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GEOMETRY OF CONFORMAL MECHANICS

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1. Introduction

Conformally invariant field theories are of use in a wide range of phenomena. Conformal models in two dimensions are of particular interest as they constitute a field-theoretical basis of strings and superstrings. They describe possible string compactifications, provide explicit field realizations of Virasoro and super-Virasoro algebras, make it easy to establish a correspondence between the string theory and the d=2 statistical systems, etc.[1]. The geometric structure of these models is expected to encode the characteristic features of the geometry underlying string and superstring dynamics and so it deserves thorough analysis.

Many aspects of d=2 field theory are well modelled by its d=1 prototype, that is the quantum mechanics. In particular, the theories of a point particle and a superparticle were intensively studied for the last years, with focusing on their similarities with the string and superstring theories. A good deal of attention was paid to supersymmetric quantum mechanics which has interesting applications in its own right[2]. In view of the important role of conformal field theory it seems instructive to apply to studying conformal[3] and superconformal [4,5] mechanics as these provide the simplest examples of such a theory. They reveal amusing analogies with the special class of d=2 conformal models, the Liouville and super-Liouville ones. The latter have profound implications in string and superstring theories[6,7] and enjoy remarkable geometric properties, such as full integrability.

In the pioneer paper by de Alfaro, Fubini and Furlan[3] as well as in the subsequent papers[4,5] devoted to supersymmetric versions of conformal mechanics the main emphasis was made on quantum-mechanical aspects of these models (spectrum, the structure of Hilbert space, etc.). At the same time, their geometric basics were not understood in full generality even at the classical level. Such an understanding might be conducive both to achieving a deeper insight into the geometry of d=2 conformal theories (e.g., the Liouville and super-Liouville ones) and to constructing higher N superextensions of conformal mechanics[1]. Up to now, only the N=2 and N=4 superconformal mechanics have been constructed /4,5/. A manifestly invariant superfield off-shell formulation was given only for N=2 case/4/.

In the present and forthcoming papers we propose a universal geometric framework for treating conformal mechanics and its superconformal extensions. These systems will be shown to be related to the geodesic motion on group manifolds of d=1 conformal and superconformal groups. The basis of our consideration is the covariant reduction method developed earlier by two of us/8/ in application to the d=2 Liouville-type systems. It proved to be an effective tool for algorithmic construction of higher N superextensions of the Liouville equation and was recently used to set up a new wide class of d=2 superconformal sigma-models with the Wess-Zumino action/9,10/. Geometrically, this method amounts to singling out certain finite-dimensional geodesic hypersurfaces in infinite-dimensional coset manifolds of d=2 conformal and superconformal groups. The Liouville and super-Liouville equations naturally emerge as the most essential conditions among those specifying these hypersurfaces. To put the method in force, one merely needs to know the structure relations of the corresponding d=2 superconformal algebra.

The equations of conformal and superconformal mechanics are generated when applying the same techniques to the group spaces of d=1 conformal and superconformal groups. These groups are finite dimensional so all the things go simpler than in the d=2 case. This makes it possible to understand more clearly the geometric meaning of covariant reduction.

In the present paper we give an account of our approach by the simplest example of bosonic (N=0) conformal mechanics. Our consideration will be purely classic. Supersymmetric case will be treated in the forthcoming paper where we will construct off-shell superfield formulations of N=4 superconformal mechanics.

The matter is organized as follows. In Sect. 2 we interpret conformal mechanics in terms of Cartan's 1-forms on the parameter space of d=1 conformal group SO(1,2) subject to a kind of covariant reduction. In Sect. 3 we explain the geometric meaning of this procedure and prove that the equation of conformal mechanics defines a class of geodesics on the group manifold. A simple geometric method of integrating this equation is also presented. It admits a straightforward extension to more complicated cases including the supersymmetric one.

In the Appendix we establish the relation with the customary description of geodesics in terms of the metric on the manifold.

2. Conformal mechanics and the nonlinear realization of group SO(1,2)

We begin with recalling the basics of conformal mechanics. It is defined by the equation/3/ (We consider the one-component case)

$$\ddot{\rho}(t) = \gamma^2 \frac{1}{\rho^3} , \quad [\gamma] = cm^{-2}, \quad [\rho] = cm^0 \tag{2.1}$$

which follows from the action

$$S = \frac{1}{\lambda^2} \int dt \left[(\dot{\rho})^2 - \frac{\gamma^2}{\rho^2} \right] , \quad [\lambda^2] = cm^{-1}. \tag{2.2}$$

The system (2.1), (2.2) respects invariance under transformations of the d=1 conformal group SO(1,2)

$$\delta t = a + bt + ct^2 \equiv f(t)$$
$$\delta \rho(t) = \frac{1}{2} \dot{f}(t) \tag{2.3}$$

where a, b, c are, respectively, infinitesimal parameters of d=1 translation (L_{-1}), dilatation(L_0) and conformal boost (L_{+1}) . The generators L_n form the algebra $SO(1,2) \sim Sl(2,R)$:

$$i [L_n, L_m] = (n-m) L_{n+m}; \quad n, m = -1, 0, 1 \tag{2.4}$$

(The simplest representation of L_n is via Pauli matrices, $L_{\pm 1} = \frac{1}{2}(\tau^1 \pm i\tau^2)$, $L_0 = \frac{i}{2}\tau^3$). This notation demonstrates that $SO(1,2)$ is a finite-dimensional prototype of d=2 conformal (i.e. Virasoro) algebra (and enters into the latter as a maximal subalgebra).

Our aim is to relate the system (2.1), (2.2) to the geometry of group SO(1,2). It will be convenient to choose the following parametrization of this group:

$$g(x^1, x^2, x^3) = e^{ix^1 L_{-1}} e^{ix^2 L_{+1}} e^{ix^3 L_0} \tag{2.5}$$

Nonlinear SO(1,2) transformations in the space of parameters $\{x^i\}$ are induced by left multiplications of group element (2.5).

$$g_0(a, b, c)\, g(x^1, x^2, x^3) = g(x^{1\prime}, x^{2\prime}, x^{3\prime}). \tag{2.6}$$

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1) By N we mean the number of real spinor generators.

$$\delta x^1 = a + b x^1 + c(x^1)^2 \equiv f(x^1)$$

$$\delta x^2 = \tfrac{1}{2} f''(x^1) - f'(x^1)\, x^2 \qquad (2.7)$$

$$\delta x^3 = f'(x^1).$$

It is seen that x^1 and $e^{\frac{1}{2}x^3}$ transform just as the quantities t and ρ in eqs. (2.3). In what follows, this will allow us to identify both sets. The property that the line submanifold $\{x^1\}$ is closed under the action of SO(1,2) is related to the fact that x^1 parametrizes the left coset of SO(1,2) over the subgroup with generators L_{+1}, L_0.

The local geometric properties of group manifold $\{x^i\}$ are specified by the left-invariant Cartan 1-forms:

$$g^{-1} dg = i\omega^n L_n \qquad (2.8)$$

$$\omega^{-1} = e^{-x^3} dx^1$$
$$\omega^0 = dx^3 - 2x^2 dx^1 \qquad (2.9)$$
$$\omega^{+1} = e^{x^3}\left[dx^2 + (x^2)^2 dx^1\right]$$

which are nothing else than the SO(1,2) convariant differentials of coordinates. The invariant line element dS^2 is constructed from these forms. Representing L_n by Pauli matrices and choosing an appropriate normalization, dS^2 can be written as

$$dS^2 = -tr\left(g^{-1}dg\, g^{-1}dg\right) = 2\omega^{-1}\omega^{+1} - \tfrac{1}{2}\omega^0\omega^0 = \qquad (2.10)$$
$$= 2dx^1 dx^2 - \tfrac{1}{2}(dx^3)^2 + 2x^2 dx^1 dx^3 \equiv$$
$$\equiv g_{ij}\, dx^i dx^j .$$

Let us now identify x^1 with the time t and consider an arbitrary curve in $\{x^i\}$.

$$t = x^1, \quad x^2 = x^2(t), \quad x^3 = x^3(t). \qquad (2.11)$$

Now the group SO(1,2) is parametrized by the time t and the Goldstone fields $x^2(t)$, $x^3(t)$ which specify the embedding of the curve in $\{x^i\}$ and correspond to the conformal boost and dilatation, respectively. Thus we are left with the nonlinear realization /11-13/ of d=1 conformal group. At this stage, it is convenient to pass to the quantities with physical dimension ($[t] = cm$, $[x^2] = cm^{-1}$, $[x^3] = cm^0$), making use of the automorphism of the algebra (2.4) $L_{-1} \to f L_{-1}$, $L_{+1} \to f^{-1} L_{+1}$, $L_0 \to L_0$ where f is an arbitrary constant (it can be dimensionful).

So far, our consideration was purely kinematical; the t-dependence of fields $x^2(t)$, $x^3(t)$ was unrestricted. Just as in the case of nonlinear realization of the d=2 conformal group /8/, the dynamics arises as a result of imposing the covariant reduction conditions on coordinates $\{t, x^2(t), x^3(t)\}$. This reduction proceeds in general as follows /8/. One sets equal to zero all the Cartan forms except for those belonging to some subalgebra of the initial algebra. In the d=2 case such a subalgebra was chosen to be either $SO(1,2)$ or the algebra of d=2 Poincaré group. For the corresponding dilaton field there appeared, respectively, either the Liouville equation or the free massless one. In the present case, we will perform the reduction to a subalgebra with the one generator

$$R_0 = L_{-1} + m^2 L_{+1} . \qquad (2.12)$$

One may check that this generator corresponds to compact $SO(2)$ subalgebra of $SO(1,2)$. Thus we impose the constraints

$$g^{-1}dg = g_R^{-1} dg_R = i\omega^{-1} R_0 \qquad (2.13)$$

that amount to the set of Pfaff's equations

$$\omega^0 = 0 \;\Rightarrow\; \tfrac{1}{2}\dot{x}^3 = x^2 \qquad (a)$$
$$\omega^{+1} = m^2\omega^{-1} \;\Rightarrow\; \dot{x}^2 + (x^2)^2 = m^2 e^{2x^3} \qquad (b) \qquad (2.14)$$

The first one is kinematical, it covariantly expresses the Goldstone field x^2 as the derivative of the dilaton, thereby realizing the inverse Higgs phenomenon /14/. Indeed, it follows from the transformation laws (2.7) that x^2 transforms just as $\frac{1}{2}\dot{x}^3$. On substitution of the expression for x^2 into eq.(2.14b) the latter becomes

$$\ddot{x}^3 + \tfrac{1}{2}(\dot{x}^3)^2 = 2m^2 e^{2x^3} \qquad (2.15)$$

which is easily recognized as the equation of conformal mechanics (2.1) after identifying $\rho(t) = e^{\frac{1}{2}x^3(t)}$, $\gamma^2 = m^2$.

Thus, we have derived eq.(2.1), starting with the group space of SO(1,2) where SO(1,2) is realized by left shifts and further constraining covariant differentials of coordinates by eqs. (2.14). The geometric meaning of this procedure will be clarified in Sect. 3. Here we would like to note that one might choose a more general combination of SO(1,2) generators than in eq. (2.12)

$$\tilde{R}_0 = L_{-1} + \tilde{m}^2 L_{+1} + 2\alpha L_0 . \tag{2.16}$$

Then, instead of eqs.(2.15), one would have the more general set of equations

$$\begin{aligned} &\omega^0 = 2\alpha\,\omega^{-1} \quad \Rightarrow \quad \tfrac{1}{2}\dot{x}^3 = x^2 + \alpha e^{-x^3} && \text{(a)} \\ &\omega^{+1} = \tilde{m}^2 \omega^{-1} \quad \Rightarrow \quad \dot{x}^2 + (x^2)^2 = \tilde{m}^2 e^{-2x^3} . && \text{(b)} \end{aligned} \tag{2.14'}$$

Substitution of eq.(2.14'a) into (2.14'b) yields again eq. (2.15), now with $m^2 = \tilde{m}^2 - \alpha^2$:

$$\ddot{x}^3 + \tfrac{1}{2}(\dot{x}^3)^2 = 2\left(\tilde{m}^2 - \alpha^2\right) e^{-2x^3} . \tag{2.17}$$

So, for $\alpha^2 < \tilde{m}^2$ we have the standard conformal mechanics while for $\alpha^2 > \tilde{m}^2$ we get the "hyperbolic" version of eq.(2.1)/3/. For $\alpha^2 = \tilde{m}^2$ the equation reduces to the free one. These three different situations correspond to three possible nonequivalent covariant reductions of the manifold $\{x^i\}$. Indeed, it is a simple exercise to check that the generators (2.16) with the parameters $\tilde{m}^2$, α^2 varying within the above three domains cannot be related to each other by any SO(1,2)-rotation and so belong to different orbits in the group space of SO(1,2). Actually, the term $\sim L_0$ in $\tilde{R}_0$ can always be removed by a proper SO(1,2) rotation :

$$R_0 = L_{-1} + (\tilde{m}^2 - \alpha^2) L_{+1} = e^{-i\alpha L_{+1}} \tilde{R}_0 \, e^{i\alpha L_{+1}} \tag{2.18}$$

that amounts to a constant right shift of $g(x^1, x^2, x^3)$ as

$$g(x^1, \tilde{x}^2, x^3) = g(x^1, x^2, x^3)\, e^{i\alpha L_{+1}} \Rightarrow \tilde{x}^2 = x^2 + \alpha e^{-x^3} \tag{2.19}$$

In terms of x^1, $\tilde{x}^2$, x^3, the set (2.14') looks just as (2.14), with $m^2 = \tilde{m}^2 - \alpha^2$. Different types of covariant reduction are thus associated with three nonequivalent one dimensional subalgebras of $so(1,2)$:

$$L_{-1} + m^2 L_{+1} , \quad L_{-1} - m^2 L_{+1} , \quad L_{-1} . \tag{2.20}$$

Recall that the first subalgebra is $so(2)$ while the second one is $so(1,1)$. As will be shown in Sect. 3, these three patterns correspond to three nonequivalent classes of geodesics on SO(1,2).

To close this Section, we present a simple invariant first-order action for the system (2.14) in terms of differential 1-forms (2.9)

$$\begin{aligned} S &= \frac{1}{\lambda^2}\int_{\ell} \left[\omega^{+1} + m^2 \omega^{-1}\right] = \\ &= -\frac{1}{\lambda^2}\int dt \left\{ e^{x^3(t)}\left[\dot{x}^2(t) + (x^2)^2\right] + m^2 e^{-x^3(t)} \right\} . \end{aligned} \tag{2.21}$$

Varying x^2 yields eq.(2.14a). Inserting this constraint back into eq.(2.14b) brings the latter into the standard second-order form (2.1) (with $\rho(t) = e^{\frac{1}{2}x^3(t)}$).

3. Geometric interpretation

Let us explain the geometric meaning of constraints (2.14). We will discuss the reduction to $so(2)$ subalgebra (2.12), keeping in mind the relation (2.18) and the fact that the generators of other possible reduction subalgebras (listed in eq.(2.10)) follow from R_0 (2.12) either by substitution $m \to im$ or by putting $m^2 = 0$.

Differential forms (2.9), being covariant differentials of SO(1,2)-coordinates x^i, specify infinitesimal shifts of x^i along three independent directions in $\{x^i\}$. The constraints (2.14) restrict this motion to the shift along a curve generated by the right action of abelian subgroup with generator R_0 . Indeed, solving eq.(2.13) for $g_R(t, x^2(t), x^3(t))$, we find that the most general solution is

$$g_R(t, x^2, x^3) = g_0(c^1, c^2, c^3)\, e^{i\tau(t)\left(L_{-1} + m^2 L_{+1}\right)} , \tag{3.1}$$

where c^i are integration constants and

$$d\tau = e^{x^3(t)}\, dt . \tag{3.2}$$

It is easy to argue that eq.(3.1) defines a geodesic on the manifold $\{x^i\}$. It is known/15/ that the geodesic motion on the coset or group manifolds is generated by the right action of the group on the coset elements. In the group space, any such element specifies a point whence some geodesic grows. The geodesic as a whole is restored by multiplying this fixed element from the right by an element of onetain abelian subgroup having as the group parameter the natural parameter along the curve (the group is assumed to be taken in the exponential parametrization).The choice of this subgroup fixes the tangent to the geodesic at the origin. Thus, the geodesic on a group space is completely defined by choosing an initial group element and some one-parameter subgroup acting on the former element from the right.

The formula (3.1) ideally fits in this general scheme. To prove that the SO(1,2)-element (3.1) defines a geodesic, we merely need to show the identity of τ with the natural parameter S . Inserting eqs.(2.14) into the definition (2.10) and taking account of eq.(3.2) one gets

$$dS^2 = 2m^2\omega^{-1}\omega^{-1} = 2m^2 e^{-2x^3}(dt)^2 = \tag{3.3}$$
$$= 2m^2(d\tau)^2 \Rightarrow \frac{dS}{d\tau} = \sqrt{2}\,m$$

i.e. τ actually coincides with S (up to a constant shift and rescaling).

Expression (3.1) provides the general solution to the constraints (2.14) and, hence, to the conformal mechanics equation (2.15) (or (2.1)) which is equivalent to the set (2.14). So we have shown that this equation describes a class of geodesics on the group SO(1,2), with choosing the coordinate $x^1 = t$ as a parameter along the geodesic. For these geodesics $dS^2 > 0$, so they can be called "time-like" ones. Two other types of geodesics on SO(1,2), which are obtained by the reduction to two other subalgebras among those listed in (2.20), correspond, respectively, to $dS^2 < 0$ and $dS^2 = 0$ .Thus they are "space-like" or "light-like". In the latter case (it is described by the free $m^2 = 0$ version of eq.(2.15)) $|S|$ cannot serve as an evolution parameter, while τ or t still can. In the Appendix we establish the explicit relation to a more familiar description of geodesics in terms of the metric g_{ij} introduced by eq.(2.10).

The geometric approach allows us to render a transparent meaning to the procedure of integrating eq.(2.1). It is reduced now to finding out the explicit expressions for the original variables $\{x^i\}$ in terms of entries of the on-shell matrix (3.1). The constant factor g_0 entering into eq.(3.1) actually involves only two independent integration constants which parametrize the coset SO(1,2)/SO(2). The third one can always be absorbed into a redefinition of τ . It is convenient to choose g_0 as

$$g_0 = e^{ic^1 L_{-1}} e^{ic^3 L_0}. \tag{3.4}$$

Substituting into eq.(3.1) the expression for $g_R(t, x^2, x^3)$ (2.5) one finds

$$t = c^1 + \frac{1}{m} e^{c^3} tg(m\tau)$$
$$x^2 = e^{-c^3} \frac{m}{2} \sin(2m\tau) \tag{3.5}$$
$$x^3 = c^3 - 2\ln\cos(m\tau)$$

that yields the explicit parametrization of geodesic in terms of proper time τ (or S). In accord with the geometric interpretation of eq.(3.1) given above, we have (the use was made of eq.(3.3)):

$$t(s=0) = c^1 \qquad \left.\frac{dt}{dS}\right|_{s=0} = \frac{1}{\sqrt{2}\,m} e^{c^3}$$
$$x^2(s=0) = 0 \qquad \left.\frac{dx^2}{dS}\right|_{s=0} = \frac{m}{\sqrt{2}} e^{-c^3} \tag{3.6}$$
$$x^3(s=0) = c^3 \qquad \left.\frac{dx^3}{dS}\right|_{s=0} = 0$$

whence it follows that the constants c^1, c^3 parametrize an initial point on the geodesic. We also see that, up to an unessential rescaling, the coupling constant m defines the components of the tangent vector to the geodesic at this point.

It is a simple exercise to extract from eqs.(3.5) the general solution of eq.(2.1)

$$\rho(t) \equiv e^{\frac{1}{2}x^3(t)} = \sqrt{A\left(1 + \frac{B}{A}t\right)^2 + A^{-1}m^2t^2}, \tag{3.7}$$

where

$$A = e^{c^3} + m^2(c^1)^2 e^{-c^3}, \quad B = -c^1 m^2 e^{-c^3}. \tag{3.8}$$

Any other form of the solution is reduced to (3.7) by a redefinition of integration constants.

One may check that the general solution (3.7) is invariant under the action of the SO(2)-subgroup generated by

$$R = L_1 + 2\frac{B}{A}L_0 + A^{-2}(m^2 + B^2)L_{-1}. \tag{3.9}$$

$$\delta^{*}\rho(t) = \tfrac{1}{2}\dot{f}_R(t)\rho(t) - f_R(t)\dot{\rho}(t) = 0 \,, \tag{3.10}$$

$$f_R(t) = a\{1 + 2\frac{B}{A}t + A^{-2}(m^2+B^2)t^2\} \,. \tag{3.11}$$

Thus, there occurs the dynamical spontaneous breaking of SO(1,2) to SO(2) $\propto R$ /3/. This phenomenon has a simple interpretation in terms of geodesics. The generator R is related to R_0(2.12) via the SO(1,2)-rotation by the element g_0 (3.4)

$$R = A^{-1} g_0 [L_{-1} + m^2 L_{+1}] g_0^{-1} .$$

So, the left action of $\exp\{iaR\}$ on $g_R(x^1,x^2,x^3)$ (3.1) merely results in the shift of proper time τ by an amount aA^{-1}, without affecting the shape of geodesic, $x^i(\tau) \to x^i(\tau + aA^{-1})$. In other words, the left action of $\exp\{iaR\}$ generates the shift along a given geodesic. The action of SO(1,2)/SO(2)-transformations changes the integration constants and so transforms one geodesic into another.

It is worthwhile to note that the integration of eq.(2.1) can be also viewed as a reparametrization of the group space of SO(1,2). Indeed, let us choose from the beginning a different parametrization of SO(1,2)

$$g(x^1,x^2,x^3) = g(c^1,c^3,\tau) = e^{ic^1L_{-1}}\, e^{ic^3L_0}\, e^{i\tau(L_{-1}+m^2L_1)} .$$

Then eqs.(3.5) give the relation between the two equivalent parametrizations of SO(1,2). The Cartan forms in this new parametrization are as follows

$$\omega^{-1} = e^{-c^3} dc^1 \cos(2m\tau) + \tfrac{1}{m} dc^3 \sin(2m\tau) + \tfrac{1}{m^2}\omega^{+1}$$

$$\omega^{0} = m e^{-c^3} dc^1 \sin(2m\tau) + dc^3 \cos(2m\tau)$$

$$\omega^{+1} = m^2\{d\tau + \tfrac{1}{2} e^{-c^3} dc^1 [1 - \cos(2m\tau)] - \tfrac{1}{2m} dc^3 \sin(2m\tau)\}$$

One is free to impose the constraints (2.14) in any parametrization. It is easy to check that in terms of new variables these constraints are reduced to

$$\frac{dc^1}{d\tau} = \frac{dc^3}{d\tau} = 0 \,, \qquad c^1, c^3 \text{ are constants} \tag{3.12}$$

Expressing c^1, c^3 via original variables, one obtains two first integrals of eq.(2.1) (it is convenient to pass to the variables A and B given by (3.8))[*]:

$$A(t) = (\rho - t\dot{\rho})^2 + \frac{m^2}{\rho^2}t^2 \,; \quad B(t) = \dot{\rho}(\rho - \dot{\rho}t) - \frac{m^2}{\rho^2}t \tag{3.13}$$

$$\dot{A} = \dot{B} = 0 .$$

Note that the variables $c^1(t)$, $c^3(t)$, $\tau(t)$ are in a sense analogous to the action-angle variables of two-dimensional integrable systems.

We would like to mention that the one more way of solving eq. (2.1) is to reduce the latter to the harmonic oscillator equation. Introducing $\tilde{\rho} = \rho^{-1}$ and going to the proper time τ by eq.(3.2) one may rewrite (2.1) as

$$\frac{d^2\tilde{\rho}}{d\tau^2} + m^2\tilde{\rho} = 0 .$$

Solving this equation and expressing τ in terms of t from the first-order eq.(3.2) one arrives again to the expression (3.7).

4. Conclusions

In this paper we have demonstrated that the covariant reduction method proposed originally for unified geometric description of the Liouville-type systems in two dimensions /8,9/, equally applies to d=1 systems, i.e. the models of particle mechanics. The foundations of the method can be clearly understood when looking at the d=1 case. The simple example we have analyzed here in detail is mostly of illustrative character, though it perhaps would be of some interest to see what are the implications of this geometric picture in the quantum case. The actual power of the covariant reduction approach will be demonstrated in our forthcoming paper where this technique is applied to N=4, d=1 superconformal group SU(1,1/2) to construct a manifestly invariant superfield formulation of N=4 superconformal mechanics.

It is worth mentioning that the model we have considered belongs to a wide class of completely integrable d=1 systems. The list of corresponding potentials can be found, e.g., in the review by Olshanetsky and Perelomov /16/. An interesting task is to reproduce in

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*) Note that the energy $H = (\dot{\rho})^2 + \frac{m^2}{\rho^2}$ is expressed through these quantities by the simple formula

$$H = A^{-1}[m^2 + B^2] .$$

our approach the remaining potentials from this list (and, perhaps, to discover the unknown ones), starting with a nonlinear realization of an appropriate group and imposing the covariant reduction constraints on the relevant Cartan 1-forms. Also, it would be desirable to understand the relationship with the general method of integrating these systems which has been proposed by two authors cited above. The method is based on relating the equation associated with a given integrable potential to the free (or geodesic) motion on a certain higher dimensinnal auxiliary space. So it bears some formal analogies to ours. We would like to emphasize once again that the main merit of our scheme should be seen in its algorithmic character. One chooses the group and the covariant reduction subgroup (the latter can be in general nonabelian), after that all the things (deducing the relevant mechanical system and finding out its general solution) go straightforwardly. The question to be answered is, of course, whether all the d=1 integrable systems can be obtained in this way.

Now let us dwell on analogies with the Liouville equation which is the simplest d=2 completely integrable system. These analogies are far-reaching, despite the fact that in the Liouville case one deals with an infinite number of degrees of freedom. The latter circumstance manifests itself in that one starts with the infinite-dimensional d=2 conformal group. Respectively, there appear infinitely many Pfaff's equations of the type (2.14a) /8/. By these equations, the infinitely many fields parametrizing a coset of d=2 conformal group are expressed via a single dilaton field. The latter is a direct analog of field $x^3(t)$. The Liouville equation arises analogously to eq.(2.14b). The d=2 counterpart of d=1 reduction subalgebra SO(2) is the subalgebra SO(1,2) of the d=2 conformal algebra. Respectively, the Cartan form surviving the covariant reduction lives on that SO(1,2). As a consequence of covariant reduction constraints and of the original Maurer-Cartan equations, the remaining form satisfies the standard zero curvature condition that expresses the fact of complete integrability of the Liouville equation.

The zero curvature conditions have no analog in the d=1 case because of lack of two-forms in one dimension. However, as we have seen, the first-order covariant reduction constraints can still be implemented and these have a transparent geometric meaning. Thus it seems that the covariant reduction scheme may bear a deeper relation to the concept of integrability than the conventional approach based on the zero curvature representation. It would be of interest to extend this scheme to other integrable d=2 systems, in particular to chiral field models. We conjecture that the latter models are asso-

ciated with geodesic hypersurfaces in group manifolds of Kac-Moody groups /17/.

As a final remark, we would like to stress that the results of this paper and of /8,9/ demonstrate a close relation between the d=2 and d=1 Liouville-type systems on the one hand and the intrinsic geometry of d=2 and d=1 conformal groups on the other. Perhaps this fact deserves a special attention in view of recent growth of interest in the geometry of coset spaces of the d=2 conformal group in the context of string field theory /18,19/. A finite-dimensional toy model of the latter based on the group SO(1,2) was recently considered in /20/.

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Appendix. Relation to the standard description of geodesics on SO(1,2)

We start with the metric g_{ij} defined by eq.(2.10)

$$g_{ij} = \begin{pmatrix} 0 & 1 & x^2 \\ 1 & 0 & 0 \\ x^2 & 0 & -\frac{1}{2} \end{pmatrix}; \quad g^{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2(x^2)^2 & 2x^2 \\ 0 & 2x^2 & -2 \end{pmatrix} . \tag{A.1}$$

The equation of geodesics corresponding to this metric is written as

$$\ddot{x}^i + \Gamma^i_{jk}\, \dot{x}^j \dot{x}^k = 0 \tag{A.2}$$

$$x^i = x^i(s)\,; \quad \dot{x}^i = \frac{dx^i}{ds}\,; \quad |g_{ij}\dot{x}^i\dot{x}^j| = 1 .$$

Here Γ^i_{jk} are Christoffel coefficients calculated by the standard rules of Riemann geometry. In components, eq.(A.2) amounts to the set

$$\begin{aligned} &\ddot{x}^1 - \dot{x}^1\dot{x}^3 = 0 && (a) \\ &\ddot{x}^2 + 2x^2\dot{x}^2\dot{x}^1 + \dot{x}^2\dot{x}^3 + 2(x^2)^2\dot{x}^1\dot{x}^3 = 0 && (b) \\ &\ddot{x}^3 - 2\dot{x}^2\dot{x}^1 - 2x^2\dot{x}^1\dot{x}^3 = 0 . && (c) \end{aligned} \tag{A.3}$$

Let us show that any solution of eqs.(2.14) solves eqs. (A.3).

Making change of variables $S \to x^1 = t$ in eqs.(A.3) and using eq.(3.3), it is easy to check that eq.(A.3a) is satisfied identically. The rest of eqs.(A.3) is checked by using repeatedly eqs.(2.14).

Conversely, one may get the set (2.14) as a result of partial integration of eqs.(A.3).

Specializing to the "time-like" case $dS^2 > 0$, one readily obtains

$$
\begin{aligned}
&\frac{dx^1}{dS} = \beta_1 e^{x^3} \\
&x^2 = \frac{1}{2}\dot{x}^3 - \beta_2 e^{-x^3} \\
&\frac{1}{2}\ddot{x}^3 + \frac{1}{4}(\dot{x}^3)^2 = \frac{1}{2\beta_1^2} e^{-2x^3},
\end{aligned}
\qquad (A.4)
$$

where β_1 , β_2 are integration constants and the derivatives are taken with respect to $t = x^1$. Upon identifying $\beta_2 = \alpha$, $m^2 = \frac{1}{2\beta_1^2}$, these equations coincide with eqs.(2.14'), (2.17) and so are equivalent to eqs.(2.14). It is worthwile to emphasize that the coupling constant m^2 appears as an integration constant in this scheme.

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