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THE ALGEBRAS OF THE INTEGRALS OF MOTION AND MODIFIED BOCHNER–KILLING–YANO STRUCTURES OF THE POINT PARTICLE DYNAMICS

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1. Introduction

The Hamiltonian mechanics (HM) is in the ground of mathematical description of the physical theories [1]. But HM is in a sense blind, e.g., it does not make difference between two opposites: the ergodic Hamiltonian systems (with just one integral of motion) and integrable Hamiltonian systems (with maximal number of the integrals of motion).

By our proposal [2] Nambu's mechanics (NM) [3, 4] is proper generalization of the HM, which makes difference between dynamical systems with different numbers of integrals of motion explicit.

In this paper we investigate the integrals of motion and corresponding algebraic structures, which are an important step in the general program [2] of the Nambu-Poisson formulation of the theory of the dynamical systems. In Sec.2 of this paper, we consider the Hamiltonian extension [5] of the general dynamical system (1). In Sec.3 we consider the Lagrangian and Hamiltonian dynamics of the geodesic motion of the point particles and construct polynomial (in the velocities and/or momentum) integrals of motion using Killing-tensor structures. In Sec.4 we introduce a Modified Bochner- Killing-Yano structures which defines the integrals of motion of the Hamiltonian extension of the general dynamical systems. In Sec.5 we define an integrals and corresponding algebras for the extended Hamiltonian formulation of the geodesic dynamics of the point particles. In Sec.6 we present our conclusions and show some perspectives.

2. Hamiltonization of the general dynamical systems

Let us consider a general dynamical system described by the following system of the ordinary differential equations [6]

$$\dot{x}_n = f_n(x), \ 1 \leq n \leq N, \tag{1}$$

or p-adic [8] and $\dot{x}_n$ stands for the total derivative with respect to the parameter t.

When the number of the degrees of freedom is even, $1 \leq n, m \leq 2M$, and

$$f_n(x) = \varepsilon_{nm} \frac{\partial H_0}{\partial x_m}, \ 1 \leq n, m \leq 2M, \tag{2}$$

the system (1) is Hamiltonian one and can be put in the form

$$\dot{x}_n = \{x_n, H_0\}_0, \tag{3}$$

where the Poisson bracket is defined as

$$\{A, B\}_0 = \varepsilon_{nm} \frac{\partial A}{\partial x_n} \frac{\partial B}{\partial x_m} = A \frac{\overleftarrow{\partial}}{\partial x_n} \varepsilon_{nm} \frac{\overrightarrow{\partial}}{\partial x_m} B, \tag{4}$$

and (the Einstein's) summation rule under repeated indices has been used (throughout this work).

Let us consider the following Lagrangian

$$L = (\dot{x}_n - f_n(x))\psi_n \tag{5}$$

and the corresponding equations of motion

$$\dot{x}_n = f_n(x), \quad \dot{\psi}_n = -\frac{\partial f_m}{\partial x_n}\psi_m. \tag{6}$$

The system (6) extends the general system (1) by linear equation for the variables ψ. The extended system can be put in the Hamiltonian form [9, 5]

$$\dot{x}_n = \{x_n, H_1\}_1, \quad \dot{\psi}_n = \{\psi_n, H_1\}_1, \tag{7}$$

where first level (order) Hamiltonian is

$$H_1 = f_n(x)\psi_n \tag{8}$$

and (first level) bracket is defined as

$$\{A, B\}_1 = A\left(\frac{\overleftarrow{\partial}}{\partial x_n}\frac{\overrightarrow{\partial}}{\partial \psi_n} - \frac{\overleftarrow{\partial}}{\partial \psi_n}\frac{\overrightarrow{\partial}}{\partial x_n}\right)B. \tag{9}$$

Note that when the Grassmann grading [7] of the conjugated variables x_n and ψ_n

$$\{x_n, \psi_m\}_1 = \delta_{nm} \tag{10}$$

are different, the bracket (9) is known as Buttin's bracket[10].

3. Geodesic motion of the point particles and integrals of motion

Geodesic motion of the particles maybe described by the following action functional

$$S = \int_1^2 L(|\dot{x}|)ds, \tag{11}$$

where

$$|\dot{x}|^2 = g_{ab}\dot{x}^a\dot{x}^b \tag{12}$$

and g_{ab} is metric tensor. The corresponding Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0 \tag{13}$$

gives the extremal trajectories of the variation of the action (11)

$$\delta S = \int_1^2 ds\left(\frac{\partial L}{\partial x^a} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^a}\right)\right)\delta x^a + \left(\frac{\partial L}{\partial \dot{x}^a}\delta x^a\right)_1^2, \tag{14}$$

with fixed ends, $\delta x^a(1) = \delta x^a(2) = 0$, and have the form

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \tag{15}$$

where

$$\dot{x}^a = \frac{dx^a}{ds} \tag{16}$$

is the proper time derivative,

$$ds^2 = g_{ab}dx^a dx^b \tag{17}$$

gives the geodesic interval and

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d}) \tag{18}$$

is the Chistoffel's symbols.

Usually considered forms of the Lagrangian are $L = |\dot{x}|$ or $\frac{1}{2}|\dot{x}|^2$. The first one gives the reparametrization invariant action, the second one is easy for Hamiltonian formulation [11]. In the following we restrict ourselves by the last form of the Lagrangian

$$L = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b. \tag{19}$$

Corresponding Hamiltonian

$$H = p_a\dot{x}^a - L \tag{20}$$

is

$$H = \frac{1}{2}g^{ab}p_a p_b, \tag{21}$$

where the momentum is

$$p_a = \frac{\partial L}{\partial \dot{x}^a} = g_{ab}\dot{x}^b \tag{22}$$

and g^{ab} is the inverse metric tensor,

$$g^{ac}g_{cb} = \delta^a_b. \tag{23}$$

The Hamilton's equations of motion are

$$\begin{aligned} \dot{x}^a &= \{x^a, H\}_0 = g^{ab}p_b, \\ \dot{p}_a &= \{p_a, H\}_0 = -\frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c, \end{aligned} \tag{24}$$

where the Poisson bracket is

$$\{A, B\}_0 = \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial x^a} = A(\overleftarrow{\partial}_{x^a}\overrightarrow{\partial}_{p_a} - \overleftarrow{\partial}_{p_a}\overrightarrow{\partial}_{x^a})B \tag{25}$$

$$= A\,\overleftarrow{\partial}_{z_n}\,\varepsilon_{nm}\,\overrightarrow{\partial}_{z_m}\,B, \tag{26}$$

and with the unifying variables z_n

$$z_n = x^n, \; z_{n+N} = p_n, \; 1 \le n \le N \tag{27}$$

the Hamilton's equations of motion (24) take the form (1).

Integrals of motion H fulfil the following equations

$$\begin{aligned} \frac{d}{ds}H(x,\dot{x}) &= \left(\dot{x}^a\frac{\partial}{\partial x^a} + \ddot{x}^a\frac{\partial}{\partial \dot{x}^a}\right)H \\ &= \left(\dot{x}^a\frac{\partial}{\partial x^a} - \Gamma^a_{bc}\dot{x}^b\dot{x}^c\frac{\partial}{\partial \dot{x}^a}\right)H \\ &= \dot{x}^a\nabla_a H = 0, \\ \frac{d}{ds}H(x,p) &= \left(g^{ab}p_b\frac{\partial}{\partial x^a} - \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c\frac{\partial}{\partial p_a}\right)H, \\ &= p_b\nabla^b H = 0, \end{aligned} \tag{28}$$

where

$$\begin{aligned} \frac{d}{ds} &= \dot{x}^a\nabla_a = p_b\nabla^b, \\ \nabla_a &= \frac{\partial}{\partial x^a} - \Gamma^b_{ac}\dot{x}^c\frac{\partial}{\partial \dot{x}^b}, \\ \nabla^b &= g^{ba}\nabla_a = g^{ba}\frac{\partial}{\partial x^a} - \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_c\frac{\partial}{\partial p_a}. \end{aligned} \tag{29}$$

For the linear in $\dot{x}$ integrals

$$H_1 = K_a(x)\dot{x}^a = K^a p_a \tag{30}$$

we have

$$\begin{aligned} \dot{H}_1 = \dot{x}^a\nabla_a H_1 &= \frac{\partial K_b}{\partial x_a}\dot{x}^a\dot{x}^b - K_c\Gamma^c_{ab}\dot{x}^a\dot{x}^b \\ &= (K_{a,b} - K_c\Gamma^c_{ab})\dot{x}^a\dot{x}^b \\ &= K_{a;b}\dot{x}^a\dot{x}^b = \frac{1}{2}(K_{a;b} + K_{b;a})\dot{x}^a\dot{x}^b \end{aligned}$$

$$= K_{(a;b)}\dot{x}^a\dot{x}^b = 0. \tag{31}$$

So, from the expression (31), we see one-to-one correspondence between the expression of the first order integrals of the motion (30) and the nontrivial solutions of the following equation for the so-called Killing vector K_a

$$K_{(a;b)} = 0. \tag{32}$$

For quadratic in $\dot{x}$ integrals

$$H_2 = K_{ab}(x)\dot{x}^a\dot{x}^b \tag{33}$$

we have

$$\begin{aligned}\dot{H}_2 &= (K_{ab,c} - K_{db}\Gamma^d_{ac} - K_{ad}\Gamma^d_{bc})\dot{x}^a\dot{x}^b\dot{x}^c \\ &= K_{ab;c}\dot{x}^a\dot{x}^b\dot{x}^c = \frac{1}{3}(K_{ab;c} + K_{bc;a} + K_{ca;b})\dot{x}^a\dot{x}^b\dot{x}^c \\ &= K_{(ab;c)}\dot{x}^a\dot{x}^b\dot{x}^c = 0.\end{aligned} \tag{34}$$

So, we have one-to-one correspondence between the existence of the second order integrals of motion (33) and the nontrivial solutions of the following equation for the tensor K_{ab}

$$K_{(ab;c)} = 0. \tag{35}$$

Now we prove the following:

Theorem 1. A necessary and sufficient condition that the following polynomials

$$H_n = K_{a_1a_2\ldots a_n}(x)\dot{x}^{a_1}\dot{x}^{a_2}\ldots\dot{x}^{a_n} = K^{a_1a_2\ldots a_n}(x)p_{a_1}p_{a_2}\ldots p_{a_n} \tag{36}$$

are integrals of the geodesic motion, (15, 24) is that the symmetric tensors $K_{a_1a_2\ldots a_n}$, fulfil the equation

$$K_{(a_1a_2\ldots a_n;a)} = 0. \tag{37}$$

In fact,

$$\begin{aligned}\dot{H}_n &= \dot{x}^a\nabla_a H_n = K_{(a_1a_2\ldots a_n;a)}\dot{x}^{a_1}\dot{x}^{a_2}\ldots\dot{x}^{a_n}\dot{x}^a = 0, \\ &= p_a\nabla^a H_n = K^{(a_1a_2\ldots a_n;a)}p_{a_1}p_{a_2}\ldots p_{a_n}p_a = 0,\end{aligned} \tag{38}$$

which proves the theorem, see [12].

The symmetric tensors, which fulfil the equation (37), is known as Killing tensors.

Note that, as the metric tensor is covariantly constant, $g_{ab;c} = 0$, there is always the second order Killing tensor

$$K_{ab} = g_{ab} \tag{39}$$

and the corresponding integral of motion, Hamiltonian, H_0,

$$2H_0 = g_{ab}\dot{x}^a\dot{x}^b. \tag{40}$$

Let us define an interesting algebra on Killing tensors.

Theorem 2. The following symmetrized product of the Killing tensors K^n and K^m

$$K^{(a_1a_2\ldots a_n}K^{a_{n+1}a_{n+2}\ldots a_{n+m})} = K^{a_1a_2\ldots a_{n+m}}, \tag{41}$$

is (reducible) Killing tensor.

In fact, let us multiply the corresponding integrals of motion

$$\begin{aligned}H_nH_m &= K^{(a_1a_2\ldots a_n}K^{a_{n+1}a_{n+2}\ldots a_{n+m})}p_{a_1}p_{a_2}\ldots p_{a_{n+m}} = \\ &= K_{(a_1a_2\ldots a_n}K_{a_{n+1}a_{n+2}\ldots a_{n+m})}\dot{x}^{a_1}\dot{x}^{a_2}\ldots\dot{x}^{a_{n+m}} = H_{n+m},\end{aligned} \tag{42}$$

which, using the Theorem 1, proves this theorem.

We have the following bracket algebra of the integrals of motion

$$\{H_n, H_m\}_0 = H_{n+m-1}. \tag{43}$$

This algebra gives another method of the construction of the Killing tensors. As an example let us calculate the bracket for the integrals $H_1 = K^a p_a$ and $H_2 = K^{ab}p_ap_b$

$$\begin{aligned}\{H_1, H_2\}_0 &= K^a\{p_a, K^{bc}\}p_bp_c + K^{bc}\{K^a, p_bp_c\}p_a \\ &= (K^{ab}K^c{}_{,a} + K^{ac}K^b{}_{,a} - K^{bc}{}_{,a}K^a)p_bp_c \\ &= \tilde{K}^{ab}p_ap_b.\end{aligned} \tag{44}$$

Let us consider another, tensor, generalization of the scalar integral of motion (30)

$$\begin{aligned}H_{a_1a_2\ldots a_{m-1}} &= A_{a_1a_2\ldots a_m}(x)\dot{x}^{a_m}, \\ H^{a_1a_2\ldots a_{m-1}} &= A^{a_1a_2\ldots a_m}(x)p_{a_m},\end{aligned} \tag{45}$$

where the tensors $A_{a_1a_2\ldots a_m}(x)$ and $A^{a_1a_2\ldots a_m}(x)$ are skew-symmetric. We have the following:

Theorem 3[1]. A necessary and sufficient condition that the tensors (45) are (covariantly) constant (parallel) along any geodesic $x^a(s)$ is that the covariant derivative of the skew-symmetric tensor $A_{a_1a_2\ldots a_m}(x)$ is also skew-symmetric

$$A_{a_1a_2\ldots a_m;a_{m+1}} + A_{a_1a_2\ldots a_{m+1};a_m} = 0. \tag{46}$$

In fact, as $x^a(s)$ is geodesic, we have

$$\begin{aligned}\frac{D\dot{x}^a}{Ds} &= \ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \\ \frac{Dp_a}{Ds} &= \dot{p}_a + \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_bp_c = g_{ab}\frac{D\dot{x}^b}{Ds} = 0\end{aligned} \tag{47}$$

and

[1] This theorem is slight modification of the corresponding theorem from [13]

$$\frac{D}{Ds}(A_{a_1a_2...a_m}(x)\dot{x}^{a_m}) = A_{a_1a_2...a_m;a_{m+1}}\dot{x}^{a_m}\dot{x}^{a_{m+1}}$$
$$= \frac{1}{2}(A_{a_1a_2...a_m;a_{m+1}} + A_{a_1a_2...a_{m+1};a_m})\dot{x}^{a_m}\dot{x}^{a_{m+1}} = 0,$$
$$\frac{D}{Ds}(A^{a_1a_2...a_m}(x)p_{a_m}) = A^{a_1a_2...a_m;a_{m+1}}p_{a_m}p_{a_{m+1}}$$
$$= \frac{1}{2}(A^{a_1a_2...a_m;a_{m+1}} + A^{a_1a_2...a_{m+1};a_m})p_{a_m}p_{a_{m+1}} = 0, \tag{48}$$

which proves the theorem. From the tensor integrals (45) we can construct the second rank Killing tensor.

Theorem 4. The following (symmetric) product of the tensors A^n and B^n gives a second rank Killing tensors

$$A^{a_1a_2...a_n(a}B_{a_1a_2...a_n}{}^{b)} = K^{ab}. \tag{49}$$

In fact, if we multiply the integrals

$$A^n = A^{a_1a_2...a_n} = A^{a_1a_2...a_na}p_a,$$
$$B_n = B^{a_1a_2...a_n} = B_{a_1a_2...a_n}{}^b p_b, \tag{50}$$

we obtain again integral

$$A^nB_n = A^{a_1a_2...a_n(a}B_{a_1a_2...a_n}{}^{b)}p_ap_b$$
$$= K^{a_nb_m}p_{a_n}p_{b_m} = H_2, \tag{51}$$

and using the Theorem 1, we prove the theorem 4.

So, if we have a nontrivial solution of the equations (46) , we can construct second integral of motion H_2

$$H_2 = K^{ab}p_ap_b, \tag{52}$$

and with original Hamiltonian (21)

$$H_1 = 2H = g^{ab}p_ap_b, \tag{53}$$

we will have bi-Hamiltonian system. So, we can apply the general method of the integration of the bi-Hamiltonian systems [14, 15, 16]. Also we can construct the Nambu-Poisson formulation [2] of this system

$$\dot{x}_n = \{x_n, H_1, H_2\}$$
$$= \omega_{nmk}(x)\frac{\partial H_1}{\partial x_m}\frac{\partial H_2}{\partial x_k}, \tag{54}$$

where the Nambu-Poisson structure tensor ω_{nmk} we identify [16] by comparision of the system (54) with the original system (24).

Note, that the skew-symmetric tensors, $A_{a_1a_2...a_n}$, which have the property, that its covariant derivative is also skew-symmetric, were considered by Bochner [17] (see [13]) and/but in literature are known as the Killing-Yano tensors [18].

4. Modified Bochner-Killing-Yano (MBKY) structures

Now we return to our extended system (6) and formulate conditions for the integrals of motion $H(x,\psi)$

$$H = H_0(x) + H_1 + ... + H_N, \tag{55}$$

where

$$H_n = A_{k_1k_2...k_n}(x)\psi_{k_1}\psi_{k_2}...\psi_{k_N}, \ 1 \le n \le N, \tag{56}$$

we are assuming Grassmann valued ψ_n and the tensor $A_{k_1k_2...k_n}$ is skew-symmetric. For integrals (55) we have .

$$\dot{H} = \{\sum_{n=0}^{N} H_n, H_1\} = \sum_{n=0}^{N}\{H_n, H_1\} = \sum_{n=0}^{N}\dot{H}_n = 0. \tag{57}$$

Now we see, that each term in the sum (55) must be conserved separately. In particular for Hamiltonian systems (2), zeroth, H_0 and first level H_1, (8), Hamiltonians are integrals of motion. For $n = 0$

$$\dot{H}_0 = H_{0,k}f_k = 0, \tag{58}$$

which reduce to the condition (28), in the case of the geodesic motion of the particle (24) and defines corresponding modifications of the polynomial integrals of motion (36).

For $1 \le n \le N$ we have

$$\dot{H}_n = \dot{A}_{k_1k_2...k_n}\psi_{k_1}\psi_{k_2}...\psi_{k_N} + A_{k_1k_2...k_n}\dot{\psi}_{k_1}\psi_{k_2}...\psi_{k_N} + ... + A_{k_1k_2...k_n}\psi_{k_1}\psi_{k_2}...\dot{\psi}_{k_N}$$
$$= (A_{k_1k_2...k_n,k}f_k - A_{kk_2...k_n}f_{k_1,k} - ... - A_{k_1...k_{n-1}k}f_{k_n,k})\psi_{k_1}\psi_{k_2}...\psi_{k_N} = 0, \tag{59}$$

and there is one-to-one correspondence between the existence of the integrals (56) and the existence of the nontrivial solutions of the following equations

$$\frac{D}{Dt}A_{k_1k_2...k_n} = \dot{A}_{k_1k_2...k_n} - f_{k_1,k}A_{kk_2...k_n} - ... - f_{k_n,k}A_{k_1...k_{n-1}k}$$
$$= A_{k_1k_2...k_n,k}f_k - A_{kk_2...k_n}f_{k_1,k} - ... - A_{k_1...k_{n-1}k}f_{k_n,k} = 0. \tag{60}$$

For $n = 1$ the system (60) gives

$$A_{k_1,k}f_k - A_kf_{k_1,k} = 0 \tag{61}$$

and this equation has at list one solution, $A_k = f_k$.

The system (60) defines a Generalization of the Bochner-Killing-Yano structures (37, 46), of the geodesic motion of the point particle, for the case of the general (1) (and extended (6)) dynamical systems.

The structures defined by the system (60) we will call the Modified Bochner-Killing-Yano structures or MBKY structures for short.

5. Extended geodesic motion of the point particles and Grassmann valued integrals of motion

Let us take the following Lagrangian

$$\begin{aligned} L &= (\dot{x}^a - g^{ab}p_b)\phi_a - (\dot{p}_a + \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c)\psi^a \\ &= (\dot{x}^a - g^{ab}p_b)\phi_a + (\dot{\psi}^a - \frac{1}{2}\frac{\partial g^{ab}}{\partial x^c}\psi^c p_b)p_a + \frac{d}{ds}(\psi^a p_a) \\ &= L_1 + \frac{d}{ds}(\psi^a p_a). \end{aligned} \tag{62}$$

New momentum variables are

$$\frac{\partial L_1}{\partial \dot{x}^a} = \phi_a, \quad \frac{\partial L_1}{\partial \dot{\psi}^a} = p_a, \tag{63}$$

(fundamental) brackets are

$$\begin{aligned} &\{x^a, \phi_b\}_1 = \delta^a_b, \\ &\{\psi^a, p_b\}_1 = \delta^b_a, \\ &\{A, B\}_1 = A(\overleftarrow{\partial}_{x^a}\overrightarrow{\partial}_{\phi_a} + \overleftarrow{\partial}_{\psi^a}\overrightarrow{\partial}_{p_a} - \overleftarrow{\partial}_{\phi_a}\overrightarrow{\partial}_{x^a} - \overleftarrow{\partial}_{p_a}\overrightarrow{\partial}_{\psi^a})B \\ &\qquad = A\,\overleftarrow{\partial}_{Z_n}\,\varepsilon_{nm}\,\overrightarrow{\partial}_{Z_m}\,B. \end{aligned} \tag{64}$$

The Hamiltonian is

$$H_1 = g^{ab}\phi_a p_b + \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}\psi^a p_b p_c, \tag{65}$$

the equations of motion are

$$\begin{aligned} \dot{x}^a &= g^{ab}p_b, \\ \dot{p}_a &= -\frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c, \\ \dot{\phi}_a &= -\frac{\partial g^{bc}}{\partial x^a}p_b\phi_c - \frac{1}{2}\frac{\partial^2 g^{bc}}{\partial x^a \partial x^e}p_b p_c \psi^e, \\ \dot{\psi}^a &= \frac{\partial g^{ab}}{\partial x^c}p_b\psi^c + g^{ab}\phi_b. \end{aligned} \tag{66}$$

Note that the extended system (66) and Hamiltonian (65) can be obtained from the system (24) and Hamiltonian (21) by the following simple shift of the variables

$$x^a \Rightarrow x^a + \theta\psi^a,$$

$$p_a \Rightarrow p_a + \theta\phi_a, \tag{67}$$

where θ- Grassmann parameter, $\theta^2 = 0$.

In fact,

$$\begin{aligned} H_0 = \frac{1}{2}g^{ab}p_a p_b &\Rightarrow \frac{1}{2}g^{ab}p_a p_b + \theta(\frac{1}{2}g^{ab}_{,c}p_a p_b\psi^c + g^{ab}\phi_a p_b) \\ &= H_0 + \theta H_1. \end{aligned} \tag{68}$$

The Lagrangian L_1 (62) can be obtained by the shift (67) from the following first order Lagrangian

$$L = -\frac{1}{2}g^{ab}p_a p_b + \dot{x}^a p_a, \tag{69}$$

which is equivalent to the Lagrangian (19).

In fact, under the shift (67) we have

$$\begin{aligned} L &= -\frac{1}{2}g^{ab}p_a p_b + \dot{x}^a p_a \\ &\Rightarrow -\frac{1}{2}(g^{ab} + g^{ab},c\theta\psi^c)(p_a + \theta\phi_a)(p_b + \theta\phi_b) + (\dot{x}^a + \theta\dot{\psi}^a)(p_a + \theta\phi_a) \\ &= L_0 + \theta L_1. \end{aligned} \tag{70}$$

Let us define (extending the zeroth level bracket (25)) the Grassmann even bracket [7]

$$\begin{aligned} &\{x^a, p_b\}_0 = \delta^a_b, \\ &\{\psi^a, \phi_b\}_0 = \delta^a_b, \\ &\{A, B\}_0 = \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a} + \frac{\partial A}{\partial \psi^a}\frac{\partial B}{\partial \phi_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial x^a} - \frac{\partial A}{\partial \phi_a}\frac{\partial B}{\partial \psi^a} \\ &\qquad = A(\overleftarrow{\partial}_{x^a}\overrightarrow{\partial}_{p_a} + \overleftarrow{\partial}_{\psi^a}\overrightarrow{\partial}_{\phi_a} - \overleftarrow{\partial}_{p_a}\overrightarrow{\partial}_{x^a} - \overleftarrow{\partial}_{\phi_a}\overrightarrow{\partial}_{\psi^a})B \\ &\qquad = A\,\overleftarrow{\partial}_{Z_n}\,\varepsilon_{nm}\,\overrightarrow{\partial}_{Z_m}\,B. \end{aligned} \tag{71}$$

An interesting problem is to construct an even Hamiltonian.

General form $H(x, \psi, p, \phi)$ of the integrals of motion of the extended system (66) fulfils the following equation

$$\begin{aligned} \dot{H} &= (\dot{x}^a\frac{\partial}{\partial x^a} + \dot{\psi}^a\frac{\partial}{\partial \psi^a} + \dot{p}_a\frac{\partial}{\partial p_a} + \dot{\phi}_a\frac{\partial}{\partial \phi_a}) \\ &= (p_b\nabla^b + \phi_b\nabla^b_1 + \psi^c\nabla_{2c})H = 0, \end{aligned} \tag{72}$$

where

$$\begin{aligned} \nabla^b &= g^{ba}\frac{\partial}{\partial x^a} - \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_c\frac{\partial}{\partial p_a}, \\ \nabla^b_1 &= g^{ba}\frac{\partial}{\partial \psi^a} - \frac{\partial g^{bc}}{\partial x^a}p_c\frac{\partial}{\partial \phi_a}, \end{aligned}$$

$$\nabla_{2c} = \frac{\partial g^{ab}}{\partial x^c} p_b \frac{\partial}{\partial \psi^a} - \frac{1}{2} \frac{\partial^2 g^{be}}{\partial x^c \partial x^a} p_b p_e \frac{\partial}{\partial \phi_a}. \quad (73)$$

6. Conclusions and perspectives

The Modified Bochner-Killing-Yano structures (equations) (60) are natural generalization of the Killing(-Yano) structures of the geodesic dynamics of the relativistic (spinning) particles [18] and we hope give an useful tool of the investigation of any dynamical system (1).

The method of Hamiltonization of this paper is applicable to the infinite dimensional systems (partial differential equations). The well known (integrable) system from the hydrodynamics, the KdV equation, (see, e.g., [i])

$$V_t = VV_x + V_{xxx} \quad (74)$$

can be put in the Hamiltonian form. Corresponding Lagrangian is

$$L = \psi(V_t - VV_x - V_{xxx}), \quad (75)$$

momentum is

$$P = \frac{\partial L}{\partial V_t} = \psi, \quad (76)$$

Hamiltonian is

$$H = \psi(VV_x + V_{xxx}), \quad (77)$$

the extended system of the equation of motion is

$$\begin{aligned} V_t &= VV_x + V_{xxx}, \\ \psi_t &= V\psi_x + \psi_{xxx}, \end{aligned} \quad (78)$$

the (fundamental bracket) is

$$\begin{aligned} &\{V(t,x), \psi(t,y)\} = \delta(x-y), \\ &\{A,B\} = \int dx A(\frac{\overleftarrow{\delta}}{\delta V(t,x)} \frac{\overrightarrow{\delta}}{\delta \psi(t,x)} - \frac{\overleftarrow{\delta}}{\delta \psi(t,x)} \frac{\overrightarrow{\delta}}{\delta V(t,x)})B. \end{aligned} \quad (79)$$

Now it is ease to see [20] that for some classical system described by the following equation

$$iV_t = -\frac{1}{2}V^2 + V_{xx}, \quad (80)$$

the companion system is given by the Schrödinger equation

$$i\psi_t = -\psi_{xx} + V\psi, \ (m = \frac{1}{2}). \quad (81)$$

As another example, let us take the important part of the theory of elementary particles, [19] the Dirac's equation for the electron's field

$$(\gamma^\mu \partial_\mu - m)\psi = 0. \quad (82)$$

The usual form [19] of the Lagrangian

$$L = \bar{\psi}(\gamma^\mu \partial_\mu - m)\psi, \quad (83)$$

momentum

$$P = \frac{\partial L}{\partial \psi_t} = \bar{\psi}\gamma^0 = \psi^+, \quad (84)$$

and Hamiltonian

$$H = \bar{\psi}(\vec{\gamma}\vec{\nabla} + m)\psi \quad (85)$$

corresponds exactly to our method of Hamiltonization. A curious possibility is given by the case when the variables $\bar{\psi}$ and ψ have different Grassmann grading.

The work on the applications of the formalism of this paper for several dynamical systems is in progress [20].

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