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LORENTZ FRAMES' PARALLELISM AND THOMAS–WIGNER ROTATION

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Параллельность лоренцевских систем
и вращение Томаса–Вигнера

Предложено и обсуждено операционное определение параллельности двух лоренцевских систем отсчета. Оно позволяет уточнить понятия буста и общего преобразования Лоренца. Показано, что эта параллельность не обладает свойством транзитивности: если триада T_1 осей системы S_1 параллельна триаде T_2 системы S_2 и $T_2 \parallel T_3$, то, вообще говоря, T_1 не параллельна T_3. Указанная нетранзитивность прямо связана с известным вращением Томаса–Вигнера. Ее учет необходим при обсуждении двух рассмотренных физических приложений вращения Томаса–Вигнера: прецессии Томаса и релятивистского фазового анализа.

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Lorentz Frames' Parallelism and Thomas–Wigner Rotation

Operational definitions of the parallelism of two Lorentz frames are proposed and discussed. They allow to refine upon the notion of boosts and general Lorentz transformations. It is shown that the parallelism lacks the property of transitivity: if the triad T_1 of a frame S_1 axes is parallel to the triad T_2 of S_2 and $T_2 \parallel T_3$, then generally T_1 is not parallel to T_3. This nontransitivity is directly related to the Thomas–Wigner rotation and clarifies the discussion of two considered physical applications of the rotation: Thomas precession and relativistic phase analysis.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1 INTRODUCTION

Before deriving the law of transformation between time-space coordinates of an event measured in two inertial frames S_1 and S_2, one must at first specify the frames themselves. In the case of the special (pure) Lorents transformation (boost), the triple T_2 of mutually orthogonal space axes of the reference frame S_2 (named triad T_2 below) is set to be parallel to the S_1 triad T_1. However, T_2 moves relative to T_1, and determination of their parallelism is more complicated as compared to the case when T_1 is at rest relative to T_2. In the latter case, it is sufficient to measure the projections of each axis of T_2 relative to T_1 and verify that T_1 axes are parallel to T_2 axes (two vectors $\vec{a}$ and $\vec{b}$ are parallel if $a_i = kb_i$, $i = 1, 2, 3$).

Moving axes parallelism is discussed in sect. 2. It is shown in sect. 3 that the notion is needed not only for the boost definition but also for refinement of the general Lorentz transformation definition.

Section 4 demonstrates that the property of the parallelism of the Lorentz triads is nontransitive (unlike the case of immovable triads), i.e., if $T_1 \parallel T_2$ and $T_2 \parallel T_3$, then T_1 is not parallel to T_3 in the general case. This is directly related to the well-known property of boosts: They do not form a group, i.e., the product B_2B_1 of two boosts is not a boost B in general. Instead, B_2B_1 is equal to a boost multiplied by a space rotation R: $B_2B_1 = RB$ e.g., see the books [1] and [2]. Two names are used for R in the literature: Thomas rotation or Wigner rotation (apropos of the terminology, see e.g. [3],[4]). I shall use the appellation "Thomas-Wigner rotation".

Physical applications of the rotation cannot be understood if transitivity property of Lorenz triads' parallelism is unconsciously assumed. This is illustrated in sect. 5 using as examples the Thomas precession and relativistic phase analysis for spinning particle scattering. The rotation is also known to be of importance when measuring the abnormal electron and muon magnetic moments, see e.g.[5].

The summary is given in the concluding section 6.

2 PARALLELISM OF LORENTZ FRAMES' TRIADS

As usual, it is supposed that in each Lorentz frame there are synchronized clocks, rulers, radars and other devices which allow one to measure lengths, angles and velocities. The information interchange between S_1 and S_2 (e.g. by radio or light signals) is also assumed to be possible.

2.1 Definition of moving triads parallelism

Fisher [6] emphasized the necessity of a particular definition of triads' parallelism and proposed the following one. An observer in S_1 measures the projections $(v_{21})_i$ $i = 1, 2, 3$ of the S_2 velocity relative to S_1 (with respect to his triad T_1). The

projections $(v_{12})_i$ of the S_1 velocity relative to S_2 (with respect to T_2) are measured in S_2. The S_1 observer informs the S_2 observer of his results. Fisher proposed to consider T_2 being parallel to T_1 if

$$(v_{21})_i = (-v_{12})_i, \quad i = 1, 2, 3 \quad or \quad \theta_i^{(1)} = \theta_i^{(2)} \tag{1}$$

where $\theta_i^{(1)}$ are the angles between the $\vec{v}_{21}$ and triad T_1 axes $\vec{e}_i^{\,(1)}$ and $\theta_i^{(2)}$ are the angles between $-\vec{v}_{12}$ and $\vec{e}_i^{\,(2)}$.

Obviously, the triads are not parallel if eq.(1) does not hold, i.e., it is a necessary condition for the triads' parallelism. But I stress that it is not a sufficient one. Indeed, eqs.(1) survive if one rotates arbitrarily T_1 around $\vec{v}_{21}$ or T_2 around $\vec{v}_{12}$. This is evident if one considers the equivalent rotation of a vector with respect to invariable axes. The vector does not change when rotating around itself. One may also use the formula for rotation around a direction $\vec{n}$ at an angle α:

$$\vec{a}' = \vec{a} + \vec{n} \times \vec{a} \sin\alpha + [\vec{n}(\vec{n}\vec{a}) - \vec{a}](1 - \cos\alpha). \tag{2}$$

If $\vec{n} = \vec{a}/a$, then $a'_i = a_i$. Therefore Fisher's proposal needs a complement.

Note that Fisher proposed a check of the T_1 and T_2 parallelism. I shall at first consider a construction of the T_2 triad which can be regarded as being parallel to T_1. Measurements or observations, which may be performed in the frames S_1 and S_2, will be used for the construction. Relative velocities are the basic supporting observables which can be used following [6].

To construct the needed triple $\vec{e}_i^{\,(2)}$, let us choose in S_2 an auxiliary spherical coordinate system with $-\vec{v}_{12}$ as an axis z. The polar angles of $\vec{e}_i^{\,(2)}$ in this system coincide with the angles $\theta_i^{(2)}$ which are required to be equal to $\theta_i^{(1)}$. Evidently, one has

$$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1, \quad \theta_i = \theta_i^{(1)} \quad or \quad \theta_i^{(2)}. \tag{3}$$

In order to construct $\vec{e}_i^{\,(2)}$, one must know in addition to θ_i the azimuthes φ_i of $\vec{e}_i^{\,(2)}$. Note that the needed $\vec{e}_i^{\,(2)}$ must be mutually orthogonal

$$(\vec{e}_i^{\,(2)} \cdot \vec{e}_j^{\,(2)}) = \cos\theta_i\cos\theta_j + \sin\theta_i\sin\theta_j\cos(\varphi_i - \varphi_j), \quad \forall i \neq j\,. \tag{4}$$

One can show that the differences $\varphi_2 - \varphi_1, \varphi_3 - \varphi_1$ (and consequently $\varphi_3 - \varphi_2 = (\varphi_3 - \varphi_1) - (\varphi_2 - \varphi_1)$ are fixed by eqs. (4) (θ_i being given) together with eq. (3) and by the requirement that $\vec{e}_1^{\,(2)}, \vec{e}_2^{\,(2)}, \vec{e}_3^{\,(2)}$ should form the right-hand triple (as $\vec{e}_i^{\,(1)}$ do). Only these differences are fixed but not the very azimuthes $\varphi_1, \varphi_2, \varphi_3$, e.g., φ_1 may be arbitrary. This arbitrariness has been discussed above when criticizing Fisher's proposal. So the given angles θ_i allow one to construct continually many triples $\vec{e}_i^{\,(2)}$. To complete the construction of the needed (parallel to T_1) triple, one must fix any of the azimuthes, e.g. φ_1.

I shall outline the following example of a measuring device which allows doing this. Let us take a luminous rod which represents the vector $\vec{e}_1^{\,(1)}$ of the triad T_1.

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The observer S_2 photographes it using a camera which is at the S_2 origin. Its optical axis is directed along $\vec{v}_{12}$ the photographic plate being orthogonal to $\vec{v}_{12}$. So a line (segment) is obtained on the plate which is the image of the (moving) rod $\vec{e}_1^{(1)}$, see the dashed line on Fig. 1.

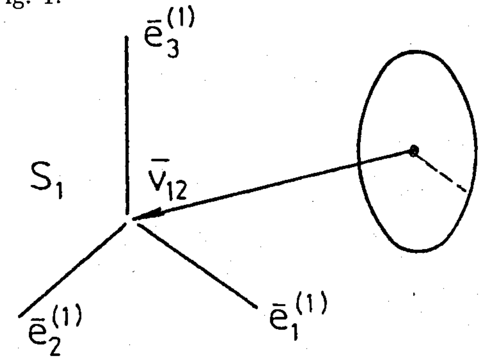

Fig. 1. Boldface dot represents the S_2 origin. The dashed line is the image of $\vec{e}_1^{(1)}$ on a photographic plate (shown by the circle).

The image and the optical axis determine a plane $\prod_1$. A unit length vector is constructed which is in $\prod_1$ and makes the angle θ_1 with $(-\vec{v}_{12})$ [1]. This vector is declared (defined) to be the axis $\vec{e}_1^{(2)}$ which is parallel to $\vec{e}_1^{(1)}$.

Let us make some subsidiary notes. We suppose that at $t_1 = t_2 = 0$ the origins of T_1 and T_2 coincide and the velocity of S_1 origin observed from S_2 origin is represented by the same vector $\vec{v}_{12}$ at all times ($\vec{v}_{12}$ does not depend on time). This means that one deals with a pure Lorentz transform but not with its combination with a translation.

The moment of photographing and the time of exposure are inessential.

The image given by the camera is not inverted or rotated (about the optical axis) as compared to its prototype. This may be verified in a separate experiment performed in S_2.

Of course, instead of the suggested T_2 construction one may check the T_1 and T_2 parallelism. The observer S_2 finds the plane $\prod_1$ and verifies that $\vec{e}_1^{(2)}$ is in the plane and makes the angle θ_1 with $(-\vec{v}_{12})$ which is equal to the angle between $\vec{e}_1^{(1)}$ and $\vec{v}_{21}$, etc.

Remark. The proposed definition of the triads' parallelism is valid for both the finite limiting (invariant) velocity (Lorentz case) and the infinite one (Galilei case). The definition allows one to obtain, in the usual manner, the Lorentz boosts in the former case and Galilei transformation in the latter case (e.g., see [2]).

The suggested construction of $T_2 \parallel T_1$ can be applied to the case when T_2 is at rest relative to T_1 but is translated by a vector $\vec{R}$. To this end one can treat $\vec{R}$ in the same way as the vector of relative velocity has been considered above. The construction can be used if there are difficulties in simple measuring of all projections of $\vec{e}_1^{(2)}, \vec{e}_2^{(2)}, \vec{e}_3^{(2)}$ with respect to T_1 (see Introduction).

[1]The described device does not measure θ_1, of course. The numerical value of θ_1 is communicated from S_1 to S_2 by radio. There exist two possible vectors in $\prod_1$ having the same angle θ_1 with $(-\vec{v}_{12})$. The right vector makes an acute angle with the image if $\vec{e}_1^{(1)}$

2.2 Other variants of the definition of parallelism

Measuring the photographic images of $\vec{e}_2^{(1)}$ and $\vec{e}_3^{(1)}$, one can obtain the planes $\prod_2$ and $\prod_3$ in the same way as $\prod_1$ has been determined above. Note that the angles between the planes $\prod_1, \prod_2, \prod_3$ are equal to the differences of azimuthes $\varphi_1^{(2)}, \varphi_2^{(2)}, \varphi_3^{(2)}$ of $\vec{e}_i^{(2)}$. Indeed, one may define $\varphi_i^{(2)}$ as the angle between the plane $\prod_1$ and an arbitrary plane containing the line connecting the S_1 and S_2 origins (it is directed along $\vec{v}_{12}$). Let us emphasize that measuring $\prod_2$ and $\prod_3$ in addition to $\prod_1$ is excessive (i.e., it would give no new information) if all polar angles θ_i of $\vec{e}_i^{(2)}$ are known. Indeed, the differences $\varphi_i - \varphi_j$ of the angles between the planes $\prod_1$ are fixed if θ_i are given, see the text below eq.(4).

But one can construct $T_2 \parallel T_1$ using chiefly the azimuthes $\varphi_1^{(2)}, \varphi_2^{(2)}, \varphi_3^{(2)}$ (measured with the help of the planes $\prod_1, \prod_2, \prod_3$) instead of $\theta_1, \theta_2, \theta_3$ and $\varphi_1^{(2)}$. It is possible to show, using eq. (4), that $\theta_i^{(2)}$ are determined by $\varphi_2^{(2)} - \varphi_1^{(2)}$ and $\varphi_3^{(2)} - \varphi_1^{(2)}$ if one knows in addition whether θ_1 (or θ_2 or θ_3) is either acute or obtuse (note that $0 \le \theta_i \le \pi$).

Besides the two stated ways of defining parallelism let us indicate the third one. It uses the particular case when the axis $\vec{e}_1^{(1)}$ of T_1 is chosen along $\vec{v}_{21}$. Then, the axis $\vec{e}_1^{(2)}$ of T_2 must be directed along $(-\vec{v}_{12})$. The directions of other axes $\vec{e}_2^{(2)}$ and $\vec{e}_3^{(2)}$ are determined as stated above, e.g. $\vec{e}_2^{(2)}$ must be directed in the plane $\prod_2$ along $\vec{e}_2^{(1)}$ image (in this case $\theta_2 = \pi/2$). The general case can be reduced to this particular one. To ascertain that T_1 and T_2 are parallel, both triads T_1 and T_2 must be rotated so that their new x-axis would coincide with the direction of the relative velocity (i.e., $\vec{v}_{21}$ for T_1 and $-\vec{v}_{12}$ for T_2). If these rotations are equal (more exactly if they differ only by rotations around relative velocity), then further verification of the parallelism proceeds as in the particular case above. If they are not, then T_1 is not parallel to T_2.

Remark 1. Aharoni [2] at the beginning of his ch. 1.11 discussed the triad parallelism. In distinction to my approach, he did not strive for a definition which would precede Lorentz transformation derivation. But he pointed out that the parallelism in the general case can be reduced to the parallelism in the case when the relative velocity is parallel to the x-axis.

Remark 2. The difficulty of the parallelism definition may be illustrated when discussing the following simple suggestion: "T_1 and T_2 are parallel if they coincide when the origins of S_1 and S_2 coincide". It is implied that the coincidence is detected in any of the frames, e.g., in S_1. The S_1 triad T_1 must coincide with an image of T_2 which is to be measured in S_1. For example, T_2 can be a triple of mutually orthogonal rods, and the measurement can be exemplified by the device for detecting moving rod lengths. In the Lorentz case, the image of T_2 turns out to be, in general, a triple of vectors which are not mutually orthogonal. Such a triple cannot coincide with the triad T_1, and the suggestion fails.

3 LORENTZ FRAMES' PARALLELISM AND DETERMINATION OF THE GENERAL LORENTZ TRANSFORMATION

As has been stated above, the definition of the triad parallelism must precede the boost determination. Let us show that the definition allows us also to refine upon the definition of the general Lorentz transformation (GLT) when the triads T_1 and T_2 are not parallel.

The boost matrix depends on three parameters, *viz.* the projections of relative velocity. One usually says that the GLT matrix depends, in addition, on three Euler rotation angles. But the triad T_2 moves relative to T_1 while the Euler angles determine the mutual orientation of two immovable triads. The following refinement of the GLT angles is proposed. In the Lorentz frame S_2, the triad $T_2^{\|}$ parallel to T_1 is constructed in the manner explained in subsect. 2.1. The triads $T_2^{\|}$ and T_2 are mutually immovable, and there are Euler rotations which turn $T_2^{\|}$ into T_2. The angles of these rotations can be taken as GLT parameters. Of course, one can define the latter using the rotation which turns the triad T_1 into the triad $T_1^{\|}$ which is parallel to T_2.

Let $r_i^{(2)}$ denote space coordinates of an event relative to $T_2^{\|}$, and $\rho_i^{(2)}$ are the event coordinates relative to T_2. We have

$$r_i^{(2)} = \Sigma_j \mathcal{D}_{ij}\rho_j^{(2)} \quad or \quad \vec{r}^{\,(2)} = \mathcal{D}\vec{\rho}^{\,(2)} \tag{5}$$

where $\mathcal{D}$ is the matrix of the rotation which turns $T_2^{\|}$ into T_2. As $T_2^{\|} \parallel T_1$ the coordinates $(r_i^{(2)}, t^{(2)})$ of the event are expressed in terms of its coordinates $(r_i^{(1)}, t^{(1)})$ relative to S_1 with the help of a boost (for the boost matrix derivation see e.g. [1],[2])

$$\begin{aligned} \vec{r}^{\,(2)} &= \vec{r}^{\,(1)} + \vec{v}_{21}(\vec{r}^{\,(1)}\cdot\vec{v}_{21})(\gamma_{21}-1)/v_{21}^2 - \vec{v}_{21}\gamma_{21}t^{(1)} \\ t^{(2)} &= \gamma_{21}[t^{(1)} - (\vec{r}^{\,(1)}\cdot\vec{v}_{21})/c^2], \quad \gamma_{21} \equiv [1 - v_{21}^2/c^2]^{-1/2} \end{aligned} \tag{6}$$

It follows from eqs.(3) and (4) that

$$\begin{aligned} \vec{\rho}^{\,(2)} = \mathcal{D}^{-1}\vec{r}^{\,(2)} &= \mathcal{D}^{-1}\vec{r}^{\,(1)} + \mathcal{D}^{-1}\vec{v}_{21}(\vec{r}^{\,(1)}\cdot\vec{v}_{21})(\gamma_{21}-1)/v_{21}^2 - \mathcal{D}^{-1}\vec{v}_{21}\gamma_{21}t^{(1)}; \\ t^{(2)} &= \gamma_{21}[t^{(1)} - (\vec{r}^{\,(1)}\vec{v}_{21})/c^2]\,. \end{aligned} \tag{7}$$

So GLT is represented as a product of a boost and a rotation.

Möller [1] gives another writing of eq.(7). If $\vec{v}_{21}$ is S_2 velocity relative to S_1 (the projections of $\vec{v}_{21}$ or T_1 are implied), then $-\vec{v}_{21}$ is S_1 velocity relative to $T_2^{\|}$. The velocity $\vec{v}_{12}$ of S_1 relative to T_2 is connected with $-\vec{v}_{21}$ by the rotation

$$-\vec{v}_{21} = \mathcal{D}\vec{v}_{12} \quad or \quad \vec{v}_{12} = -\mathcal{D}^{-1}\vec{v}_{21}. \tag{8}$$

Using eq. (8), one can rewrite eq.(7) in the form of eq. (2.28b) from [1]

$$\begin{aligned} \vec{\rho}^{\,(2)} &= \mathcal{D}^{-1}\vec{r}^{\,(1)} - \vec{v}_{12}(\vec{r}^{\,(1)}\cdot\vec{v}_{21})(\gamma_{21}-1)/\vec{v}_{21}^{\,2} - \vec{v}_{12}\gamma_{21}t^{(1)} \\ t^{(2)} &= \gamma_{21}[t^{(1)} - (\vec{r}^{\,(1)}\cdot\vec{v}_{21})/c^2]\,. \end{aligned} \tag{9}$$

4 NONTRANSITIVITY OF THE LORENTZ TRIADS' PARALLELISM AND THOMAS - WIGNER ROTATION

The property "if $a = b$ and $b = c$ then $a = c$" may serve as an example of the notion of transitivity. The equality of elements a, b and c and can be replaced by other binary relations, e.g., by the property of being parallel in the case when a, b and c are vectors. It is natural to ask whether the binary relation "Lorentz triads T_1 and T_2 are parallel" is transitive, i.e., is it true "if $T_1 \parallel T_2$ and $T_2 \parallel T_3$ then $T_1 \parallel T_3$"? Of course, if one draws on a sheet of paper these three triads, as one usually does it (see, e.g., Figure 2), then all triads turn out to be parallel pairwise. But really we ascertain in this manner that this is the property of immovable triads. It will be shown in this section that triads' parallelism (defined above in sect. 2) has no transitive property when invariant (limiting) velocity is finite (the Lorentz case).

4.1 Nontransitivity of the Lorentz triads parallelism

The nontransitivity results from the following reasoning. If $T_1 \parallel T_2$, then the transformation from S_1 to S_2 is a boost B_{21}. If $T_2 \parallel T_3$, then the transformation $S_3 \leftarrow S_2$ is also a boost B_{32}. If $T_1 \parallel T_3$, then the transformation $S_3 \leftarrow S_1$ must also be a boost according to the boost definition, see Introduction (1). However, the transformation $S_3 \leftarrow S_1$ may be determined as the product $B_{32}B_{21}$. It is not difficult to show that the product is not a boost in the general case, e.g., see [1],[2]. So T_1 cannot be parallel to T_3, and transitivity does not hold.

Let us demonstrate using the following simple example [2], that $B_{32}B_{21}$ is not a boost. Let B_{21} be the boost with the relative velocity $\vec{v}_{21} \parallel \vec{e}_x^{\,(1)}$ and B_{32} be the boost with $\vec{v}_{32} \parallel \vec{e}_y^{\,(2)}$, see Fig. 2.

Using eq. (6) we can write the matrices B_{32} and B_{21}, calculate their product $B_{32}B_{21}$ and compare it with the matrix of a boost B corresponding to a velocity $\vec{v}$, see again eq. (6). It turns out that presupposed equalities of the matrix elements $(B_{32}B_{21})_{\mu\nu}$ and $B_{\mu\nu}, \mu, \nu = 1,2,3,0$ are mutually exclusive. In particular, the equality of the last columns of $B_{32}B_{21}$ (i.e., elements $\mu, 0$) give some values for v_x, v_y while the equality of the last row (i.e., elements $0, \mu$) gives differing values for v_x, v_y.

A simpler proof of the nontransitivity is given in [1]. Suppose that $T_1 \parallel T_3$ along with $T_1 \parallel T_2$ and $T_2 \parallel T_3$. Then the projection of the velocity $\vec{v}_{31}$ of S_3 relative to S_1 and of the velocity $\vec{v}_{13}$ of S_1 relative to S_3 must satisfy

$$(\vec{v}_{31}) = -(\vec{v}_{13})_i, \quad i = 1,2,3\,. \tag{10}$$

See subsect. 2.1, eq.(1). I remind that $(\vec{v}_{31})_i$ are projections onto T_1 while $(v_{13})_i$ are projections onto T_3. Both the velocities $\vec{v}_{31}$ and $\vec{v}_{13}$ can be computed as functions of $\vec{v}_{21}$ and $\vec{v}_{32}$. For example the velocity $\vec{v}_{31}$ of S_3 relative to S_1 can be determined with the help of the boost B_{21} (the corresponding velocity being $\vec{v}_{21}$) and the velocity $\vec{v}_{32}$ of S_3 relative to S_2, e.g. see [1] ch. 2.7, eq. (2.55). Let us record the result as $\vec{v}_{31} = s(\vec{v}_{32}/\vec{v}_{21})$, for explicit expression see [1], eq. (2.55). Analogously $\vec{v}_{13}$ can be determined as

$$\vec{v}_{13} = s(\vec{v}_{12}/\vec{v}_{23}) = s(-\vec{v}_{21}/-\vec{v}_{32}) \tag{11}$$

see [1], eq. (2.55'). It turns out that eq. (10) does not hold in general. This means that the triads T_1 and T_3 are not parallel.

Note. The function $s(\vec{v}/\vec{v}\,')$ is referred usually to as "sum of $\vec{v}$ and $\vec{v}\,'$)". This name seems to be inappropriate in the general case because $s(\vec{v}/\vec{v}\,')$ depends upon $\vec{v}$ and $\vec{v}\,'$ in different ways and does not have the property $s(\vec{v}/\vec{v}\,') = s(\vec{v}\,'/\vec{v})$ except for the case $\vec{v} \parallel \vec{v}\,'$.

4.2 Thomas-Wigner rotation as the measure of the difference of T_1 vs. T_3 orientations

Let us consider quantitative difference of the T_1 and T_3 orientations provided that $T_1 \parallel T_2$ and $T_2 \parallel T_3$.

Lorentz transformations form a group; therefore, the product $B_{32}B_{21}$ of two boosts is generally speaking a GLT i.e., the product RB of a boost B and of a rotation R, see sect. 3. Here, R denotes the 4×4 matrix such that its time-space and space-time elements are zero, time-time element is unity and space-space elements form a 3×3 rotation matrix, see sect. 3. The parameters of B and R must be determined from the equation

$$B_{32}B_{21} = RB\,. \tag{12}$$

This is a hard algebraic work, which has been carried out, e.g., see [7]-[11]. It turns out that the boost B velocity is equal to $\vec{v}_{31} = s(\vec{v}_{32}/\vec{v}_{21})$. The rotation R is

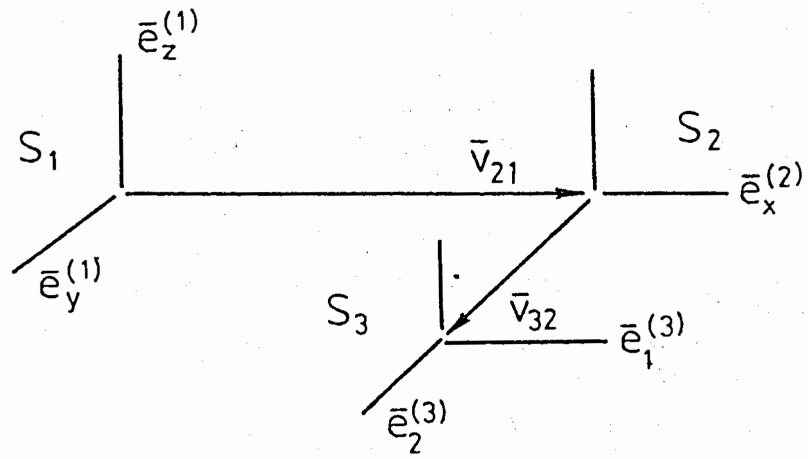

Fig. 2. The sequence of two boosts with nonparallel relative velocities $\vec{v}_{21}$ and $\vec{v}_{32}$.

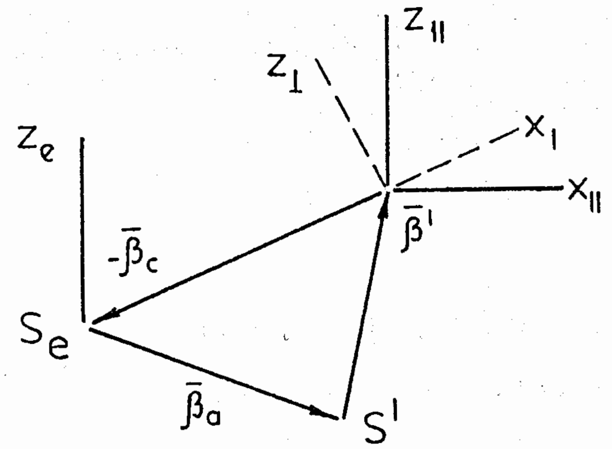

Fig. 3. Succession of boosts connecting the particle c rest systems S_{II} and S_I. Y-axes of all Lorentz frames are perpendicular to the plane of Fig. The axes x_{II} and z_{II} of S_{II} are drawn by the solid lines; the dashed lines show the S_I axes x_I and z_I.

also determined as a function of $\vec{v}_{32}$ and $\vec{v}_{21}$; R is called here the Thomas-Wigner rotation, see Introduction.

Being the boost, B must transform from the Lorentz frame S_1, into the frame $S_3^{\|}$ whose triad is parallel to T_1 ($S_3^{\|}$ velocity relative to S_1 being equal to $\vec{v}_{31}$). The triad has been denoted earlier as $T_3^{\|}$. So the space-space part of R must be the rotation which turns $T_3^{\|}$ into T_3 thereby determining the latter

$$T_2 = RT_3^{\|} \quad or \quad T_3 = RT_1 \,. \tag{13}$$

We see that the Thomas-Wigner rotation specifies the T_3 orientation relative to T_1 provided that $T_1 \parallel T_2$ and $T_2 \parallel T_3$.

R allows as to express $(-\vec{v}_{13})_j, j = 1,2,3$ ($-\vec{v}_{13}$ projections on T_3) in terms of $(v_{31})_i, i = 1,2,3$ (projections on T_1)

$$(-\vec{v}_{13})_j = \Sigma_i (R^{-1})_{ji} (\vec{v}_{31})_i \quad or \quad \vec{v}_{31} = R(-\vec{v}_{13}). \tag{14}$$

Vice versa, if $(\vec{v}_{31})_i$ and $\vec{v}_{13})_j$ are given, then one can find the simplest notation R_{sim} which satisfies eq.(14): the R_{sim} axis $\vec{n}$ is parallel to the vector product $\vec{v}_{31} \times \vec{v}_{13}$ (which is parallel to $\vec{v}_{31} \times \vec{v}_{21}$) and the R_{sim} angle α is such that

$$\sin\alpha = \vec{v}_{31} \times \vec{v}_{13} / |\vec{v}_{31} \parallel \vec{v}_{13}| \,.$$

See eq. (2).

Möller in his ch. (2.8) identifies R with R_{sim}. However, their equality is not evident because R might differ from R_{sim} by additional rotations around the vectors $\vec{v}_{31}$ or $\vec{v}_{13}$ (the rotation around $\vec{v}_{31}$ does not change $\vec{v}_{31}$). Nevertheless, actually the rotation R determined by eq. (10) does coincide with R_{sim}.

One can obtain from eq. (12) its following modification

$$B_{31} B_{21} = B(s(\vec{v}_{21}/\vec{v}_{32}))R = B(-\vec{v}_{13})R, \tag{15}$$

see eq. (14a) in [9].

5 EXAMPLES OF PHYSICAL APPLICATIONS

5.1 Kinematic origin of the Thomas precession

Consider an electron which moves with acceleration in the laboratory system (e.g., in the Coulomb field of a nucleus). Let us introduce the non inertial frame of reference A (Accelerated frame) which moves together with the electron so that the electron at each moment of time is at rest in A. Assume that the A triad $T(\tau)$ at the moment τ is parallel to the A triad $T(\tau + d\tau)$ at the infinitesimal close moment $\tau + d\tau$ (A can be called accompanying frame of reference).

Remember that we deal with inertial systems when considering the Thomas-Wigner rotation. To apply this consideration to the case of an accelerated electron,

let us introduce a set of inertial frames $S(\tau), -\infty < \tau < \infty$ such that at each moment τ one frame of the set coincides with $A(\tau)$. In the case of classical electron, one may assume that its own instantaneous inertial Lorentz frame is introduced at each point of the electron trajectory.

Let us consider two inertial frames $S_2 = S(\tau_2)$ and $S_3 = S(\tau_3)$, $\tau_3 = \tau_2 + d\tau$. According to the A and $S(\tau)$ definitions, the triads of S_2 and S_3 are parallel. I may refer to Fig. 2 where S_1 denotes the laboratory frame whose axes are parallel to those of S_2. The evident reservation is that the electron velocity increment $\vec{v}_{32} = \vec{v}(\tau_3) - \vec{v}(\tau_2)$ is now infinitesimally small and need not be perpendicular to $\vec{v}(t_2)$.

Suppose that the force acting on the electron is torqueless so that the electron spin direction is the same relative to the triads T_2 and T_3 (in the non inertial system A, the spin does not rotate). In the quantum case, the "spin direction" is defined as a mean value of the spin vector operator (the polarization vector). Under the described conditions the Thomas-Wigner rotation means that the triads T_1 and T_3 orientations are different. The electron polarization vector is the same relative to T_2 and T_3 but it has different projections with respect to the laboratory triad T_1 at the times τ_2 and τ_3. This is the origin of the Thomas precession. For its quantitative description and physical significance see, e.g., [12] and [2] and references therein.

5.2 The Thomas-Wigner rotation and phase analysis

Let us consider a reaction $a + b \to c + d$ (reaction I) involving particles with spins. Its phase analysis requires not only differential cross section measurement but also information on particle polarizations. In particular, one needs to measure the particle c polarization. Let its spin be equal to 1/2. The polarization vector may be determined by measuring angular asymmetries in another reaction II which may be scattering: the particle c scatters on a target e: $c + e \to c + e$. I am going to show that c polarization measured in II cannot be used immediately for the phase analysis of the first reaction I: the polarization vector must be transformed beforehand in a specific manner.

The phase analysis is simplified if one uses the following way of describing the state of the spin particle. One specifies its linear momentum $\vec{p}$ along with the spin wave function χ_m in the particle rest system (and not in the Lorentz system where particle momentum is $\vec{p}$), m being eigenvalues of the projection of the spin vector operator $\vec{s}$ along $\vec{p}$ (helicity). The operator $\vec{s}$ is defined as the particle total angular momentum in the particle rest system. The orbital part of the momentum is then equal to zero, the total angular momentum being reduced to its spin part. For details see, e.g. [13]-[15]. Particle c polarization vector is defined as the mean value of $\vec{s}$ in the state χ. Just this vector can be determined using the reaction II. The corresponding particle rest system is denoted by S_{II}. There may be distinct particle rest systems which differ by orientations of their triads. I shall now show that there is one more particle rest system S_I when considering reactions I and II.

The system S_{II} is linked with the laboratory system S_l with the help of the boost with velocity $\vec{\beta}_c$, parallel to the particle c linear momentum $\vec{p}_c$ in S_l. One can obtain another particle rest system starting from S_l. To this end let us perform first the Lorentz transformation from S_l to the center-of-mass system S' of the reaction I (the corresponding velocity $\vec{\beta}_a$ being parallel to the linear momentum $\vec{p}_a$ of the reaction I incident particle a). Further, one transforms from S' to the particle c rest system S_I using the velocity $\vec{\beta}\,'$ parallel to the momentum p_c' of the particle c in S'. In order to show that the S_I triad differs from the S_{II} triad let us write out the succession of boosts which allows one to pass from S_{II} to S_I:

1) $B(S_l \leftarrow S_{II}; -\vec{\beta}_c)$; $-\vec{\beta}_c$ is the S_l velocity relative to S_{II}, $\vec{\beta}_c \parallel \vec{p}_c$.
2) $B(S' \leftarrow S_l; \vec{\beta}_a)$; $\vec{\beta}_a \parallel \vec{p}_a$
3) $B(S_I \leftarrow S'; \vec{\beta}\,')$.

The corresponding velocity $\vec{\beta}\,' \parallel \vec{p}\,'_c$ is determined by the velocities $\vec{\beta}_c$ and $\vec{\beta}_a$. Indeed, $\vec{\beta}\,' = s(\vec{\beta}_c / -\vec{\beta}_a)$, $\vec{\beta}_c$ being the particle c velocity with respect to S_l and $-\vec{\beta}_a$ being the velocity of S_l with respect to S', see subsect. 4.1. Let us note that it is the velocities $\vec{\beta}_c$ and $\vec{\beta}_a$ (or momenta $\vec{p}_c$ and $\vec{p}_a$) which are measured directly; the velocity $\vec{\beta}\,'$ is to be computed using the former ones. The product

$$B(\vec{\beta}\,')B(\vec{\beta}_a)B(-\vec{\beta}_c) \tag{16}$$

of the boosts 1), 2), 3) is a rotation. Indeed, let us rewrite eq. (15) as

$$B^{-1}(S(\vec{v}_{21}|v_{32})B(v_{32})B(v_{21}) = R \tag{17}$$

and compare the l.h.s. of eq. (17) with the product (16) where

$$\vec{\beta}\,' = s(\vec{\beta}_c / -\vec{\beta}_a) = -s(-\vec{\beta}_c/\vec{\beta}_a)\,. \tag{18}$$

Equation (18) can be verified using eq. (2.55) from [1] for the "sum of velocities" s. Being a boost, $B(\beta')$ satisfies the equation

$$B^{-1}(\vec{\beta}\,') = B(-\vec{\beta}\,') \quad or \quad B(\vec{\beta}\,') = B^{-1}(-\vec{\beta}\,')\,.$$

Inserting into eq. (16) the equalities $\vec{v}_{21} = -\vec{\beta}_c; \vec{v}_{32} = \vec{\beta}_a$, we can rewrite the l.h.s. of eq. (17) as the product

$$B^{-1}(s(-\vec{\beta}_c/\vec{\beta}_a))B(\vec{\beta}_a)B(-\vec{\beta}_c) = B(\vec{\beta}\,')B(\vec{\beta}_a)B(-\vec{\beta}_c). \tag{19}$$

We see that the product (16) is equal to the Thomas-Wigner rotation R which turns the S_{II} triad into the S_I triad. Fig. 3 illustrated the foregoing succession of boosts

The polarization vector projections, which can be determined using reaction II, are referred to the S_{II} triad. Using R, one can calculate the vector projections with respect to the S_I triad. Just these projections are needed for the reaction I phase analysis.

The rotation R is complementary to the obvious rotation caused by the change of the quantization axes when helicities are used: One must obtain the polarization vector projections on the vector $\vec{p}\,'_c$ starting with the polarization vector projections with respect to the direction $\vec{p}_c$ (and other two axes orthogonal to $\vec{p}_c$).

6 CONCLUSION

The construction of parallel moving triads (or verification of their parallelism) meets difficulties which are exemplified by the Remark 2 in subsect. 2.2 and at the beginning of subsect. 2.1.

The measuring and structural methods of the operational definitions of the parallelism are proposed and discussed.

It is shown that the parallelism does not possess the transitivity property when the invariant (limiting) velocity is finite (Lorentz case). This nontransitivity is directly related to the Thomas-Wigner rotation which can be considered as its quantitative measure. Taking the nontransitivity into account is essential when discussing physical applications of the Thomas-Wigner rotation.

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