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N.P.Konopleva*

VI HILBERT'S PROBLEM AND S.LIE'S INFINITE GROUPS

*Permanent address: All-Russian Scientific Research Institute of Electromechanics, Horomnyi tup. 4, 101000, Moscow, Russia
E-mail: nelly@thsun1.jinr.dubna.su

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2 Introduction

Among the problems formulated by Hilbert at the turn of the XX century, there is the sixth problem: the mathematical formulation of the axioms of physics. Hilbert wrote:

To construct the physical axioms according to the model of the axioms of geometry, one must first try to encompass the largest possible class of physical phenomena by means of a small number of axioms and then, by adding each subsequent axiom, to arrive at more special theories, after which there may arise a classification principle which can make use of the deep theory of infinite Lie groups of transformations. Moreover, as is done in geometry, the mathematician must bear in mind not only the facts of actual reality, but also all the logically possible theories, and must be particularly careful to obtain the most complete survey of the totality of consequences which follow from the adopted systematization.

In this talk the progress in Hilbert's sixth problem solving is demonstrated. That became possible thanks to the gauge field theory in physics and to the geometrical treatment of the gauge fields. It is shown that the fibre bundle spaces geometry is the best basis for solution of the problem being discussed.

Usually Hilbert's sixth problem is citing in connection with an axiomatic formulation of probability theory. But this Hilbert's idea is only one of the ways of this problem realization. Moreover now the probability theory is not regarded as a chapter of physics.

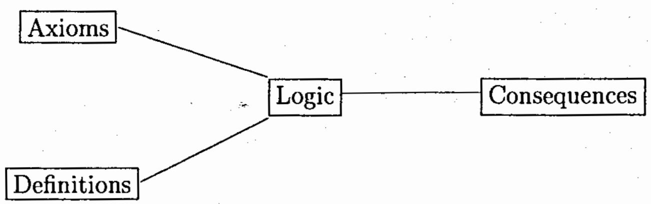
The modern physics is very spacious and ramous science. In principle different ramifications of the physics tree can have their own axiom systems. Is there the axiom system covering all physics branches that is the question. But classical field theory, mechanics and, partially, elementary particle physics can be axiomatically formulated by analogy with geometry as Hilbert supposed. The base of such axiom system is, really, Sophus Lie's finite and infinite group theory in accordance with Hilbert's hypothesis. In his VIth problem Hilbert suggested also that all theorems of solid body motion would be obtained by passage to the limit from the axiom system being based on idea of undergoing continuous change state of matter, which fills continuously the whole space. In the gauge field theory the equations of particle motion followed from the equations of field. Hence, mechanics and solid body motion theory can be obtained from the field theory by a process of passage to the limit as Hilbert supposed.

This progress in solving of Hilbert's sixth problem became possible thanks to some new branches in physics and mathematics: gauge field theory, fibre bundle space geometry and development of variational methods mentioned in Hilbert's 23rd problem.

3 Structure of Mathematical and Physical Theories

In order to discuss the means of solving Hilbert's sixth problem, we compare structure of mathematical theory with physical one. It is necessary to take note of a distinction between purely mathematical inferences and the usually employed physical inferences. Mathematical inference is analytic, i.e. it is done in accordance with definite logical rules on the basis of the adopted definition and axioms. No additional information which is not contained in the initial definitions and axioms is admitted in the process of logical deduction. Otherwise, it would be possible to obtain arbitrary consequences. Mathematical propositions are valid for the abstract objects introduced by means of the definitions, these being logical atoms of the theory.

The reasoning scheme of mathematical (analytic) inference is as follows:

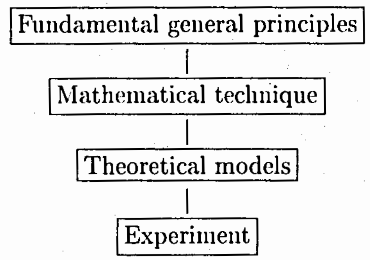


As a rule, a physical theory is based on concepts which are poorly defined from the point of view of mathematics. These have descriptive nature and bear the marks of the various methods of experimental study of physical objects, as well as the

sense perception of them by the experimenter. Therefore, for the axiomatization of physics it is necessary, first and foremost, to go over from concrete ideas to general concepts. The general concepts usually reflect a small part of the properties of real objects, but then the distinguished properties are inherent in many real objects, so that arguments based on the general concepts have a certain degree of generality, which is necessary for scientific inferences.

Thus, the logical atoms of a physical theory are abstract objects which possess properties that are common to some class of real physical objects. Consequently, under different conditions one and the same physical object can serve as a model of the logical atoms of physical theories which differ from their mathematical technique. Conversely, one and the same mathematical technique can be used to describe phenomena which are completely different to their physical nature (for example, d'Alembert's equations and all possible periodic processes). A mathematical theory becomes physical if a physical realization of its basic concepts has been found.

Let us classify the various types of physical propositions according to their degree of generality. The following scheme presents the result:



The proposition of each level is valid for the classes of concrete propositions of the level below it and is general in relation to them. Thus, using exactly the same fundamental general principles, it is possible to construct different forms of mathematical technique. At the present time, the following are known in theoretical physics: 1) the Lagrangian formalism; 2) the Hamiltonian formalism; 3) the axiomatic approach in quantum field theory; 4) the geometrical formulation of gauge field theory. The principles of invariance and symmetry are being used as the fundamental general principles for construction a physical theory. A symmetry can be local, valid in the neighborhood of a point, or global, valid over all space-time. Local symmetry is source of infinite S.Lie's groups appearance in physics and geometrical treatment of interactions.

4 S.Lie's and F.Klein's concepts: Erlangen Program

S.Lie and F.Klein were first who understand the role of symmetry principles in geometry axiomatics. The ideas were formulated by Klein in 1872 in his "Erlangen Program". In this lecture Klein proposed to regard as geometrical only that properties of figures which are invariable under space transformations forming a Lie group. He implied the finite Lie group which transformations depend on finite number of parameters. Consequently, space symmetry properties became the main subject of geometry axiomatics. In this case the properties of geometrical figures are described by a set of Lie group invariants. Later the spaces admitting any Lie group of symmetry was named homogeneous (or Klein's) spaces.

Geometry can be regards as physics and as mathematics. Geometry as physics study the extension properties of material bodies. Its statements can and must be proved by experiments. Geometry as mathematics is only interesting in the logical dependences between its statements and the process of obtaining them from the axioms. Describing by geometry a motion of matter, we unify the space and time into a single extension and unify geometry with physics. Axiomatic physics is a part of mathematical or theoretical physics.

Structure of any physical theory reflects the process of obtaining information of the external world by experimental investigation. A distinctive feature of such investigation is the requirement of reproducibility of the results. This means that it is implicitly assumed that there exist a class of mutually identical objects of investigation, a class of identical frames of reference and instruments by means of which the measuring procedure is implemented. Regardless of how the identity of the studied objects or frames of reference is established in practice, the identity relation has the structure of a group. The measuring procedure consists in comparision of a studied object and a standard. Independent on the choice of the frame of reference results are formulated in terms of invariants of the symmetry Lie groups. So the symmetry group specifies a principle of relativity of the theory.

An analogous situation also exists in geometry. Euclidean geometry investigates the properties of figures independent of its position in 2-dimentional space. To determine these properties we have to move the figures in space and to compare them with standards. Geometrical properties of figures will be that are invariable under these movements. So, two-dimentional Euclidean geometry can be regarded as a theory of the invariants of the group of motions of the plane. At the same time, two-dimentional rotations and displacements constitute the group of motions of the implements used to construct geometrical figures, to prove the congruence of some of them, and to prove theorems. These implements are the compass and ruler without divisions. The use of other implements (for example, a ruler with divisions) would take us beyond the scope of Euclidean geometry. In this case the conformal group would become the group of motions of the implements and the symmetry group of the theory.

So, every physical theory contains in the structure of its axioms the properties which the instruments used to test it must possess. Conversely, the choice of the instruments and scheme of an experiment predetermines the possible type of symmetry of the theory describing the given experiment. Although in the physical experiment it is often difficult to determine directly an experimentally adequate type of symmetry, the logical connection between experimental and theoretical methods of investigation the world is the same in physics as in geometry. As Heisenberg said, we must remember that what we observe is not Nature itself, but Nature which appears in the form in which it reveals itself as a result of our manner of asking the questions.

5 Symmetry Groups and Axiomatics of Physics

Until the beginning of the twentieth century, the traditional method of the physical theory construction was the inductive method, which proceeds from experiment. Individual fields of physics (Newton's mechanics, Maxwell's electrodynamics) were axiomatized only after having been sufficiently well studied experimentally. An understanding of the final form to be taken by a physical theory and of the rules for construction any theory makes it possible to costruct a physical theory axiomatically, as Hilbert wanted to do. As is well known, Hilbert attempted to construct a unified theory of gravitation and electromagnetism on the base of a few axioms. The equations, obtained by him but without electrodynamic part, coincide with gravitational Einstein's equations. Unfortunately, this unified theory was not further development by Hilbert. However it was rediscovered by J.A. Weeler and C.W. Misner in 50's years. At the present time, a physical theory exists which was constructed axiomatically prior to experiment and afterwards found its physical realization. This is the theory of gauge fields which covers all fundamental interactions. It generalized the Hilbert-Weeler-Misner theory and include Maxwell's and Einstein's theories. Newtonian mechanics can be obtained from it by integration and passage to limit. As Hilbert predicted, it makes use of the deep theory of infinite Lie groups to classify interactions. Moreover, it admits a purely geometrical formulation, in which the analogy between axiomatics of geometry and physics becomes clear.

The physical theory is based on the principles of invariance and symmetry like geometry.

6 Types of Geometries and Types of Physical Theories

Three methods of construction geometry are used in physics: 1) Klein's approach, which assumes that space is homogeneous; all properties of geometrical objects in Klein's geometry are described by sets of invariants of the space symmetry group; 2) Riemann's approach, which does not consider any symmetry of the space; in this case, the characteristics of geometrical objects are constructed step by step from local differential expressions; connection coefficients are required for the construction of the space as a whole; 3) Cartan's approach, in which the space as a whole constitutes a set of local homogeneous Klein spaces associated with each point of a Riemannian space and interrelated by generalized connection coefficients.

The geometrical approaches in physics can be classified in a natural way in accordance with the foregoing conceptions of geometry. Klein's point of view is used, for example, in classical and relativistic mechanics. The images of Riemannian geometry - the metric, connection coefficients, and curvature - were used in general relativity. Cartan's approach, which was developed in the modern geometry of fiber bundle spaces, made it possible to geometrize the theory of gauge fields.

The connection between physics and geometry is determined by Poincare's symbolic formula: $\mathbf{G}=\mathbf{G}_0+\mathbf{F}$, where **G** represents the dynamical geometry, $\mathbf{G}_0$ is the geometry of the "background," and **F** - the forces of interaction. The meaning of this formula is that physics and geometry do not occur separately in experiment; only the combination of geometry and physical laws is subject to experimental verification. This idea was first expressed by Kant. Poincare understand that the decomposition of the sum **G** into a purely geometrical background and an interaction **F** depends on us. Now we can formulate it more precisely: this decomposition depends on the our choice of the means of measurement.

As long as physical phenomena are described as occurring at some place and time, space-time ideas cannot be excluded from the theoretical description of experiment. But the idea of forces which produce an interaction is not essential. A forcefree description of interactions renders the theory purely geometrical. The actually observed bending of trajectories of particles is described by means of the concept of connection coefficients of a nonholonomic space, which replaces the concept of force. If one and the same phenomenon is described in two different ways, there must exist a "principle of equivalence" which permits the transition from one description to the other. But in view of the relation between the form of the theory and the choice of the means of measurement, we must remember that the scheme of an experiment to test the geometrical theory must be different from one to test the ordinary theory of interactions in terms of forces. A geometrical description equivalent to a description in terms of forces always exists, but for an experimental verification of the geometrical form of the theory the test bodies and instruments must be correctly chosen. Any geometrical theory of physical phenomena is a theory of the motion of test bodies.

Cartan's approach is development of "Erlangen Program" using the geometrical idea of a space which points are arbitrary elements.

7 Summary

So, the Hilbert idea being formulated in his VIth problem is realized in the gauge field theory. Really, the classification principle which use of infinite Lie groups of transformations arises in this theory. This principle classifies the forces carring the interactions into effect in the field theory (classical and quantum), and in elementary particle physics. Newtonian mechanics can be obtained from the gauge field theory by integration and passage to limit as Hilbert proposed. In its geometrical form the gauge field theory (i.e. the physics) is the theory of connection coefficients of fibre bundle space. Consequently, the axiomatic theory of corresponding class of physical phenomena is possible in accordance with axiomatization of geometry. But the relevant axiom system will describe physics as geometry. Axiomatics of physics as physics is also possible, but it is other than axiomatics of physics as geometry. The common axioms in both cases will be Lie group symmetries.

The next step in VIth Hilbert's problem solution consists in answer the question: what is the largest possible class of physical phenomena admitting a pure geometrical theory as its description?

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Коноплева Н.П. E2-99-301
Шестая проблема Гильберта и бесконечные группы С.Ли

Демонстрируется прогресс в решении шестой проблемы Гильберта, который стал возможен благодаря теории калибровочных полей в физике и геометрической трактовке калибровочных полей. Показано, что геометрия расслоенных пространств является наилучшим базисом для решения обсуждаемой проблемы.

Работа была доложена на Международном семинаре «100 лет после Софуса Ли» (Лейпциг, Германия).

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна, 1999

Konopleva N.P. E2-99-301
VI Hilbert's Problem and S.Lie's
Infinite Groups

The progress in Hilbert's sixth problem solving is demonstrated. That became possible thanks to the gauge field theory in physics and to the geometrical treatment of the gauge fields. It is shown that the fibre bundle spaces geometry is the best basis for solution of the problem being discussed.

This talk has been reported at the International Seminar «100 Years after Sophus Lie» (Leipzig, Germany).

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