(\frac{\dot{x}_\mu^2}{e} + em^2\right)\,. \tag{20}$$

The independent configuration variables are particle wordline coordinates $x_\mu(\tau)$ and the additional "vielbein" determinant $e(\tau)$.

Invariance of the action (20) under the reparametrization of time $\tau \to \tau' = f(\tau)$ spoils the uniqueness of the Cauchy problem for the corresponding equations of motion. Therefore the problem is to fix the part of the variables whose dynamics will be unique and whose initial conditions are free from any constraints. The usual way to deal with this problem consists in choosing of a gauge which tights the parameter of evolution with the configuration variables. For example, the proper time gauge fixing $x_0(\tau) = \tau$ leads to the instant form of the dynamics for a relativistic particle. However, let us act in spirit of the previous section and try to reproduce the results of the instant form of particle dynamics without introduction of gauge conditions.

According to the Dirac prescription the generalized Hamiltonian dynamics for the system (20) takes place on the phase space spanned by five canonical pairs (e, p_e) and (x_μ, p_μ) resticted by the primary constraint $p_e = 0$ and the secondary constraint

$$p_\mu p^\mu - m^2 = 0 \,. \tag{21}$$

To take into account these constraints and to derive equations of motion one can consider the Poincare-Cartan 1-form

$$\Theta := p_e de + p_\mu dx^\mu - H_T d\tau \,, \tag{22}$$

with the total Hamiltonian

$$H_T := \frac{1}{2} e(p_x^2 - m^2) + \lambda(\tau) p_e \,. \tag{23}$$

The equation of motion together with both constraints follow from functional

$$W[e, p_e; x, p, ; \lambda] := \int \Theta \,, \tag{24}$$

using independent variation of the canonical pairs $(e, p_e), (x, p)$ and the Lagrange multiplier λ

$$\dot{x}_\mu = e p_\mu \,, \qquad \dot{p}_\mu = 0 \,, \tag{25}$$

$$\dot{e} = \lambda \,, \tag{26}$$

$$p^2 - m^2 = 0 \,, \qquad p_e = 0. \tag{27}$$

Let us now convince ourselves that performing certain canonical transformations one can put the equation in such form that the Lagrange multiplier function $\lambda(\tau)$ enters only in the equation for one canonical pair. According to the general scenario described in previous section each canonical transformation

$$\begin{pmatrix} e & p_e \\ x^\mu & p_\mu \end{pmatrix} \longmapsto \begin{pmatrix} e & p_e \\ X^\mu & \Pi_\mu \end{pmatrix} ,$$

that identifies one canonical momentum with the energy constraint (27), say Π_0

$$\Pi_0 = \frac{1}{2}(p_x^2 - m^2) \tag{28}$$

leads to this pattern. One possible way to complete the canonical transformations is [***]

$$\begin{array}{ll} \Pi_0 = \frac{1}{2}(p_x^2 - m^2) \,, & X_0 = \frac{x_0}{p_0} \,, \\ \Pi_i = p_i \,, & X_i = x_i - \frac{p_i}{p_0} x_0 \,. \end{array} \tag{29}$$

and the inverse transformation is

$$\begin{array}{ll} p_0 = \sqrt{2\Pi_o + \vec{\Pi}^2 + m^2} \,, & x_0 = X_0 \sqrt{2\Pi_o + \vec{\Pi}^2 + m^2} \,, \\ p_i = \Pi_i \,, & x_i = X_i + \Pi_i X_0 \,. \end{array} \tag{30}$$

In terms of the new variables the total Hamiltonian is

$$H_T = e\Pi_o + \lambda p_e \,. \tag{31}$$

and the equations of motion separate into two parts; one for the canonical pairs (e, p_e) and X_0, Π_0, with dependence from the Lagrange multiplier $\lambda(\tau)$

$$\dot{X}_0 = e \,, \quad \dot{e} = \lambda \,, \tag{32}$$

$$\dot{p}_e = -\Pi_0 \,, \quad \dot{\Pi}_0 = 0 \,, \tag{33}$$

constrained by $\Pi_0 = 0$ and the equations of mothion for the variables (X_i, Π_i)

$$\dot{X}_i = 0 \qquad \dot{\Pi}_i = 0 \,, \tag{34}$$

which have a unique solution with arbitrary initial values free from any restriction. One can construct the reduced Poincare -Cartan

[***]Different possibilities to complete the canonical transformations for remaining variables will lead to another forms of dynamics, or to equivalent form but in another frame of reference.

1-form for physical unconstrained variables X_i, Π_i from (22), rewritten in terms of the new cannical variables

$$\Theta = \Pi_0 dX^0 - \Pi_i dX_i + p_e de - (e\Pi_o + \lambda p_e)dt + d(X_0(\Pi_0 + m^2)) \,, \tag{35}$$

by considering the projection onto the constraint shell

$$\Theta^* = \Theta|_{\Pi_0=0,\ p_e=0} = -\Pi_i dX_i + d(X_o)m^2 \,. \tag{36}$$

Thus we have convinced ourselves that the variables Π_i, X_i – are Jacobi's coordinates for the obtained unconstrained theory with zero Hamiltonian. Now we shall show how to reconstruct the unconstrained Hamiltonian in terms of initial variables using the generating function to new set of canonical pairs (29) and Hamilton-Jacobi equation. To find the unconstrained system whose Jacobi's coordinates are Π_i, X_i let us write down the generating function $S(\Pi, x)$ of the canonical transformation $(x, p) \to (X, \Pi)$ (29)

$$p = \frac{\partial S(\Pi, x)}{\partial x}, \qquad X = \frac{\partial S(\Pi, x)}{\partial \Pi} \,. \tag{37}$$

One can easily verify from the condition

$$\Pi dX - p dx = d(X_o(\Pi_o + m^2)), \tag{38}$$

that the function

$$S(\Pi, x) = x_0\sqrt{2\Pi_0 + \vec{\Pi}^2 + m^2} - x_i\Pi_i \tag{39}$$

generates the above canonical transformations (29). Restriction the generating function by the condition $\Pi_o = 0$ leads to the function

$$S^*(\Pi_i, x_i, x_0) = S(\Pi, x)|_{\Pi_o=0} = x_0\sqrt{\vec{\Pi}^2 + m^2} - x_i\Pi_i \,, \tag{40}$$

which we shall now treat as generating function defined on the unconstrained phase space (x_i, p_i) and depended explicitely on some parameter x_0, which has the meaning of evolution parameter for the obtained reduced system. To verify this, one can use the generating function $S^*(\Pi_i, x_i, x_0)$ to write down the inverse transformation

for variables in the reduced Poincare -Cartan form directly on the constraint shell

$$\Theta^* = -\Pi_i dX_i + m^2 dX_o\Big|_{\left\{\begin{array}{l} X_i = \frac{\partial S^*}{\partial \Pi_i} = x_i \\ p_i = \frac{\partial S^*}{\partial x_i} = \Pi_i \end{array}\right.} \tag{41}$$

$$= -p_i dx_i + \sqrt{\vec{p}^2 + m^2} dx_o \,.$$

From this form it follows that we get the Hamiltonian system for a relativistic particle

$$\frac{dx_i}{dt} = \{x_i, h\} = \frac{2p_i}{\sqrt{\vec{p}^2 + m^2}} \tag{42}$$

$$\frac{dp_i}{dt} = \{p_i, h\} = 0 \,, \tag{43}$$

in the instant form of the dynamics with the parameter $t := x_0$ and the Hamiltonian defined from the reduced generating function

$$h =: \frac{\partial S^*}{\partial x_0} = \sqrt{\vec{p}^2 + m^2}. \tag{44}$$

IV. HAMILTONIAN REDUCTION OF FRW COSMOLOGICAL MODELS

A. Scalar field with minimal coupling to gravity

After performing the Legandre transformation on the Lagrangian in the action (4) describing the the dynamics of a homogeneous scalar field with minimal coupling to FRW space time one finds that the phase space spanned by the canonical pairs $(a, p_a), (N_c, P_N)$ and (Φ, P_Φ) is restricted by the primary constraint

$$P_N = 0 \tag{45}$$

and secondary constraint

$$C = \frac{\kappa p_a^2}{12} + \frac{3ka^2}{\kappa r_0^2} - \frac{P_\Phi^2}{2a^2} . \tag{46}$$

Exploting the nondegenerate character of the metric ($a \neq 0$) the secondary constraint (46) can be rewritten in the equivalent form

$$\tilde{C} = a^2 C = a^2 \left(\frac{\kappa p_a^2}{12} + \frac{3ka^2}{\kappa r_0^2} \right) - P_\Phi^2/2 , \tag{47}$$

which shows the separability of the gravitational and the matter source part in constraint. To obtain the reduced Hamiltonian dess-cribing the evolution of cosmic scalar factor a one can introduce the new canonical coordinates for scalar field

$$\Pi_\Phi := P_\Phi^2/2 , \quad T_\Phi := \Phi/P_\Phi . \tag{48}$$

After this redefinition the corresponding Poincare-Cartan form

$$\Theta = p_a da + \Pi_\Phi dT_\Phi - \frac{N}{a^2}\tilde{C}dt + d(\Pi_\Phi T_\Phi) , \tag{49}$$

projected onto the constraint shell reduces to

$$\Theta^* = p_a da + H(a)dT_\Phi + d(H(a)T_\Phi) , \tag{50}$$

where the reduced Hamiltonian that govers the scale factor a evo-lution in time T_Φ is

$$H(a) := a^2 \left(\frac{\kappa p_a^2}{12} + \frac{3ka^2}{\kappa r_0^2} \right) . \tag{51}$$

Note, that there is another possibility to reduce the theory. The reduced theory can be formulated in terms of a scalar field. To find the dynamics of the scalar field we perform the canonical transfor-mation on the scale factor

$$\Pi_a := a^2 \left(\frac{\kappa p_a^2}{12} + \frac{3ka^2}{\kappa r_0^2} \right) \tag{52}$$

$$T_a := \int_{a_o}^{a} a^2 da \left(\frac{\kappa}{3}\Pi_a - ka^4 r_o^{-2} \right)^{-1/2} \tag{53}$$

and as a result the reduced Poincare-Cartan form in terms of scalar fiel variables is

$$\Theta^* = P_\Phi d\Phi - H(P_\Phi)dT_a + d(S(a, \Pi_a) - T_a \Pi_a) , \tag{54}$$

where the reduced Hamiltonian that describes the evolution of scalar field Φ in time T_a is

$$H(P_\Phi) := \frac{1}{2}P_\Phi^2 , \tag{55}$$

and the function $S(a, \Pi_a)$ is the generating function of the canonical transformation (52).

B. Scalar field with conformal coupling to gravity

In the case of a homogeneous scalar field conformally coupled to the FRW space time (6) the phase space spanned by the canoni-cal pairs $(a, p_a), (N_c, P_N)$ and (φ, p_φ) is restricted by the primary constraint

$$P_N = 0 , \tag{56}$$

and secondary constraint

$$C := \Pi_\varphi - \Pi_a , \tag{57}$$

where

$$\Pi_a := \frac{\kappa p_a^2}{12} + \frac{3ka^2}{\kappa r_o^2} , \qquad \Pi_\varphi := \frac{p_\varphi^2}{2} + \frac{k\varphi^2}{2r_o^2} . \tag{58}$$

The total Hamiltonian $H_T := N_c C + \lambda(t) P_N$ contains the arbitrary function $\lambda(t)$ and thus the Hamilton-Dirac equations

$$\begin{array}{lll} \dot{a} = -N_c \kappa p_a/6 & \dot{N}_c = \lambda & \dot{\varphi} = N_c p_\varphi \\ \dot{p}_a = N_c 6ka/(\kappa r_o^2) & \dot{P}_N = C & \dot{p}_\varphi = -N_c k\varphi/r_o^2 \end{array} \tag{59}$$

cannot be solved in a unique way. According to the scheme de-scribed in the preceding sections to implement the Hamiltonian reduction one can search for a transformation to a new set of canonical variables in terms of which the equations of motion separate into independent parts: the physical (independent of the arbitrary function) and the unphysical one with unpredictable evolution. To achieve this let us perform the canonical transformation from (p_a, a) and (p_φ, φ) to the new canonical pairs such that matter part of the constraint Π_φ becomes one of the new canonical momenta

$$\Pi_\varphi = \frac{p_\varphi^2}{2} + \frac{k\varphi^2}{2r_o^2} . \tag{60}$$

Using the generating function

$$S(\Pi_\varphi, \varphi) := \int_{a_o}^{a} da \sqrt{2\Pi_\varphi - \frac{k}{2r_o^2}\varphi^2} , \tag{61}$$

the corresponding canonical conjugated coordinate T_φ is

$$T_\varphi = \int_{a_o}^{a} \frac{da}{\sqrt{2\Pi_a - \frac{k}{2r_o^2}\varphi^2}} \tag{62}$$

and the reduced action reads [†††]

$$W^*[a] = \tag{63}$$

$$\int p_a da + (\frac{\kappa}{12}p_a^2 + \frac{3k}{\kappa r_o^2}a^2)dT_\varphi + d(S(\Pi_\varphi, \varphi) - \Pi_\varphi T_\varphi) ,$$

It is worth mentioning that if instead of matter part the gravitational part of constraint Π_a will be used for the construction of the new canonical momenta then the reduced action describing the evolution of scalar field is

$$W^*[\varphi] = \int p_\varphi d\varphi - \frac{1}{2}(p_\varphi^2 + \frac{k\varphi^2}{r_o^2})dT_\varphi . \tag{64}$$

[†††] Based on this action one can derive the Hubble parameter $H^2 = \frac{1}{a^4(T_c)}\frac{\kappa}{3}\Pi_\varphi - \frac{k}{r_o^2 a^2(T_c)}$ and convince ourselves that it corresponds to the radiation dominanted Fridmann model with the constant Π_φ.

C. Spinor field as source field for FRW Universe

The Hamiltonian reduction of this model is achieved along the same lines as in the previous section. However, dealing with fermion fields there are some specific features due to the presence of the second class constraints.

The action (8) for the homogeneous spinor field in FRW Universe is degenerate and the corresponding primary constraints are

$$\begin{aligned} C_N &:= p_N = 0 \\ C_\psi &:= p_\psi + \tfrac{i}{2}\bar\psi\gamma^o = 0 \\ C_{\bar\psi} &:= p_{\bar\psi} + \tfrac{i}{2}\gamma^o\psi = 0 . \end{aligned} \tag{65}$$

They satisfy the algebra

$$\{C_N, C_\psi\} = 0 , \quad \{C_N, C_{\bar\psi}\} = 0 , \quad \{C_\psi^{(1)}, C_{\bar\psi}^{(1)}\} = -i\gamma^o . \tag{66}$$

According to Dirac prescription the evolution in time is governed by the total Hamiltonian

$$H_T = H_c + \lambda_N C_N + C_\psi\lambda_\psi + \lambda_{\bar\psi}C_{\bar\psi} , \tag{67}$$

with the arbitrary functions λ and the canonical Hamiltonian H_c

$$H_c = N_c\left[-\left(\frac{\kappa p_a^2}{12} + \frac{3ka^2}{\kappa r_o^2}\right) + a\mathcal{H}_D\right] . \tag{68}$$

The requirement to conserve the constraints during the evolution fixes the functions λ_ψ and $\lambda_{\bar\psi}$

$$\lambda_{\bar\psi} = iN_c ma\bar\psi\gamma^o, \qquad \lambda_\psi = iN_c ma\gamma^o\psi . \tag{69}$$

but leaves the function λ_N unspecified it and leads to the existence of the secondary constraint

$$C := \frac{\kappa p_a^2}{12} + \frac{3ka^2}{\kappa r_o^2} - a\mathcal{H}_D = 0 . \tag{70}$$

Due to the algebra of constraints (66) and the Poisson brackets of secondary constraint C with any other

$$\{C,C_\psi\} = ma\bar{\psi}\,, \quad \{C,C_{\bar{\psi}}\} = -ma\psi\,, \quad \{C,C_N\} = 0\,, \qquad (71)$$

one can verify that no additional constraints emerge. ‡‡‡ The algebra (66) and (71) shows that the constraints represent a mixed system of first and second class constraints. In order to perform the Hamiltonian reductuion we will start with rewriting the constraints into an equivalent form such that the first class constraints form the ideal of algebra and the algebra of second class constraints is canonical. This equivalent set of constraints $\bar{C}_\psi, \bar{C}_{\bar{\psi}}$ is given in the Appendix B. The canonical character of the new algebra $\{\bar{C}_\psi, \bar{C}_{\bar{\psi}}\} = -1$ allows to perfom the canonical transformation that converts the new second class constraints $\bar{C}_\psi, \bar{C}_{\bar{\psi}}$ to the pair of canonical variables

$$\begin{aligned} \bar{\Pi}_\psi &= \bar{C}_\psi\,, \quad & \Pi_\psi &= ip_\psi\gamma^o + \tfrac{1}{2}\bar{\psi}\,, \\ \bar{Q}_\psi &= \bar{C}_{\bar{\psi}}\,, \quad & Q_\psi &= p_{\bar{\psi}} - \tfrac{i}{2}\gamma^o\psi\,. \end{aligned} \qquad (72)$$

This means that the dynamics of phase space variables $\bar{Q}_\psi, \bar{\Pi}_\psi$ is completely "frozen" and other canonical pairs change in time independently of them. In other words we can everywhere in the formulas omit this variables without destroying the dynamics of the physically relevant quantities. Turning to the reduction due to the first class constraints let us pass to the new Hamiltonian constraint $\mathcal{C}$

$$\mathcal{C} := \frac{1}{a}C = \frac{1}{a}\left(\frac{\kappa p_a^2}{12} + \frac{3ka^2}{\kappa r_o^2}\right) - im\Pi_\psi\gamma^o Q_\psi\,, \qquad (73)$$

assuming that the metric is nondegenerate $a \neq 0$. In order to achive the reduction for first class constraint we perform the canonical transformation from the (p_a, a) to the new variables (Π_a, Q_a) such that

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‡‡‡Secondary constraint C is conserved in weak sense

$$\dot{C} = iNmp_a(C_\psi\gamma^o\psi + \bar{\psi}\gamma^o C_{\bar{\psi}}) \approx 0.$$

$$\Pi_a = \frac{1}{a}\left(\frac{\kappa p_a^2}{12} + \frac{3ka^2}{\kappa r_o^2}\right). \qquad (74)$$

Using the generating function $S(a,\Pi_a)$

$$S(a,\Pi_a) = \frac{6}{\kappa}\int_{a_o}^{a} da\sqrt{\frac{\kappa}{3}a\Pi_a - ka^2r_o^{-2}}\,, \qquad (75)$$

one can find the variable canonically conjugated to Π_a

$$T_a = \int_{a_o}^{a} ada\left(\frac{\kappa}{3}a\Pi_a - ka^2r_o^{-2}\right)^{-1/2}, \qquad (76)$$

and after projection onto the constraint shell $\mathcal{C} = 0$, $\bar{C}_\psi = 0$, $\bar{\Pi}_\psi = 0$, $\bar{C}_{\bar{\psi}} = 0$, $\bar{Q}_\psi = 0$ the reduced action is

$$W^*[Q_\psi] = \int dQ_\psi\Pi_\psi + m\Pi_\psi\gamma^o Q_\psi dT_a\,. \qquad (77)$$

Thus we have derived the standard Dirac Hamiltonian for reduced spinor field and this matter source corresponds to the case of the dust filled Universe; the Hubble constant behaves as

$$H^2 = \left(\frac{1}{a}\frac{da}{dT_a}\right)^2 = \frac{\kappa}{3}\frac{M_D}{a^3} - \frac{k}{a^2r_o^2} \qquad (78)$$

with the constant M_D.

We will finish with one remark concerning the simple generalization of the above result to a more complex system. It is interesting to note that if one includes the interaction of massive spinor with the scalar masssless one with an action of the following type

$$\begin{aligned} W[g,\Phi,\Psi] = \int d^4x\sqrt{-g}\Big[&-\frac{1}{16\pi G}{}^{(4)}R \\ &+\frac{1}{12}{}^{(4)}R\Phi^2 + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \\ &+\frac{i}{2}\left(\bar{\Psi}\gamma^\mu(x)\nabla_\mu\Psi - \nabla_\mu\bar{\Psi}\gamma^\mu(x)\Psi\right) \\ &-m\bar{\Psi}\Psi - \mu\Phi\bar{\Psi}\Psi\Big]\,, \end{aligned} \qquad (79)$$

then the action obtained after supposition of the FRW Universe

$$W[a,N_c,\varphi,\psi]=\int dt\left[-\frac{3}{\kappa}\frac{\dot{a}^2}{N_c}+\frac{1}{2}\frac{\dot{\varphi}^2}{N_c}+\frac{i}{2}(\bar{\psi}\gamma^o\dot{\psi}-\dot{\bar{\psi}}\gamma^o\psi)\right. \\ \left.-N_c\left(-\frac{3}{\kappa}\frac{ka^2}{r_o^2}+\frac{k\varphi^2}{2r_o^2}+(ma+\mu\varphi)\bar{\psi}\psi\right)\right], \tag{80}$$

can be connected with the action describing the interaction of fermion field and massles scalar field. Let us consider two possible cases.

a). $\kappa m^2<6\mu^2$. One can convince ourself that after introduction of the new scalar field ϕ and the scale factor α

$$ma+\mu\varphi=\mu\phi\sqrt{1-\frac{\kappa}{6}\frac{m^2}{\mu^2}}\,;\quad a+\frac{m}{\mu}\frac{\kappa}{6}\varphi=A\sqrt{1-\frac{\kappa}{6}\frac{m^2}{\mu^2}} \tag{81}$$

we get the action for the massless spinor interacting with the field ϕ

$$W[A,N_c,\phi,\psi]=\int dt\left[-\frac{3}{\kappa}\frac{\dot{A}^2}{N_c}+\frac{1}{2}\frac{\dot{\phi}^2}{N_c}+\frac{i}{2}(\bar{\psi}\gamma^o\dot{\psi}-\dot{\bar{\psi}}\gamma^o\psi)\right. \\ \left.-N_c\left(-\frac{3}{\kappa}\frac{kA^2}{r_o^2}+\frac{k\varphi^2}{2r_o^2}+\tilde{\mu}\phi\bar{\psi}\psi\right)\right], \tag{82}$$

and the new coupling constant

$$\tilde{\mu}=\mu\sqrt{1-\frac{\kappa}{6}\frac{m^2}{\mu^2}}\,. \tag{83}$$

b). $\kappa m^2>6\mu^2$. In this case one can use another transformation,

$$\varphi=\frac{1}{\sqrt{1-\frac{6}{\kappa}\frac{\mu^2}{m^2}}}\left(\phi-\frac{6}{\kappa}\frac{\mu}{m}A\right)\;;\qquad a=\frac{1}{\sqrt{1-\frac{6}{\kappa}\frac{\mu^2}{m^2}}}\left(A-\frac{\mu}{m}\phi\right), \tag{84}$$

and get the action

$$W[A,N_c,\phi,\psi]=\int dt\left[-\frac{3}{\kappa}\frac{\dot{A}^2}{N_c}+\frac{1}{2}\frac{\dot{\phi}^2}{N_c}+\frac{i}{2}(\bar{\psi}\gamma^o\dot{\psi}-\dot{\bar{\psi}}\gamma^o\psi)\right. \\ \left.-N_c\left(-\frac{3}{\kappa}\frac{kA^2}{r_o^2}+\frac{k\phi^2}{2r_o^2}+\tilde{m}A\bar{\psi}\psi\right)\right], \tag{85}$$

with the new mass for the fermion field $\tilde{m}=m\sqrt{1-\frac{6}{\kappa}\frac{\mu^2}{m^2}}$. One can verify that these two actions are related by the field redefinition

$$A\to i\frac{\kappa}{6}\phi,\qquad \phi\to i\frac{6}{\kappa}A, \tag{86}$$

and thus it is enough to reduce one of the actions (82),(85).

For the action (82) the energy constraint

$$C=-\frac{\kappa p_a^2}{12}-\frac{3ka^2}{\kappa r_o^2}+\frac{p_\phi^2}{2}+\frac{k\phi^2}{2r_o^2}+\tilde{\mu}\phi\mathcal{H}_D\,, \tag{87}$$

again has separable contributions from the gravitational and the matter part. After introduction of the new canonical momentum

$$\Pi_a:=\frac{\kappa p_a^2}{12}-\frac{3ka^2}{\kappa r_o^2} \tag{88}$$

and the corresponding conjugated coordinate T_a in the same manner as for the case of the conformal scalar field the following action for the physical scalar and spinor fields can be derived

$$W^*[\phi,\psi]=\int dQ_\psi\Pi_\psi+p_\phi d\phi-HdT_a, \tag{89}$$

with physical Hamiltonian describing the system of interacting spinor and scalar fields

$$H:=\frac{p_\phi^2}{2}+\frac{k\phi^2}{2r_o^2}+\tilde{\mu}\phi\mathcal{H}_D\,. \tag{90}$$

V. CLASSICAL AND QUANTUM OBSERVABLES FOR FRW UNIVERSE

A. Extended quantization: Wheeler-deWitt equation

According to the Dirac prescription in the extended quantization scheme one considers the classical constraints to be the conditions on the state vector Ψ [25], [26] , §§§

$$P_N \Psi = 0\,, \tag{91}$$

$$H_T \Psi = 0\,. \tag{92}$$

Quantum observable in this quantization scheme are constructed with analogy to that of the two-dimensional relativistic spin zero bosonic Klein-Gordon field as expectation value

$$\langle O \rangle = \int d\phi \left(\Psi^* O \partial_a(\Psi) - \partial_a(\Psi^*) O \Psi^*\right)\,. \tag{93}$$

However, as it has been analyzed by Kaup and Vitello [27] this conventional interpretation cannot be used without violating the correspondence principle. More precisely, it was been shown that the expectation values for the scalar fields and the cosmic scale factor do not correspond to the classical values; their evolution describe the expansion phase of Friedmann evolution, but then instead of contraction, the expectation values tunnel through the barrier and continue to expand. In the following section it will be demonstrated that opposite to this scheme of quantization obtained in the preceding part of the paper, the unconstrained system leads to the fulfillment of the correspondence principle..

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§§§The standard procedure of letting $P_N \to -i\partial_N$, $p_a \to -i\partial_a$, $P_\Phi \to -i\partial_{P_\Phi}$ is assumed.

B. Reduced quantization: Heisenberg equation

To analyse the correspondence principle let us consider the case of a conformal scalar field in the closed Friedmann Universe. As it has been shown the evolution of scale factor a in conformal time t is governed by the harmonic oscilator Hamiltonian which after conventional quantization reads

$$\hat{H} = \frac{\kappa}{12}\hat{p}^2 + \frac{3}{\kappa r_o^2}\hat{a}^2\,. \tag{94}$$

Assuming the quantum state in the form

$$\Psi = \frac{1}{(\alpha^2\pi)^{1/4}} \exp\left[\frac{i}{\hbar}p_o a - \frac{(a-a_o)^2}{2\alpha^2}\right] \tag{95}$$

where a_o and p_o are the mean values of the coordinate and the momentum respectively (real parameter α characterizes the mean square deviation of a) and using the solution of Heisenberg equations for the operators $\hat{a}(t)$ and $\hat{p}(t)$

$$\hat{a}(t) = \hat{a}(0)\cos\frac{t}{r_o} - \frac{\kappa r_o}{6}\hat{p}(0)\sin\frac{t}{r_o}\,, \tag{96}$$

$$\hat{p}(t) = \frac{6}{\kappa r_o}\hat{a}(0)\sin\frac{t}{r_o} + \hat{p}(0)\cos\frac{t}{r_o}\,, \tag{97}$$

one can find the time dependence of the mean values of $\hat{a}(t)$ and $\hat{p}(t)$,

$$\overline{a(t)} = \int_{-\infty}^{+\infty} \Psi^* \hat{a}(t)\Psi da = a_o\cos\frac{t}{r_o} - \frac{\kappa r_o}{6}p_o\sin\frac{t}{r_o}\,, \tag{98}$$

$$\overline{p(t)} = \int_{-\infty}^{+\infty} \Psi^* \hat{p}(t)\Psi da = \frac{6}{\kappa r_o}a_o\sin\frac{t}{r_o} + p_o\cos\frac{t}{r_o}\,. \tag{99}$$

This means that we have the correspondence with the classical formulae

$$a(t) = r_o\sqrt{\frac{\kappa}{3}|H|}\sin\left(\frac{Q-T_c}{r_o}\right) \qquad (100)$$

$$p(t) = \sqrt{\frac{12}{\kappa}|H|}\cos\left(\frac{Q-T_c}{r_o}\right) \qquad (101)$$

when constants are taken as

$$\overline{a(0)} = a_o = r_o\sqrt{\frac{\kappa}{3}|H|}\sin\frac{Q}{r_o}\,, \quad \overline{p(0)} = p_o = \sqrt{\frac{12}{\kappa}|H|}\cos\frac{Q}{r_o}\,. \qquad (102)$$

At the end we note that there is no wave packet diffusion when the mean square deviation

$$\overline{(\Delta a(t))^2} = \frac{\alpha^2}{2}\left(\cos^2\frac{t}{r_o} + \left(\frac{\kappa r_o}{6}\right)^2\frac{\hbar^2}{\alpha^4}\sin^2\frac{t}{r_o}\right) \qquad (103)$$

is time independent. This holds for the special value of α

$$\alpha^2 = \hbar\frac{\kappa r_o}{6}\,. \qquad (104)$$

VI. CONCLUDING REMARKS

In the present paper the method of Hamiltonian reduction for reparametrization invariant mechanical systems have been elaborated. This approach is based on the choice of adapted coordinates using the generating function of the canonical transformation that is a solution of the corresponding Hamilton – Jacobi equation. We have derived the reduced Hamiltonians for the Friedmann cosmological models with homogeneous scalar and spinor field matter sources and find the corresponding observable time. The reduced Hamiltonians are the generators of evolution with respect to this time and at the same time represent the conserved quantities which can be treated as the energy of the reduced systems. The conservation of the conformal matter Hamiltonian with respect to conformal time translations follows from conformal symmetry of the Robertson – Walker space-time. The representation for the Hubble parameter

and the red shift is founded in terms of the Dirac observables in the frame of the generalized Hamiltonian dynamics. A correspondence between field Friedmann models and perfect fluid Friedmann models with different equations of state has been established.

The extended quantization with the Wheeler-deWitt equation and the canonical quantization of unconstrained system are compared. In reduced system a Schödinger type equation corresponds to the Wheeler-deWitt equation, the wave function is normalizble with respect to the physical variables. It is shown that quantum observables treated as expectation values of the Dirac observables properly describe the original classical theory.

Finally we make a remark concerning the relation to the conventionol gauge-fixing method. Certainly, the results derived in the present note by reduction without introduction of gauge functions can be reproduced by the gauge fixing method. However, from our derivation it is clear that due to the complicated relations between the initial variables and the observable time the gauge functions depend on the initial variables in a complex way which is difficult to guess.

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APPENDIX A: DIRAC EQUATION IN FRW SPACE TIME

To describe a spinor field on a Rimanian manifold the vierbein fields $h^\mu_a(x)$ $\mu, \nu, a, b = 0, 1, 2, 3$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}(h^a_\mu dx^\mu)(h^b_\nu dx^\nu)\,; \qquad \eta_{ab} := (+\,-\,-\,-),$$

and the Dirac γ-matricies with a specific dependence on space time coordinates are introduced

$$\gamma^\mu(x) = h^\mu_a(x)\gamma^a.$$

The following relations between the vierbein fields and the metric tensor $g_{\mu\nu}$ hold

$$\begin{aligned} & h^\mu_a h_{b\mu} = \eta_{ab} \;; \quad h^a_\mu h_{a\nu} = g_{\mu\nu} \;; \\ & h^a_\mu h^\mu_b = \delta^a_b \;; \quad h^a_\mu h^\nu_a = \delta^\nu_\mu \;; \quad h_{a\mu} = \eta_{ab} h^b_\mu = g_{\mu\nu} h^\nu_a \,. \end{aligned} \tag{A1}$$

The Dirac equation for spinors in curved space time reads

$$(i\gamma^\mu(x) \nabla_\mu - m)\Psi(x) = 0 \,, \tag{A2}$$

whith the covariant derivative

$$\nabla_\mu \Psi(x) = [\partial_\mu + \frac{1}{4} C_{abc} h^c_\mu \gamma^b \gamma^a]\Psi(x) \,, \tag{A3}$$

where Ricci coefficients

$$C_{abc} \equiv (\nabla_\mu h^\nu_a) h_{b\nu} h^\mu_c \;; \quad \nabla_\mu h^\nu_a = (\Gamma^\nu_{\mu\lambda} - h^\nu_b \partial_\mu h^b_\lambda) h^\lambda_a \,, \tag{A4}$$

are introduced. For the specific case of the Robertson – Walker metric,

$$\begin{aligned} ds^2 &= a^2(t) \tilde{d}s^2 \\ &= a^2(t) \left[(N(t)dt)^2 - \frac{\left(dr^2 + r^2(d\xi^2 + \sin^2 \xi d\zeta^2)\right)}{\left(1 + \frac{kr^2}{4r_o^2}\right)^2} \right] , \end{aligned} \tag{A5}$$

the following vierbein fields

$$\begin{cases} h^{\underline{o}}_o = aN & h^{\underline{1}}_1 = a \left(1 + \frac{kr^2}{4r_o^2}\right)^{-1} \\ h^{\underline{2}}_2 = ar \left(1 + \frac{kr^2}{4r_o^2}\right)^{-1} & h^{\underline{3}}_3 = ar \sin\zeta \left(1 + \frac{kr^2}{4r_o^2}\right)^{-1} \end{cases} \tag{A6}$$

are used in the main text. Here the vierbein indices are underlined. The Dirac equation then looks

$$\frac{i}{a}\left[\gamma^o \frac{1}{N}\frac{\partial}{\partial t} + \gamma^1 \left(1 + \frac{kr^2}{4r_o^2}\right)\frac{\partial}{\partial r} + \gamma^2 \frac{1 + \frac{kr^2}{4r_o^2}}{r}\frac{\partial}{\partial \zeta} + \gamma^3 \frac{1 + \frac{kr^2}{4r_o^2}}{r \sin\zeta}\frac{\partial}{\partial \xi} \right.$$
$$\left. + \frac{3\dot{a}}{2aN}\gamma^o + \frac{1 - \frac{kr^2}{4r_o^2}}{r}\gamma^1 + \frac{\cot\zeta}{2r}\left(1 + \frac{kr^2}{4r_o^2}\right)\gamma^2 \right] \Psi(x) - m\Psi(x) = 0 \,.$$

To maintain the space homogeniety of the Friedmann Universe we suppose that the spinor field is only time dependent. In the main text the FRW Universe with the spinor matter source is formulated in terms of the fermion variable ψ

$$\psi(t) = a^{3/2}(t)\Psi(t). \tag{A7}$$

APPENDIX B: SEPARATION OF FIRST AND SECOND CLASS CONSTRAINTS IN MODEL WITH SPINOR FIELD

The set of constraints $C_A = (C_\psi, C_{\bar{\psi}}, C)$ represent a mixed system of first and second class constraints; the rank of the Poisson matrix $\mathcal{M} = \{C_A, C_B\}$ is equal to two. The explicit form of the Poissson matrix is

$$\mathcal{M} = \begin{pmatrix} \triangle & K \\ -K^T & 0 \end{pmatrix} ,$$

where and $\triangle$ and K denote

$$\triangle = \begin{pmatrix} 0 & -i\gamma^o \\ -i\gamma^o & 0 \end{pmatrix} \qquad K = \begin{pmatrix} -ma\bar{\psi} \\ ma\psi \end{pmatrix} .$$

In order to perform the reduction procedure it is useful to separate first and second class constraints. One can easily verify that applying the similarity transformation T

$$T = \begin{pmatrix} 1 & 0 \\ K^T \triangle^{-1} & 1 \end{pmatrix} , \qquad \mathrm{Sdet} T \neq 0$$

to the constraints C_A

$$\tilde{C} = T \cdot C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ima\gamma^o\psi & -iam\bar{\psi}\gamma^o & 1 \end{pmatrix} \cdot \begin{pmatrix} C_\psi \\ C_{\bar{\psi}} \\ C \end{pmatrix}$$

we achieve the separation of the constraints on the surface defined by the second class constraints

$$\{\tilde{C}, \tilde{C}_\psi\} = ima\tilde{C}_\psi\gamma^o \qquad \{\tilde{C}, \tilde{C}_{\bar{\psi}}\} = -ima\gamma^o\tilde{C}_{\bar{\psi}} .$$

To have this separation on the whole phase space one can pass to the new set of constraints

$$\begin{aligned} \bar{C} &= \tilde{C} + ma\tilde{C}\psi\tilde{C}_{\bar{\psi}} \\ &= \Pi + ma\left(p_\psi p_{\bar{\psi}} - \frac{i}{2}[p_\psi\gamma^o\psi + \bar{\psi}\gamma^o p_{\bar{\psi}}]\right) - \frac{1}{4}a\mathcal{H}_D, \end{aligned} \tag{B1}$$

$$\bar{C}_\psi = -i\tilde{C}_\psi\gamma^o = -ip_\psi\gamma^o + \frac{1}{2}\bar{\psi}, \tag{B2}$$

$$\bar{C}_{\bar{\psi}} = \tilde{C}_{\bar{\psi}} = p_{\bar{\psi}} + \frac{i}{2}\gamma^o\psi . \tag{B3}$$

In this new set $\bar{C}$ belongs to the ideal of the algebra of constraints

$$\{\bar{C}, \bar{C}_\psi\} = \{\bar{C}_N, \bar{C}_{\bar{\psi}}\} = 0 , \tag{B4}$$

and second class constraints $\bar{C}_\psi, \bar{C}_{\bar{\psi}}$ obey the canonical algebra

$$\{\bar{C}_\psi, \bar{C}_{\bar{\psi}}\} = -1 . \tag{B5}$$

APPENDIX C: REDUCED HAMILTONIAN AS CONSERVED QUANTITY FROM CONFORMAL SYMMETRY

In this Appendix we discuss the existence of time independed reduced Hamiltonians from the geometrical standpoint. The Friedmann – Robertson – Walker space-time is conformally flat

$$ds^2_{FRW} = A^2(x)ds^2_{Minkowski} . \tag{C1}$$

In the flat Friedmann Universe the conformal factor $A(x)$ is simple scale factor $a(T_c)$ and it is easy to veify that the conformal time translation is a conformal symmetry

$$\pounds_{\partial_{T_c}} g_{\mu\nu} = \pounds_{\partial_{T_c}}(a^2(T_c)\eta_{\mu\nu}) = \eta_{\mu\nu}\partial_{T_c}a^2(T_c) = g_{\mu\nu}2\frac{\dot{a}}{a}. \tag{C2}$$

It is well-known that if space time possesses the conformal Killing vector and matter energy-momentum tensor is traceless, then one can construct the conserved quantity as follows. Considering the covariant derivative of contraction of the stress tensor and the confomal Killing vector :

$$\begin{aligned} \nabla_\mu P^\mu &= \nabla_\mu(\xi_\nu T^{\mu\nu}) = \xi_\nu \nabla_\mu T^{\mu\nu} + T^{\mu\nu}\frac{1}{2}(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu) \\ &= \xi_\nu \nabla_\mu T^{\mu\nu} + \frac{1}{n}T^\mu_\mu \nabla^\nu \xi_\nu , \end{aligned}$$

and assuming the covariant conservation of the traceless matter energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0 , \qquad T^\mu_\mu = 0 , \tag{C3}$$

we have the conservation law for four-vector P^μ in the covariant differential form

$$\nabla_\mu P^\mu = 0. \tag{C4}$$

To get the global conserved quantity one can integrate this equality over the whole space-time and use Gauss theorem

$$\begin{aligned} \int_V d^4x\sqrt{-g}\,\nabla_\mu\,(\xi_\nu T^{\mu\nu}) &= \int_V d^4x\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\xi_\nu T^{\mu\nu}\right) \\ &= \int_{T_c'} (\xi_{T_c})_\nu T^{\mu\nu}\sqrt{-g}d^3x - \int_{T_c''} (\xi_{T_c})_\nu T^{\mu\nu}\sqrt{-g}d^3x \end{aligned} \tag{C5}$$

where in the last line we specify the Killing vector corresponding to the conformal translation in Robertson – Walker space-time. For the conformal scalar field with Lagrangian

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{12}\ {}^{(4)}\!R\Phi^2\right) , \tag{C6}$$

the canonical stress tensor

$$T^C_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - g_{\mu\nu}\frac{1}{\sqrt{-g}}\mathcal{L} \tag{C7}$$

has nonzero trace $T^{C\mu}_{\mu} \neq 0$. However one can pass to the improved [28] tensor

$$T_{\mu\nu} = T^C_{\mu\nu} - \frac{1}{6}\left[-\ {}^{(4)}\!R_{\mu\nu} + \partial_\mu\partial_\nu - g_{\mu\nu}\partial^\mu\partial_\mu\right]\Phi^2 , \tag{C8}$$

which is traceless $T^\mu_\mu = 0$. Thus for a conformal time Killing vector in adapted coordinates $\xi_{T_c} = (1,0,0,0)$ and for homogeneous scalar field $\varphi(T_c) = a(T_c)\Phi(T_c)$ from eq. (C5) it follows that

$$H = \int_{T_c}(\xi_{T_c})_o T^{oo}\sqrt{-g}d^3x = V_{(3)}\left(\frac{p^2_\varphi}{2} + \frac{k\varphi^2}{2r^2_o}\right) \tag{C9}$$

is conserved charge that coincides with the reduced Hamiltonian derived in the main text.

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