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THE INTEGRALS OF MOTION AND MODIFIED BOCHNER–KILLING–YANO STRUCTURES OF THE DYNAMICAL SYSTEMS

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1. Introduction

The Hamiltonian mechanics (HM) is in the ground of mathematical description of the physical theories [1]. But HM is in a sense blind, e.g., it does not make difference between two opposites: the ergodic Hamiltonian systems (with just one integral of motion) and integrable Hamiltonian systems (with maximal number of the integrals of motion).

By our proposal [2] Nambu's mechanics (NM) [3, 4] is proper generalization of the HM, which makes difference between dynamical systems with different numbers of integrals of motion explicit.

In this paper* we investigate the integrals of motion and corresponding structures which are an important step in the general program [2] of the Nambu-Poisson formulation of the theory of the dynamical systems. In Sec.2 of this paper we consider the Hamiltonian extension [5] of the general dynamical system (1). In Sec.3 we consider the Lagrangian and Hamiltonian dynamics of the geodesic motion of the point particles and construct polynomial in the momentum integrals of motion using Killing-tensor structures. In Sec.4 we introduce Modified Bochner- Killing-Yano structures which defines the integrals of motion of the Hamiltonian extension of the general dynamical systems. In Sec.5 we present our conclusions.

2. Hamiltonization of the general dynamical systems

Let us consider a general dynamical system described by the following system of the ordinary differential equations [6]

$$\dot{x}_n = f_n(x),\ 1 \leq n \leq N, \tag{1}$$

where some components of the state vector x may be Grassmann valued [7], others take value from some number fields, real, complex or p-adic [8] and $\dot{x}_n$ stands for the total derivative with respect to the parameter t.

When the number of the degrees of freedom is even, $1 \leq n \leq 2M$, and

$$f_n(x) = \varepsilon_{nm}\frac{\partial H_0}{\partial x_m},\ 1 \leq n, m \leq 2M, \tag{2}$$

the system (1) is Hamiltonian one and can be put in the form

$$\dot{x}_n = \{x_n, H_0\}_0, \tag{3}$$

where the Poisson bracket is defined as

$$\{A, B\}_0 = \varepsilon_{nm}\frac{\partial A}{\partial x_n}\frac{\partial B}{\partial x_m} = A\frac{\overleftarrow{\partial}}{\partial x_n}\varepsilon_{nm}\frac{\overrightarrow{\partial}}{\partial x_m}B, \tag{4}$$

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and (the Einstein's) summation rule under repeated indices has been used (throughout this work).

Let us consider the following Lagrangian

$$L = (\dot{x}_n - f_n(x))\psi_n \tag{5}$$

and the corresponding equations of motion

$$\begin{aligned} \dot{x}_n &= f_n(x), \\ \dot{\psi}_n &= -\frac{\partial f_m}{\partial x_n}\psi_m. \end{aligned} \tag{6}$$

The system (6) extends the general system (1) by linear equation for the variables ψ. The extended system can be put in the Hamiltonian form [9, 5]

$$\begin{aligned} \dot{x}_n &= \{x_n, H_1\}_1, \\ \dot{\psi}_n &= \{\psi_n, H_1\}_1, \end{aligned} \tag{7}$$

where first level (order) Hamiltonian is

$$H_1 = f_n(x)\psi_n \tag{8}$$

and (first level) bracket is defined as

$$\{A, B\}_1 = A(\frac{\overleftarrow{\partial}}{\partial x_n}\frac{\overrightarrow{\partial}}{\partial \psi_n} - \frac{\overleftarrow{\partial}}{\partial \psi_n}\frac{\overrightarrow{\partial}}{\partial x_n})B. \tag{9}$$

Note that when the Grassmann grading, [7] of the conjugated variables x_n and ψ_n

$$\{x_n, \psi_m\}_1 = \delta_{nm} \tag{10}$$

are different, the bracket (9) is known as Buttin's bracket [10].

3. Geodesic motion of the point particles and integrals of motion

Geodesic motion of the particles maybe described by the following action functional

$$S = \int_1^2 L(|\dot{x}|)ds, \tag{11}$$

where

$$|\dot{x}|^2 = g_{ab}\dot{x}^a\dot{x}^b \tag{12}$$

and g_{ab} is metric tensor. The corresponding Euler-Lagrange equation

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}^a}) - \frac{\partial L}{\partial x^a} = 0, \tag{13}$$

gives the extremal trajectories of the variation of the action (11)

$$\delta S = \int_1^2 ds(\frac{\partial L}{\partial x^a} - \frac{d}{ds}(\frac{\partial L}{\partial \dot{x}^a}))\delta x^a + (\frac{\partial L}{\partial \dot{x}^a}\delta x^a)_1^2 \tag{14}$$

with fixed ends, $\delta x^a(1) = \delta x^a(2) = 0$, and have the form

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \tag{15}$$

where

$$\dot{x}^a = \frac{dx^a}{ds} \tag{16}$$

is the proper time derivative,

$$ds^2 = g_{ab}dx^a dx^b \tag{17}$$

gives the geodesic interval and

$$\Gamma^a_{bc} = g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d}) \tag{18}$$

is the Chistoffel's symbols.

Usually considered forms of the Lagrangian are $L = |\dot{x}|$ or $\frac{1}{2}|\dot{x}|^2$. The first one gives the reparametrization invariant action, the second one is easy for Hamiltonian formulation [11]. In the following we restrict ourselves by the last form of the Lagrangian.

Corresponding Hamiltonian

$$H = p_a\dot{x}^a - L \tag{19}$$

is

$$H = \frac{1}{2}g^{ab}p_a p_b, \tag{20}$$

where the momentum is

$$p_a = \frac{\partial L}{\partial \dot{x}^a} = g_{ab}\dot{x}^b \tag{21}$$

and g^{ab} is the inverse metric tensor,

$$g^{ac}g_{cb} = \delta^a_b. \tag{22}$$

The Hamilton's equations of motion are

$$\begin{aligned} \dot{x}^a &= \{x^a, H\}_0 = g^{ab}p_b, \\ \dot{p}_a &= \{p_a, H\}_0 = -\frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c, \end{aligned} \tag{23}$$

where the Poisson bracket is

$$\{A,B\}_0 = \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial x^a} = A(\overleftarrow{\partial}_{x^a}\overrightarrow{\partial}_{p_a} - \overleftarrow{\partial}_{p_a}\overrightarrow{\partial}_{x^a})B \tag{24}$$

$$= A\,\overleftarrow{\partial}_{z_n}\,\varepsilon_{nm}\,\overrightarrow{\partial}_{z_m}\,B, \tag{25}$$

and with the unifying variables z_n

$$z_n = x^n,\ z_{n+N} = p_n,\ 1 \le n \le N \tag{26}$$

the Hamilton's equations of motion (23) takes the form (1).

Integrals of motion, $H(x,\dot{x})$, fulfil the following equation

$$\begin{aligned}\dot{H} &= (\dot{x}^a\frac{\partial}{\partial x^a} + \ddot{x}^a\frac{\partial}{\partial \dot{x}^a})H \\ &= (\dot{x}^a\frac{\partial}{\partial x^a} - \Gamma^a_{bc}\dot{x}^b\dot{x}^c\frac{\partial}{\partial \dot{x}^a})H \\ &= \dot{x}^a\nabla_a H = p_a\nabla^a H = 0,\end{aligned} \tag{27}$$

where

$$\begin{aligned}\frac{d}{ds} &= \dot{x}^a\nabla_a = p_a\nabla^a, \\ \nabla_a &= \frac{\partial}{\partial x^a} - \Gamma^b_{ac}\dot{x}^c\frac{\partial}{\partial \dot{x}^b}.\end{aligned} \tag{28}$$

For the linear in $\dot{x}$ integrals

$$H_1 = K_a(x)\dot{x}^a = K^a p_a \tag{29}$$

we have

$$\begin{aligned}\dot{H}_1 = \dot{x}^a\nabla_a H_1 &= \frac{\partial K_b}{\partial x_a}\dot{x}^a\dot{x}^b - K_c\Gamma^c_{ab}\dot{x}^a\dot{x}^b \\ &= (K_{a,b} - K_c\Gamma^c_{ab})\dot{x}^a\dot{x}^b \\ &= K_{a;b}\dot{x}^a\dot{x}^b = \frac{1}{2}(K_{a;b} + K_{b;a})\dot{x}^a\dot{x}^b \\ &= K_{(a;b)}\dot{x}^a\dot{x}^b = 0.\end{aligned} \tag{30}$$

So, from the expression (30), we see one-to-one correspondence between the expression of the first order integrals of the motion (29) and the nontrivial solutions of the following equation for the so-called Killing vector K_a

$$K_{(a;b)} = 0. \tag{31}$$

For quadratic in $\dot{x}$ integrals

$$H_2 = K_{ab}(x)\dot{x}^a\dot{x}^b, \tag{32}$$

we have

$$\begin{aligned}\dot{H}_2 &= (K_{ab,c} - K_{db}\Gamma^d_{ac} - K_{ad}\Gamma^d_{bc})\dot{x}^a\dot{x}^b\dot{x}^c \\ &= K_{ab;c}\dot{x}^a\dot{x}^b\dot{x}^c = \frac{1}{3}(K_{ab;c} + K_{bc;a} + K_{ca;b})\dot{x}^a\dot{x}^b\dot{x}^c \\ &= K_{(ab;c)}\dot{x}^a\dot{x}^b\dot{x}^c = 0.\end{aligned} \tag{33}$$

So, we have one-to-one correspondence between the existence of the second order integrals of motion (32) and the nontrivial solutions of the following equation for the Killing tensor K_{ab}

$$K_{(ab;c)} = 0. \tag{34}$$

Higher order Killing tensors $K_{a_1a_2\dots a_n}$ fulfil the equation

$$K_{(a_1a_2\dots a_n;a)} = 0 \tag{35}$$

and give the following integrals of motion

$$H_n = K_{a_1a_2\dots a_n}(x)\dot{x}^{a_1}\dot{x}^{a_2}\dots\dot{x}^{a_n} = K^{a_1a_2\dots a_n}(x)p_{a_1}p_{a_2}\dots p_{a_n}. \tag{36}$$

In fact,

$$\begin{aligned}\dot{H}_n &= \dot{x}^a\nabla_a H_n = K_{(a_1a_2\dots a_n;a)}\dot{x}^{a_1}\dot{x}^{a_2}\dots\dot{x}^{a_n}\dot{x}^a \\ &= p_a\nabla^a H_n = K^{(a_1a_2\dots a_n;a)}p_{a_1}p_{a_2}\dots p_{a_n}p_a = 0.\end{aligned} \tag{37}$$

Note that, there is always the second order Killing tensor

$$K_{ab} = g_{ab} \tag{38}$$

and the corresponding integral of motion, Hamiltonian, H_0,

$$2H_0 = g_{ab}\dot{x}^a\dot{x}^b. \tag{39}$$

4. Modified Bochner-Killing-Yano (MBKY) structures

Now we return to our extended system (6) and formulate conditions for the integrals of motion $H(x,\psi)$

$$H = H_0(x) + H_1 + \dots + H_N, \tag{40}$$

where

$$H_n = A_{k_1k_2\dots k_n}(x)\psi_{k_1}\psi_{k_2}\dots\psi_{k_N},\ 1 \le n \le N \tag{41}$$

and we are assuming Grassmann valued ψ_n. For integrals (40) we have

$$\dot{H} = \{\sum_{n=0}^{N} H_n, H_1\} = \sum_{n=0}^{N}\{H_n, H_1\} = \sum_{n=0}^{N}\dot{H}_n = 0. \tag{42}$$

In particular for Hamiltonian systems (2), zeroth, H_0 and first level H_1 (8) Hamiltonians are integrals of motion. Now we see, that each term in the sum (40) must be conserved separately.

For $n = 0$

$$\dot{H}_0 = H_{0,k} f_k = 0, \tag{43}$$

which reduce to the condition (27), in the case of the geodesic motion of the particle (23) and define corresponding modifications of the polynomial integrals of motion (36).

For $1 \le n \le N$ we have

$$\begin{aligned}\dot{H}_n &= \dot{A}_{k_1 k_2 \ldots k_n} \psi_{k_1} \psi_{k_2} \ldots \psi_{k_N} + A_{k_1 k_2 \ldots k_n} \dot{\psi}_{k_1} \psi_{k_2} \ldots \psi_{k_N} + \ldots + A_{k_1 k_2 \ldots k_n} \psi_{k_1} \psi_{k_2} \ldots \dot{\psi}_{k_N} \\ &= (A_{k_1 k_2 \ldots k_n, k} f_k - A_{k k_2 \ldots k_n} f_{k_1, k} - A_{k_1 k \ldots k_n} f_{k_2, k} - \\ &\ldots - A_{k_1 \ldots k_{n-1} k} f_{k_n, k}) \psi_{k_1} \psi_{k_2} \ldots \psi_{k_N}\end{aligned} \tag{44}$$

and there is one-to-one correspondence between the existence of the integrals (41) and the existence of the nontrivial solutions of the following equations

$$\begin{aligned}&A_{k_1 k_2 \ldots k_n, k} f_k - A_{k k_2 \ldots k_n} f_{k_1, k} - A_{k_1 k \ldots k_n} f_{k_2, k} - \\ &\ldots - A_{k_1 \ldots k_{n-1} k} f_{k_n, k} = 0.\end{aligned} \tag{45}$$

For $n = 1$ the system (45) gives

$$A_{k_1, k} f_k - A_k f_{k_1, k} = 0 \tag{46}$$

and this equation has at list one solution, $A_k = f_k$.

The system (45) defines a Generalization of the Bochner-Killing-Yano structures of the geodesic motion of the point particle, for the case of the general (1) (and extended (6)) dynamical systems.

The structures defined by the system (45) we will call the Modified Bochner-Killing-Yano structures or MBKY structures for short.

5. Conclusions

The Modified Bochner-Killing-Yano structures (equations) (45) are natural generalization of the Killing(-Yano) structures of the geodesic dynamics of the relativistic (spinning) particles, [12] and we hope give an useful tool of the investigation of any dynamical system (1). Some applications of this formalism see in [17].

The method of Hamiltonization of this paper is applicable also to the infinite dimensional systems (partial differential equations). As an example, let us consider [15] many-field Korteveg-de Vries (KdV) equations [16]

$$V^i_t = V^i_{xxx} + a^i_{jk} V^j V^k_x. \tag{47}$$

Corresponding Lagrangian is

$$L = \psi_i (V^i_t - V^i_{xxx} - a^i_{jk} V^j V^k_x), \tag{48}$$

momentum is

$$P = \frac{\partial L}{\partial V^i_t} = \psi_i, \tag{49}$$

Hamiltonian is

$$H = \psi_i (V^i_{xxx} + a^i_{jk} V^j V^k_x), \tag{50}$$

the extended system of the equation of motion is

$$\begin{aligned}V^i_t &= V^i_{xxx} + a^i_{jk} V^j V^k_x, \\ \psi_{it} &= \psi_{ixxx} + a^k_{ij} V^j \psi_{kx},\end{aligned} \tag{51}$$

the (fundamental bracket) is

$$\begin{aligned}&\{V^i(t,x), \psi_j(t,y)\} = \delta^i_j \delta(x-y), \\ &\{A, B\} = \int dx A \left(\frac{\overleftarrow{\delta}}{\delta V^i(t,x)} \frac{\overrightarrow{\delta}}{\delta \psi_i(t,x)} - \frac{\overleftarrow{\delta}}{\delta \psi_i(t,x)} \frac{\overrightarrow{\delta}}{\delta V^i(t,x)} \right) B.\end{aligned} \tag{52}$$

The work on the applications of this formalism for several dynamical systems is in progress [17].

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Балеану Д., Махалдиани Н. E2-99-338
Интегралы движения и модифицированные структуры Бохнера–Киллинга–Яно для динамических систем

Для любой динамической системы с конечным числом степеней свободы мы определяем модифицированную структуру Бохнера–Киллинга–Яно, которая является естественной модификацией соответствующей структуры для геодезической динамики релятивистской частицы со спином.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Baleanu D., Makhaldiani N. E2-99-338
The Integrals of Motion and Modified Bochner–Killing–Yano Structures of the Dynamical Systems

For any finite dimensional dynamical system we define a modified Bochner–Killing–Yano structures, which are a natural modification of the corresponding structures of the geodesic motion of the relativistic spinning particles.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna, 1999