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ELECTROMAGNETIC FIELDS OF ELECTRIC, MAGNETIC, AND TOROIDAL DIPOLES MOVING IN MEDIUM

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Электромагнитные поля электрических, магнитных
и тороидальных диполей, движущихся в среде

Найдены точные выражения для электромагнитных полей бесконечно малых электрических и магнитных диполей и тороидального соленоида, движущихся в бездисперсной среде. Исследование выполнено для произвольных скоростей и ориентаций диполей. При движении со скоростью, меньшей скорости света в веществе, электромагнитное поле отлично от нуля во всем пространстве (даже для тороидального соленоида). При скорости, большей скорости света в веществе, электромагнитное поле отлично от нуля только внутри и на поверхности черенковского конуса.

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Electromagnetic Fields of Electric, Magnetic,
and Toroidal Dipoles Moving in Medium

We found the exact electromagnetic fields of point-like electric, magnetic dipoles and toroidal solenoid moving in a non-dispersive medium. The investigation is performed for arbitrary velocities and orientations of dipoles. For the motion with a velocity smaller than the light velocity in medium, the electromagnetic field differs from zero everywhere (even for toroidal solenoids). When the velocity is greater than the light velocity in medium, the electromagnetic field is confined to the interior and surface of the Cherenkov cone.

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1 Introduction

To our best knowledge, the electromagnetic field (EMF) arising from the motion of electric and magnetic dipoles in medium was first considered by Frank in 1942 [1]. He solved Maxwell's equations in the laboratory frame (LF) with electric and magnetic polarizations obtained from the corresponding polarizations in the rest frame of a moving dipole by the Lorentz transformations. It was, therefore, suggested there that transformations laws between electric and magnetic moments moving in medium are the same as in vacuum (see, e.g., [2] where the nice exposition of transformation properties of the polarizabilities in two reference frames is given). The magnetic dipole considered in [1] was an elementary (i.e., infinetisimally small) current loop. Formulae describing the intensity radiation for a moving magnetic dipole did not satisfy Frank, as the intensity radiation did not disappear for the case when the dipole velocity coincided with the phase velocity in medium (the vanishing of the above radiation is intuitively expected and is satisfied, e.g., for a moving electric dipole).
10 years later, in 1952, another Frank's publication [3] on the same subject appeared. In it, he treated the magnetic dipole as consisting of two magnetic poles and obtained a correct expression (in the sense mentioned above) for the intensity radiation of a magnetic dipole moving in medium. To reconcile the results of [1] and [3] Frank suggested that transformation laws between the electric and magnetic moment moving in medium should differ from that in vacuum.
This Frank guess has been confirmed by Ginzburg [4] who, starting with the Maxwell equations in a moving medium and writing the corresponding constitutive relations between the EMF strengths and inductions, obtained the correct vector of magnetic polarization.
In 1984, two further publications by Frank [5] and Ginzburg [6] appeared. The difference between [1] and [3] was attributed to different definitions of magnetic dipoles used there: the electric current magnetic dipole and magnetic dipole composed of magnetic poles was used in [1] and [3], resp. These two models of magnetic dipole possess different properties as to their interactions with magnetic medium ([7]). At present, both experiment and theory definitely support that magnetic moments of elementary particles are of the electric-current type [8,9].
In Ref. [10], the radiation of toroidal moments (i.e., the elementary (infinesimally small) toroidal solenoids (TS)) was considered. It was shown that the EMF of the TS moving in medium extends beyond its boundaries. This seemed to be surprising since the EMF of TS resting either in medium (or vacuum) or moving in vacuum is confined to its interior. In one of the latest life-time publications [11], Frank returned to the initial premise ([1]) that the transformation laws between the dipole electric and magnetic moments moving in medium should be the same as in vacuum.

The goal of this consideration is to obtain exact EMF potentials and strengths for the point-like electric and magnetic dipoles and elementary TS moving in a non-dispersive medium with an arbitrary velocity v which may be greater or smaller than the light velocity in medium c_n. The method is in a straghtforward solution of the wave equations for the EMF potentials with charge-current densities in their r.h.s. and in a subsequent evaluation of the EMF strengths.
The question arises why not to use Frank's idea for the evaluation of EMF of the moving

dipole. In our translation from Russian, it may be formulated as follows ([3], p. 190):

> It is suggested that a moving electric dipole p'_1 is equivalent to some dipoles at rest, namely, to the electric p_1 and magnetic m_1 placed at the point coinciding with the instantaneous position of a moving dipole. The same is suggested for a magnetic dipole.

The reason for not using the transformation formulae for the electric and magnetic dipole moments moving in media is that there are different formulations of the moving media electrodynamics leading to different transformation laws for electric and magnetic polarizabilities and, therefore, for the electric and magnetic moments (which are the space integrals of polarizabilities). The nice brief exposition of the moving media electrodynamics may be found in [12,13].
Even more confusing is the situation with toroidal moments (see Refs. [14-17] for their definition) for which the transformations formulae are not known.

The idea of this treatment is exactly opposite to Frank's idea. Evaluating the EMF produced by moving dipoles and elementary toroidal solenoids in the $\vec{r}, t$ representation, we try to identify their electromagnetic moments and obtain their transformation laws. Earlier, the EMF of electric and magnetic dipoles moving in medium was found in Refs. [1-6, 10, 18- 20] yet in the ω representation. The sole exception is Ref. [10] where the scalar electric potential for elementary TS with its symmetry axis directed along the motion axis was obtained in configuration $\vec{r}, t$ space.
The plan of this exposition is as follows. In section 2, which is essentially a quintessence of Refs. [21], we give the necessary mathematical details which will be used in a subsequent exposition. In section 3, we find the EMF of the magnetic moment moving in a non-dispersive medium with an arbitrary velocity (which may be greater or smaller the light velocity in medium) and with different orientations relative to the motion axis. The exact EMFs of elementary TS and the electric dipole are presented in Sects. 4 and 5, resp. The EMFs of moving dipoles evaluated according to Frank's prescription are compared with the exact ones in Sect. 6. Short discussion of the results obtained is given in Sect. 7.

Where the obtained exact expressions for EMF's can be applied to? First, any particle having either electric or magnetic dipole moments should radiate when its velocity exceeds the light velocity in medium. Then, exact results obtained here show how the arising EMF's are distributed in space-time. Second, EMF's obtained in Sect. 4 can be observed in neutrino experiments. As far as we know, the neutrino possesses both dipole and toroidal magnetic moments ([22-25]). In the massless limit only the toroidal moment survives. This is valid, in particular, for Majorana neutrino.

2 Mathematical preliminairies: equivalent sources of electromagnetic field

As we have mentioned, this section is essentially an extract of Refs. [21]. It is needed for the understanding of subsequent exposition.

2.1 A pedagogical example: circular current

According to the Ampere hypothesis, the distribution of magnetic dipoles $\vec{M}(\vec{r})$ is equivalent to the current distribution $\vec{J}(\vec{r}) = curl\vec{M}(\vec{r})$. For example, a circular current flowing in the $z = 0$ plane

$$\vec{J} = I\vec{n}_\phi\delta(\rho - d)\delta(z) \tag{2.1}$$

is equivalent to the magnetization (see Fig. 1)

$$\vec{M} = I\vec{n}\Theta(d - \rho)\delta(z) \tag{2.2}$$

different from zero in the same plane and directed along its normal $\vec{n}$ ($\Theta(x)$ is a step function). When the radius d of the circumference along which the current flows tends to zero, the current $\vec{J}$ becomes ill-defined (it is not clear what does the vector $\vec{n}_\phi$ mean at the origin). On the other hand, the vector $\vec{M}$ is still well-defined. In this limit, the elementary current (2.1) turns out to be equivalent to the magnetic dipole oriented normally to the plane of this current:

$$\vec{M} = I\pi d^2\vec{n}\delta^3(\vec{r}), \quad (\delta^3(\vec{r}) = \delta(\rho)\delta(z)/2\pi\rho) \tag{2.3}$$

and

$$\vec{J} = I\pi d^2 curl(\vec{n}\delta^3(\vec{r})) \tag{2.4}$$

Equations (2.3) and (2.4) define the magnetization and current density corresponding to the elementary magnetic dipole.

2.2 The elementary toroidal solenoid

The case next in complexity is the poloidal current flowing in the winding of TS (Fig.2):

$$\vec{j} = -\frac{gc}{4\pi}\vec{n}_\psi\frac{\delta(R - \tilde{R})}{d + \tilde{R}\cos\psi}. \tag{2.5}$$

The coordinates $\tilde{R}, \psi$ and ϕ are related to the Cartesian ones as follows:

$$x = (d + \tilde{R}\cos\psi)\cos\phi, \quad y = (d + \tilde{R}\cos\psi)\sin\phi, \quad z = \tilde{R}\sin\psi. \tag{2.6}$$

The condition $\tilde{R} = R$ defines the surface of a particular torus (Fig. 3). For $\tilde{R}$ fixed and ψ, ϕ varying, the points x, y, z given by (2.6) fill the surface of torus $(\rho - d)^2 + z^2 = R^2$. The choice $\vec{j}$ in the form (2.5) is convenient, because in the static case a magnetic field H equals g/ρ inside the torus and vanishes outside it. In the static case, g may be also expressed either through the magnetic flux Φ penetrating the torus or through the total number N of turns in toroidal winding and the current I in a particular turn;

$$g = \frac{\Phi}{2\pi(d - \sqrt{d^2 - R^2})} = \frac{2NI}{c}.$$

We write out differential operators div and $curl$ in $\tilde{R}, \psi$, and ϕ coordinates:

$$div\vec{A} = \frac{1}{\tilde{R}(d + \tilde{R}\cos\psi)}[\frac{\partial}{\partial\tilde{R}}\tilde{R}(d + \tilde{R}\cos\psi)A_{\tilde{R}} + \frac{\partial}{\partial\psi}(d + \tilde{R}\cos\psi)A_\psi + \frac{\partial}{\partial\phi}\tilde{R}A_\phi],$$

$$(curl\vec{A})_{\tilde{R}} = \frac{1}{\tilde{R}(d+\tilde{R}\cos\psi)}[\frac{\partial}{\partial\phi}(\tilde{R}A_\psi) - \frac{\partial}{\partial\psi}(d+\tilde{R}\cos\psi)A_\phi],$$

$$(curl\vec{A})_\phi = \frac{1}{\tilde{R}}[\frac{\partial}{\partial\psi} - \frac{\partial}{\partial\tilde{R}}(\tilde{R}A_\psi)],$$

$$(curl\vec{A})_\psi = \frac{1}{d+\tilde{R}\cos\psi}[\frac{\partial}{\partial\tilde{R}}(d+\tilde{R}\cos\psi)A_\phi - \frac{\partial A_{\tilde{R}}}{\partial\phi}]. \qquad (2.7)$$

As $div\vec{j} = 0$, the current $\vec{j}$ can be presented as $curl$ of a certain vector $\vec{M}$:

$$\vec{j} = curl\vec{M}. \qquad (2.8)$$

Or, in a manifest form:

$$-\frac{gc}{4\pi}\frac{\delta(R-\tilde{R})}{d+\tilde{R}\cos\psi} = \frac{1}{d+\tilde{R}\cos\psi}[\frac{\partial}{\partial\tilde{R}}(d+\tilde{R}\cos\psi)M_\phi - \frac{\partial M_{\tilde{R}}}{\partial\phi}].$$

Due to the axial symmetry of the problem, the term involving ϕ differentiation drops out, and one gets

$$-\frac{gc}{4\pi}\frac{\delta(R-\tilde{R}}{d+\tilde{R}\cos\psi} = \frac{1}{d+\tilde{R}\cos\psi}\frac{\partial}{\partial\tilde{R}}(d+\tilde{R}\cos\psi)M_\phi.$$

Contracting by the factor $d+\tilde{R}\cos\psi$ one has

$$-\frac{gc}{4\pi}\delta(R-\tilde{R}) = \frac{\partial}{\partial\tilde{R}}(d+\tilde{R}\cos\psi)M_\phi.$$

It follows from this that

$$M_\phi = \frac{gc}{4\pi}\frac{\Theta(R-\tilde{R})}{d+\tilde{R}\cos\psi}, \qquad (2.9)$$

i.e., M_ϕ is confined to the interior of the torus (Fig. 4).
We rewrite M_ϕ in cylindrical coordinates:

$$M_\phi = \frac{gc}{4\pi\rho}\Theta[R-\sqrt{(\rho-d)^2+z^2}]. \qquad (2.10)$$

Since $div\vec{M} = 0$, the magnetization vector $\vec{M}$ can in its turn be presented as a $curl$ of a certain vector $\vec{T}$. It turns out that only the z component of $\vec{T}$ differs from zero:

$$T_z = -\frac{gc}{4\pi}[\Theta(d-\sqrt{R^2-z^2}-\rho)\ln\frac{d-\sqrt{R^2-z^2}}{d+\sqrt{R^2-z^2}}+$$

$$+\Theta(d+\sqrt{R^2-z^2}-\rho)\Theta(\rho-d+\sqrt{R^2-z^2})\ln\frac{\rho}{d+\sqrt{R^2-z^2}}]. \qquad (2.11)$$

Thus, T_z differs from zero in two space regions:
a) Inside the torus hole defined as $0 \le \rho \le d-\sqrt{R^2-z^2}$, where T_z does not depend on ρ:

$$T_z = -\frac{gc}{4\pi}\ln\frac{d-\sqrt{R^2-z^2}}{d+\sqrt{R^2-z^2}}. \qquad (2.12)$$

b) Inside the torus itself $(d-\sqrt{R^2-z^2} \le \rho \le d+\sqrt{R^2-z^2})$ where

$$T_z = -\frac{gc}{4\pi}\ln\frac{\rho}{d+\sqrt{R^2-z^2}}. \qquad (2.13)$$

In other space regions, $T_z = 0$.

Now let the minor radius R of a torus tend to zero (this corresponds to an infinitely thin torus). Then, the second term in (2.11) drops out, while the first one reduces to

$$T_z \to \frac{gc}{2\pi d}\Theta(d-\rho)\sqrt{R^2-z^2}. \qquad (2.14)$$

For infinitesimal R

$$\sqrt{R^2-z^2} \to \frac{1}{2}\pi R^2\delta(z).$$

Therefore, in this limit,

$$\vec{j} = curlcurl\vec{T}, \quad \vec{T} = \vec{n}_z\frac{gcR^2}{4d}\delta(z)\Theta(d-\rho). \qquad (2.15)$$

i.e., the vector $\vec{T}$ is confined to the equatorial plane of a torus and perpendicular to it.

Let now $d \to 0$ (in addition to $R \to 0$). In this limit,

$$\frac{1}{d}\Theta(d-\rho) \to \frac{d}{2\rho}\delta(\rho)$$

and the current of an elementary (i.e., infinitely small) TS is

$$\vec{j} = curlcurl\vec{T}, \quad \vec{T} = \frac{1}{4}\pi cgdR^2\delta^3(\vec{r})\vec{n}_z. \qquad (2.16)$$

Then, the elementary current flowing in the winding of the elementary TS is given by

$$j_1 = f curl^{(2)}(\vec{n}\delta^3(\vec{r})) \qquad (2.17)$$

where $curl^{(2)} = curlcurl$, $\vec{n}$ means the unit vector normal to the equatorial plane of TS and $f = \pi cgdR^2/4$.

Physically, Eqs. (2.5), (2.8) and (2.15)-(2.17) mean that the poloidal current $\vec{j}$ given by Eq.(2.5) is equivalent (i.e., produces the same magnetic field) to the toroidal tube with the magnetization $\vec{M}$ defined by (2.9) and to the toroidization $\vec{T}$ given by (2.11). This illustrates Fig. 4.

Another remarkable property of these configurations is that they interact in the same way with the time-dependent magnetic or electric field ([21]). For example, the usual current loop interacts with an external magnetic field in the same way as the magnetic dipole orthogonal to it. The poloidal current shown in the upper part of Fig. 4, the magnetized ring corresponding to the magnetization M in its middle part and the toroidal distribution T in its lower part, all of them interact in the same way with the external electromagnetic field. Obviously, the equivalence between current distributions and magnetizations (toroidizations) is a straightforward generalization of the original Ampere hypothesis.

In what follows, we need the Lorentz transformation formulae for the charge-current densities and for electromagnetic strengths. They may be found in any textbook on electrodynamics (see, e.g., [26,27]). Let ρ' and $\vec{j}'$ be charge and current densities in the rest frame S' which moves with a constant velocity $\vec{v}$ relative to the laboratory frame (LF) S. Then,

$$\rho = \gamma(\rho' + \vec{\beta}\vec{j}'/c), \quad \vec{j} = \gamma(\vec{j}' + \vec{v}\rho') \tag{2.18}$$

Here $\gamma = (1-\beta^2)^{-1/2}$, $\vec{\beta} = \vec{v}/c$. If there is no charge density in S', then

$$\rho = \gamma\vec{\beta}\vec{j}'/c, \quad \vec{j} = \gamma\vec{j}'. \tag{2.19}$$

If there is no current density in S', then

$$\rho = \gamma\rho', \quad \vec{j} = \gamma\vec{v}\rho'. \tag{2.20}$$

Let $\vec{E}, \vec{D}, \vec{H}, \vec{B}$ and $\vec{E}', \vec{D}', \vec{H}', \vec{B}'$ be electromagnetic strengths and inductions in the LF and in S', resp. Then,

$$\vec{E} = \gamma(\vec{E}' - \vec{\beta}\times\vec{B}') - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\vec{E}'), \quad \vec{B} = \gamma(\vec{B}' + \vec{\beta}\times\vec{E}') - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\vec{B}'),$$

$$\vec{D} = \gamma(\vec{D}' - \vec{\beta}\times\vec{H}') - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\vec{D}'), \quad \vec{H} = \gamma(\vec{H}' + \vec{\beta}\times\vec{D}') - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\vec{H}'). \tag{2.21}$$

We need also constituitive relations ([13]) in the reference frame which moves with the velocity $\vec{v}$ relative to the laboratory frame (in the latter the surrounding matter is at rest)

$$\vec{D}' = \frac{1}{1-\beta_n^2}\{\epsilon[\vec{E}'(1-\beta^2) + \vec{\beta}(\vec{\beta}\vec{E}')(1-n^2)] + \vec{\beta}\times\vec{H}'(1-n^2)\},$$

$$\vec{B}' = \frac{1}{1-\beta_n^2}\{\mu[\vec{H}'(1-\beta^2) + \vec{\beta}(\vec{\beta}\vec{H}')(1-n^2)] - \vec{\beta}\times\vec{E}'(1-n^2)\}, \tag{2.22}$$

where $\beta_n = v/c_n$, $c_n = c/n$ is the light velocity in medium, $n = \sqrt{\epsilon\mu}$ is its refractive index, ϵ and μ are electric permittivity and magnetic permeability, resp.
For the sake of completeness, we write out Maxwell equations and wave equations for the electromagnetic potentials corresponding to charge $\rho(\vec{r},t)$ and current $\vec{j}(\vec{r},t)$ densities imbedded into a nondispersive medium with constant ϵ and μ:

$$div\vec{D} = 4\pi\rho, \quad \vec{D} = \epsilon\vec{E}, \quad div\vec{B} = 0, \quad \vec{B} = \mu\vec{H},$$

$$curl\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \quad curl\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t} + \frac{4\pi}{c}\vec{j},$$

$$\vec{B} = curl\vec{A}, \quad \vec{E} = -grad\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad div\vec{A} + \frac{\epsilon\mu}{c}\frac{\partial\Phi}{\partial t} = 0,$$

$$(\Delta - \frac{1}{c_n^2}\frac{\partial^2}{\partial t^2})\Phi = -\frac{4\pi}{\epsilon}\rho_{Ch}, \quad (\Delta - \frac{1}{c_n^2}\frac{\partial^2}{\partial t^2})\vec{A} = -\frac{4\pi\mu}{c}\vec{j}. \tag{2.23}$$

In what follows, by the term 'magnetic moment' we mean the magnetic moment carried by an infinitesimal circular loop. The alternative to this is the magnetic moment composed of two magnetic poles. We have mentioned, these two different realizations of magnetic moments differently interact with magnetic media.

We also use the fields of electric $\vec{p}$ and magnetic $\vec{m}$ dipoles which rest at the origin ([27])

$$\vec{E} = -\frac{\vec{p}}{r^3} + 3\vec{r}\frac{\vec{r}\vec{p}}{r^5}, \quad \vec{B} = -\frac{\vec{m}}{r^3} + 3\vec{r}\frac{\vec{r}\vec{m}}{r^5}. \tag{2.24}$$

3 Electromagnetic field of moving point-like current loop

3.1 The velocity is along the loop axis

Consider a conducting loop $\mathcal{L}$ moving uniformly in a nondispersive medium with the velocity v directed along the loop axis. Let in this loop a constant current I flows. In the reference frame attached to the moving loop, the current density is equal to

$$\vec{j} = I\vec{n}_\phi\delta(\rho' - d)\delta(z'), \quad \rho' = \sqrt{x'^2 + y'^2}.$$

In accordance with (2.19), one gets in the LF

$$j = I\vec{n}_\phi\delta(\rho - d)\delta(\gamma(z - vt)) = \frac{I}{\gamma}\vec{n}_\phi\delta(\rho - d)\delta(z - vt).$$

Here $\vec{n}_\phi = \vec{n}_y\cos\phi - \vec{n}_x\sin\phi$, $\gamma = 1/\sqrt{1-\beta^2}$. Since the current direction is perpendicular to the velocity, no charge density arises in the LF.
The solution of Eq.(2.23) for electromagnetic potentials is given by

$$\Phi = \frac{1}{\epsilon}\int\frac{1}{R}\rho_{Ch}(\vec{r}\,',t')\delta(t' - t + \frac{R}{c_n})dV'dt',$$

$$\vec{A} = \frac{\mu}{c}\int\frac{1}{R}\vec{j}(\vec{r}\,',t')\delta(t' - t + \frac{R}{c_n})dV'dt', \quad R = |\vec{r} - \vec{r}\,'| \tag{3.1}$$

Like for a charge at rest, the current $\vec{j}$ may be expressed through the magnetization

$$\vec{j} = curl\vec{M}. \tag{3.2}$$

The magnetization $\vec{M}$ is perpendicular to the plane of a current loop:

$$M_z = \frac{I_0}{\gamma}\Theta(d - \rho)\delta(z - vt). \tag{3.3}$$

Substituting this into (3.1) and integrating by parts, one finds

$$\vec{A} = \frac{\mu}{c}curl\int\frac{1}{R}\delta(t' - t + \frac{R}{c_n})\vec{M}dV'dt. \tag{3.4}$$

The electric scalar potential is zero.
Now let the loop radius d tend to zero. Then [21],

$$\Theta(d - \rho) \to \pi d^2\delta(x)\delta(y) \quad \text{and} \quad M_z \to \frac{I_0\pi d^2}{\gamma}\delta(x)\delta(y)\delta(z - vt)$$

Substituting this into (3.4) and integrating over the space variables one gets

$$A_\phi = -\frac{\mu I_0\pi d^2}{c\gamma}\frac{\partial\alpha}{\partial\rho}, \tag{3.5}$$

where

$$\alpha = \int \frac{1}{R}\delta(t'-t+R/c_n)dt', \quad R=\sqrt{\rho^2+(z-vt')^2}. \tag{3.6}$$

This integral can be taken in a closed form (see, e.g. [28]):

$$\alpha=\frac{1}{r_m} \quad \text{for} \quad v<c_n \quad \text{and} \quad \alpha=\frac{2}{r_m}\Theta(vt-z-\rho/\gamma_n) \quad \text{for} \quad v>c_n. \tag{3.7}$$

Here $r_m=[(z-vt)^2+\rho^2(1-\beta_n^2)]^{1/2}$, $\gamma_n=|1-\beta_n^2|^{-1/2}$, $\beta_n=v/c_n$. The equality $vt-z-\rho/\gamma_n=0$ defines the surface of the so-called Cherenkov cone attached to the moving magnetic dipole. Therefore, for $\beta_n<1$, α differs from zero everywhere, while for $\beta_n>1$ it differs from zero only inside the Cherenkov cone where $vt-z-\rho/\gamma_n>0$. Performing differentiation in (3.5), one gets

$$A_\phi=\frac{\mu m(1-\beta_n^2)\rho}{\gamma r_m^3} \quad \text{for} \quad \beta<\beta_n \quad \text{and}$$

$$A_\phi=\frac{2\mu m(1-\beta_n^2)\rho}{\gamma r_m^3}\Theta(vt-z-\rho/\gamma_n)+\frac{2\mu m}{\gamma\gamma_n r_m}\delta(vt-z-\rho/\gamma_n). \tag{3.8}$$

for $\beta_n>1$. Here $m=I_0\pi d^2/c$. Therefore, for $\beta_n<1$, A_ϕ differs from zero everywhere vanishing in the $\theta=\pi/2$ plane ($\rho=r\sin\theta$) and decreasing like r^{-2} at large distances. For $\beta_n>1$, A_ϕ vanishes outside the Cherenkov cone, being infinite on its surface and falling as r^{-2} inside it. Electromagnetic field strengths are obtained by differentiating A_ϕ:

$$E_x=\frac{\mu\beta m}{\gamma}\frac{\partial^2\alpha}{\partial z\partial y}, \quad E_y=-\frac{\mu\beta m}{\gamma}\frac{\partial^2\alpha}{\partial z\partial x}, \quad E_z=0,$$

$$B_x=\frac{\mu m}{\gamma}\frac{\partial^2\alpha}{\partial z\partial x}, \quad B_y=\frac{\mu m}{\gamma}\frac{\partial^2\alpha}{\partial z\partial y},$$

$$B_z=-\frac{\mu m}{\gamma}[\Delta-(1-\beta_n^2)\frac{\partial^2\alpha}{\partial z^2}], \quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+(1-\beta_n^2)\frac{\partial^2}{\partial z^2}. \tag{3.9}$$

The action of Δ and $\frac{\partial^2}{\partial z^2}$ on α gives for $\beta_n<1$:

$$\Delta\alpha=-4\pi\delta(x)\delta(y)\delta(z-vt), \quad (1-\beta_n^2)\frac{\partial^2\alpha}{\partial z^2}=-\frac{1-\beta_n^2}{r_m^3}[1-3\frac{(z-vt)^2}{r_m^2}]-\frac{4\pi}{3}\delta^3(\vec r).$$

Here $\delta^3(\vec r)=\delta(x)\delta(y)\delta(z-vt)$. These relations result from the identity (see, e.g.,[29])

$$\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{r}=-\frac{1}{r^3}(\delta_{ij}-3\frac{x_ix_j}{r^2})-\frac{4\pi}{3}\delta_{ij}\delta^3(\vec r). \tag{3.10}$$

Higher derivatives of $1/r$ are obtained by differentiating (3.10).
For $\beta_n<1$, the EMF strengths of a moving point-like current loop are given by

$$E_x=3m\mu\beta\frac{\gamma_n^3}{\gamma}\frac{y(z-vt)}{r^5}, \quad E_y=-3m\beta\mu\frac{\gamma_n^3}{\gamma}\frac{y(z-vt)}{r^5},$$

$$B_x=3m\mu\frac{\gamma_n^3}{\gamma}\frac{x(z-vt)}{r^5}, \quad B_y=3m\mu\frac{\gamma_n^3}{\gamma}\frac{y(z-vt)}{r^5},$$

$$B_z=\frac{m\mu}{\gamma}\{\frac{8\pi}{3}\delta^3(\vec r)-\frac{\gamma_n}{r^3}[1-3\gamma_n^2\frac{(z-vt)^2}{r^2}]\}, \tag{3.11}$$

where $m=I_0\pi d^2/c$, $r^2=x^2+y^2+(z-vt)^2\gamma_n^2$, $\delta^3(\vec r)=\delta(x)\delta(y)\delta(z-vt)$. In what follows, in order not to overload exposition, we drop the δ-functions terms corresponding to the current position of a moving dipole. They are easily restored from Eq.(3.10).
It is seen that $\vec B$ in (3.11) strongly resembles the field of magnetic dipole. On the other hand, the electric field $\vec E$ having only two Cartesian components, cannot be reduced to the field of electric dipole.

We conclude: for $\beta_n<1$, the EMF strengths differ from zero everywhere, falling like r^{-3} at large distances. For $\beta_n>1$, they equal zero outside the Cherenkov cone ($vt-z-\rho/\gamma_n<0$), infinite on its surface, and fall like r^{-3} inside the Cherenkov cone ($vt-z-\rho/\gamma_n>0$). As a result, only the moving EMF singularity coinciding with the Cherenkov cone will be observed in the wave zone.
In the rest frame of the magnetic dipole, the EMF is given by

$$\vec E'=0, \quad B'_x=3\frac{m\mu\gamma_n^3}{\gamma^3}\frac{x'z'}{r'^5}, \quad B'_y=3\frac{m\mu\gamma_n^3}{\gamma^3}\frac{y'z'}{r'^5}, \quad B'_z=-m\mu\frac{\gamma_n}{\gamma r'^3}(1-3\frac{\gamma_n^2}{\gamma^2}\frac{z'^2}{r'^2}),$$

$$H'_x=3m\frac{\gamma_n}{\gamma}\frac{x'z'}{r'^5}, \quad H'_y=3m\frac{\gamma_n}{\gamma}\frac{y'z'}{r'^5}, \quad H'_z=-m\frac{\gamma_n}{\gamma r'^3}(1-3\frac{\gamma_n^2}{\gamma^2}\frac{z'^2}{r'^2}),$$

$$D'_x=3m(n^2-1)\frac{\gamma_n^3\beta}{\gamma}\frac{y'z'}{r'^5}, \quad D'_y=-3m(n^2-1)\frac{\gamma_n^3\beta}{\gamma}\frac{x'z'}{r'^5}, \tag{3.12}$$

where $r'^2=(x'^2+y'^2)+\gamma_n^2z'^2/\gamma^2$ and $x'=x$, $y'=y$, $z'=\gamma(z-vt)$. Since in this reference frame the medium has the velocity $-\vec v$, the familiar constituitive relations $\vec B'=\mu\vec H'$, $\vec D'=\epsilon\vec E'$ are not longer valid. Instead, Eqs.(2.22) should be used.
In vacuum, Eqs.(3.11) and (3.12) reduce to

$$E_x=3m\gamma^2\frac{y(z-vt)}{r_0^5}, \quad E_y=-3m\gamma^2\frac{x(z-vt)}{r_0^5},$$

$$H_x=3m\gamma^2\frac{x(z-vt)}{r_0^5}, \quad H_y=3m\gamma^2\frac{y(z-vt)}{r_0^5}, \quad H_z=-\frac{m}{r_0^3}[1-3\frac{\gamma^2(z-vt)^2}{r_0^2}], \tag{3.13}$$

$$\vec E'=0, \quad H'_x=3m\frac{x'z'}{r'^5}, \quad H'_y=3m\frac{y'z'}{r'^5}, \quad H'_z=-\frac{m}{r'^3}(1-3\frac{z'^2}{r'^2}), \tag{3.14}$$

where $r_0^2=\gamma^2(z-vt)^2+x^2+y^2$ and $r'^2=x'^2+y'^2+x'^2$. Equations (3.13) and (3.14) are connected by the Lorentz transformation.

3.2 The velocity is in the plane of loop

Let a circular loop move in the direction perpendicular to the symmetry axis (say, along the x axis, see Fig.5,b). Then, in LF, one gets

$$j_x=-I_0\delta(z)\frac{y\gamma}{d}\delta(\rho_1-d), \quad j_y=I_0\delta(z)\frac{(x-vt)\gamma}{d}\delta(\rho_1-d), \quad \rho_{Ch}=-I_0\delta(z)\frac{yv\gamma}{c^2d}\delta(\rho_1-d).$$

Here $\rho_1 = [(x-vt)^2\gamma^2 + y^2]^{1/2}$. The charge density arises because on a part of the loop, the current has a non-zero projection on the direction of motion. It is easy to check that

$$j_x = I_0\gamma\delta(z)\frac{\partial}{\partial y}M_s, \quad j_y = -I_0\frac{1}{\gamma}\delta(z)\frac{\partial}{\partial x}M_s, \quad \rho_{Ch} = I_0\frac{v\gamma}{c^2}\delta(z)\frac{\partial}{\partial y}M_s,$$

where $M_s = \Theta(d-\rho_1)$. In the limit of an infinitesimal loop,

$$M_s = \Theta(d-\rho_1) \to \delta(x-vt)\delta(y)\pi d^2/\gamma. \tag{3.15}$$

For the electromagnetic potentials, one easily finds

$$\Phi = \frac{m\beta}{\epsilon}\frac{\partial\alpha_1}{\partial y}, \quad A_x = m\mu\frac{\partial\alpha_1}{\partial y}, \quad A_y = -\frac{m\mu}{\gamma^2}\frac{\partial\alpha_1}{\partial x}. \tag{3.16}$$

Here

$$\alpha_1 = \int dt'\frac{1}{R_1}\delta(t'-t+R_1/c_n), \quad R_1 = [(x-vt')^2 + y^2 + z^2]^{1/2}.$$

Again, this integral can be taken in a closed form:

$$\alpha_1 = \frac{1}{r_m^{(1)}} \quad \text{for} \quad \beta < \beta_n \quad \text{and}$$

$$\alpha_1 = \frac{2}{r_m^{(1)}}\Theta(vt - x - \frac{1}{\gamma_n}\sqrt{y^2+z^2}) \quad \text{for} \quad \beta > \beta_n. \tag{3.17}$$

Here $r_m^{(1)} = [(x-vt)^2 + (y^2+z^2)(1-\beta_n^2)]^{1/2}$. Therefore,

$$\Phi = -\frac{m\beta}{\epsilon}\frac{y}{(r_m^{(1)})^3}(1-\beta_n^2), \quad A_x = -m\mu\frac{y}{(r_m^{(1)})^3}(1-\beta_n^2), \quad A_y = \frac{m\mu}{\gamma^2}\frac{x-vt}{(r_m^{(1)})^3}$$

for $\beta_n < 1$ and

$$\Phi = -2\frac{m\beta}{\epsilon}\frac{y}{(r_m^{(1)})^3}(1-\beta_n^2)\Theta(vt - x - \frac{1}{\gamma_n}) - \frac{2m\beta}{\epsilon\gamma_n}\frac{y}{r_m^{(1)}\sqrt{y^2+z^2}}\delta(vt - x - \frac{1}{\gamma_n}\sqrt{y^2+z^2}),$$

$$A_x = -2m\mu\frac{y}{(r_m^{(1)})^3}(1-\beta_n^2)\Theta(vt - x - \frac{1}{\gamma_n}) - \frac{2m\mu}{\gamma_n}\frac{y}{r_m^{(1)}\sqrt{y^2+z^2}}\delta(vt - x - \frac{1}{\gamma_n}\sqrt{y^2+z^2}),$$

$$A_y = \frac{2m\mu}{\gamma^2}\frac{x-vt}{(r_m^{(1)})^3} + \frac{2m\mu}{\gamma^2}\frac{1}{r_m^{(1)}}\delta(vt - x - \frac{1}{\gamma_n}\sqrt{y^2+z^2}). \tag{3.18}$$

Electromagnetic field strengths are given by

$$E_x = -\frac{m\beta}{\epsilon}(1-n^2)\frac{\partial^2\alpha_1}{\partial x\partial y}, \quad E_y = -\frac{m\beta}{\epsilon}(\frac{\partial^2\alpha_1}{\partial y^2} + \frac{n^2}{\gamma^2}\frac{\partial^2\alpha_1}{\partial x^2}), \quad E_z = -\frac{m\beta}{\epsilon}\frac{\partial^2\alpha_1}{\partial z\partial y},$$

$$B_x = \frac{m\mu}{\gamma^2}\frac{\partial^2\alpha_1}{\partial z\partial x}, \quad B_y = m\mu\frac{\partial^2\alpha_1}{\partial z\partial y}, \quad B_z = -m\mu(\frac{1}{\gamma^2}\frac{\partial^2\alpha_1}{\partial x^2} + \frac{\partial^2\alpha_1}{\partial y^2}). \tag{3.19}$$

For $\beta_n < 1$, the EMF falls like r^{-3} at large distances. For $\beta_n > 1$, the EMF strengths vanish outside the Cherenkov cone ($vt - x - \frac{1}{\gamma_n}\sqrt{y^2+z^2} < 0$), they decrease like r^{-3} at

large distances inside the Cherenkov cone ($vt - x - \frac{1}{\gamma_n}\sqrt{y^2+z^2} > 0$), and they are infinite on the Cherenkov cone. Thus, in the wave zone the electomagnetic field is confined to the Cherenkov cone ($vt - x - \frac{1}{\gamma_n}\sqrt{y^2+z^2} = 0$) where it is infinite.
We write out EMF in the manifest form for $\beta_n < 1$:

$$E_x = -3m\frac{\beta}{\epsilon}(1-n^2)\gamma_n^3\frac{(x-vt)y}{r'^5}, \quad E_z = -3m\frac{\beta}{\epsilon}\gamma_n\frac{yz}{r'^5},$$

$$E_y = \frac{m\beta\gamma_n}{\epsilon r'^3}\{(1-3\frac{y^2}{r'^2}) + n^2\frac{\gamma_n^2}{\gamma^2}[1-3\gamma_n^2\frac{(x-vt)^2}{r'^2}]\},$$

$$B_x = 3m\mu\frac{\gamma_n^3}{\gamma^2}\frac{(x-vt)z}{r'^5}, \quad B_y = 3m\mu\gamma_n\frac{yz}{r'^5},$$

$$B_z = \frac{m\mu\gamma_n}{r'^3}\{(1-3\frac{y^2}{r'^2}) + \frac{\gamma_n^2}{\gamma^2}[1-3\gamma_n^2\frac{(x-vt)^2}{r'^2}]\}, \tag{3.20}$$

where $r'^2 = y^2 + z^2 + (x-vt)^2\gamma_n^2$. For the motion in vacuum, this reduces to

$$E_x = 0, \quad E_z = -3\frac{m\beta\gamma}{c}\frac{yz}{r_1^5}, \quad E_y = -\frac{m\beta\gamma}{cr_1^3}(1-3\frac{z^2}{r_1^2}),$$

$$H_x = 3m\gamma\frac{z(x-vt)}{r_1^5}, \quad H_y = 3m\gamma\frac{yz}{r_1^5}, \quad H_z = -m\gamma\frac{1}{r_1^3}(1-3\frac{z^2}{r_1^2}). \tag{3.21}$$

Here $r_1^2 = y^2 + z^2 + (x-vt)^2\gamma^2$. Again, these expressions may be obtained by applying a suitable Lorentz transformation to EMF strengths in the dipole rest frame.

4 Electromagnetic field of a moving point-like toroidal solenoid

Consider the poloidal current (Fig.2) flowing on the surface of a torus

$$(\rho - d)^2 + z^2 = R_0^2$$

(R_0 and d are the minor and large radii of torus). It is convenient to introduce coordinates $\rho = d + R\cos\psi$, $z = R\sin\psi$ (Fig. 3). In these coordinates, the poloidal current flowing on the torus surface is given by

$$\vec{j} = j_0\frac{\delta(R_0 - R)}{d + R_0\cos\psi}\vec{n}_\psi.$$

Here $\vec{n}_\psi = \vec{n}_z\cos\psi - \vec{n}_\rho\sin\psi$ is the vector lying on the torus surface and defining the current direction, $R = \sqrt{(\rho-d)^2 + z^2}$. The cylindrical components of $\vec{j}$ are

$$j_z = j_0\frac{\delta(R_0 - R)}{d + R_0\cos\psi}\cos\psi, \quad j_\rho = -\frac{\delta(R_0 - R)}{d + R_0\cos\psi}\sin\psi.$$

4.1 The velocity is along the torus axis

Let this current distribution move uniformly along the z axis (directed along the torus symmetry axis) with the velocity v (Fig. 5a). In the LF the nonvanishing charge and current components are

$$\rho_{Ch} = j_0\gamma\beta\frac{\rho-d}{c\rho R_0}\delta(R_0-R_2),\quad j_\rho = -j_0\gamma\frac{z-vt}{\rho R_0}\delta(R_0-R_2),$$

$$j_z = j_0\gamma\frac{\rho-d}{\rho R_0}\delta(R_0-R_2). \quad (4.1)$$

Here $R_2 = \sqrt{(\rho-d)^2+(z-vt)^2\gamma^2}$. These components may be represented in the form

$$j_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho M_\phi),\quad j_\rho = -\frac{1}{\gamma^2}\frac{\partial M_\phi}{\partial z},\quad \rho_{Ch} = \frac{\beta}{c\rho}\frac{\partial}{\partial\rho}(\rho M_\phi). \quad (4.2)$$

Here

$$M_\phi = -j_0\gamma\frac{1}{\rho}\Theta(R_0-R).$$

The Cartesian components of $\vec M$ are

$$M_x = j_0\gamma\frac{y}{\rho^2}\Theta(R_0-R),\quad M_y = -j_0\gamma\frac{x}{\rho^2}\Theta(R_0-R). \quad (4.3)$$

Let the minor radius R_0 tend to zero. Then,

$$\Theta[R_0-\sqrt{(\rho-d)^2+(z-vt)^2\gamma^2}] \to \frac{\pi R_0^2}{\gamma}\delta(\rho-d)\delta(z-vt)$$

and

$$M_x = -\frac{j_0}{d}\pi R_0^2\frac{\partial}{\partial y}\Theta(d-\rho)\delta(z-vt),\quad M_y = \frac{j_0}{d}\pi R_0^2\frac{\partial}{\partial x}\Theta(d-\rho)\delta(z-vt). \quad (4.4)$$

Therefore,

$$j_x = -\frac{1}{\gamma^2}\frac{\partial M_y}{\partial z} = -\frac{j_0\pi R_0^2}{\gamma^2 d}\frac{\partial^2}{\partial z\partial x}\Theta(d-\rho)\delta(z-vt),$$

$$j_y = \frac{1}{\gamma^2}\frac{\partial M_x}{\partial z} = -\frac{j_0\pi R_0^2}{\gamma^2 d}\frac{\partial^2}{\partial z\partial y}\Theta(d-\rho)\delta(z-vt),$$

$$j_z = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} = \frac{j_0\pi R_0^2}{d}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Theta(d-\rho)\delta(z-vt),$$

$$\rho_{Ch} = \frac{\beta}{c}\left(\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}\right) = \frac{\beta j_0\pi R_0^2}{cd}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Theta(d-\rho)\delta(z-vt). \quad (4.5)$$

Let the major torus radius also tend to zero. Then,

$$\Theta(d-\rho) = \pi d^2\delta(x)\delta(y)$$

and

$$j_x = -\frac{j_0\pi^2R_0^2d}{\gamma^2}\frac{\partial^2}{\partial z\partial x}\delta(x)\delta(y)\delta(z-vt),\quad j_y = -\frac{j_0\pi R_0^2 d}{\gamma^2 d}\frac{\partial^2}{\partial z\partial y}\delta(x)\delta(y)\delta(z-vt),$$

$$j_z = j_0\pi^2R_0^2d\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta(x)\delta(y)\delta(z-vt),$$

$$\rho_{Ch} = \frac{\beta j_0\pi^2R_0^2d}{c}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta(x)\delta(y)\delta(z-vt). \quad (4.6)$$

From this one easily obtains the electromagnetic potentials

$$\Phi = \frac{\beta m_t}{\epsilon}[\Delta-(1-\beta_n^2)\frac{\partial^2}{\partial z^2}]\alpha,\quad A_x = -\frac{m_t\mu}{\gamma^2}\frac{\partial^2}{\partial z\partial x}\alpha,$$

$$A_y = -\frac{m_t\mu}{\gamma^2}\frac{\partial^2}{\partial z\partial y}\alpha,\quad A_z = m_t\mu[\Delta-(1-\beta_n^2)\frac{\partial^2}{\partial z^2}]\alpha, \quad (4.7)$$

where α is the same as in Eqs. (3.6) and (3.7) and $m_t = \pi^2 j_0 d R_0^2/c$. Being written in a manifest form, the electromagnetic potentials are

$$\Phi = \frac{\beta m_t}{\epsilon}(1-\beta_n^2)\frac{1}{r_m^3}[1-3\frac{(z-vt)^2}{r_m^2}],\quad A_z = \mu m_t(1-\beta_n^2)\frac{1}{r_m^3}[1-3\frac{(z-vt)^2}{r_m^2}],$$

$$A_x = -3\frac{m_t\mu}{\gamma^2}(1-\beta_n^2)\frac{x(z-vt)}{r_m^5},\quad A_y = -3\frac{m_t\mu}{c\gamma^2}(1-\beta_n^2)\frac{y(z-vt)}{r_m^5}$$

for $\beta_n < 1$ and

$$\Phi = \frac{2\beta m_t}{\epsilon}\{\frac{1-\beta_n^2}{r_m^3}[1-3(1-\beta_n^2)\frac{(z-vt)^2}{r_m^2}]\Theta(vt-z-\rho/\gamma_n)$$

$$+2(1-\beta_n^2)\frac{\rho}{\gamma_n r_m^3}\delta(vt-z-\rho/\gamma_n)+\frac{1}{r_m}[\frac{1}{\gamma_n^2}\dot\delta(vt-z-\rho/\gamma_n)-\frac{1}{\gamma_n\rho}\delta(vt-z-\rho/\gamma_n)]\},$$

$$A_z = 2\mu m_t\{\frac{1-\beta_n^2}{r_m^3}[1-3(1-\beta_n^2)\frac{(z-vt)^2}{r_m^2}]\Theta(vt-z-\rho/\gamma_n)$$

$$+2(1-\beta_n^2)\frac{\rho}{\gamma_n r_m^3}\delta(vt-z-\rho/\gamma_n)+\frac{1}{r_m}[\frac{1}{\gamma_n^2}\dot\delta(vt-z-\rho/\gamma_n)-\frac{1}{\gamma_n\rho}\delta(vt-z-\rho/\gamma_n)]\},$$

$$A_\rho = -\frac{2m_t\mu\rho}{\gamma^2}[3(1-\beta_n^2)\frac{z-vt}{r_m^5}\Theta(vt-z-\rho/\gamma_n)$$

$$+\frac{1}{r_m^3}(\frac{z-vt}{\rho}+1-\beta_n^2)\delta(vt-z-\rho/\gamma_n)+\frac{1}{r_m\rho\gamma_n}\dot\delta(vt-z-\rho/\gamma_n)]. \quad (4.8)$$

for $\beta_n > 1$ (the dot above delta function means a derivative over its argument). Earlier, the scalar electric potential Φ for $\beta_n < 1$ was found in Ref.[10].
The electromagnetic field strengths are equal to

$$E_x = -\frac{\beta m_t}{\epsilon}[\Delta+(n^2-1)\frac{\partial^2}{\partial z^2}]\frac{\partial\alpha}{\partial x},\quad E_y = -\frac{\beta m_t}{\epsilon}[\Delta+(n^2-1)\frac{\partial^2}{\partial z^2}]\frac{\partial\alpha}{\partial y},$$

$$E_z = \frac{\beta m_t}{\epsilon}(n^2-1)[\Delta+(\beta_n^2-1)\frac{\partial^2}{\partial z^2}]\frac{\partial\alpha}{\partial z},$$

$$B_x = m_t\mu[\Delta+\beta^2(n^2-1)\frac{\partial^2}{\partial z^2}]\frac{\partial\alpha}{\partial y},\quad B_y = -m_t\mu[\Delta+\beta^2(n^2-1)\frac{\partial^2}{\partial z^2}]\frac{\partial\alpha}{\partial x},$$

$$B_z = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (1-\beta_n^2)\frac{\partial^2}{\partial z^2}. \tag{4.9}$$

For $\beta_n < 1$, the EMF falls like r^{-4} at large distances. For $\beta_n > 1$, the EMF field strengths equal zero outside the Cherenkov cone; inside this cone, they fall like r^{-4} for $r \to \infty$ and they are infinite on the Cherenkov cone.
We write out the EMF in the manifest form for $\beta_n < 1$:

$$E_x = -\frac{\beta m_t}{\epsilon}\frac{3x}{r^5}(n^2-1)\gamma_n^3[1-5\frac{\gamma_n^2(z-vt)^2}{r^2}], \quad E_y = -\frac{\beta m_t}{\epsilon}\frac{3y}{r^5}(n^2-1)\gamma_n^3[1-5\frac{\gamma_n^2(z-vt)^2}{r^2}],$$

$$E_z = -\frac{\beta m_t}{\epsilon}(n^2-1)[\frac{3(z-vt)}{r^5}\gamma_n^3(1-5\frac{\gamma_n^2(z-vt)^2}{r^2})],$$

$$B_x = m_t\mu\frac{3y}{r^5}\gamma_n^3\beta^2(n^2-1)[1-5\frac{\gamma_n^2(z-vt)^2}{r^2}],$$

$$B_y = -m_t\mu\frac{3x}{r^5}\beta^2\gamma_n^3(n^2-1)[1-5\frac{\gamma_n^2(z-vt)^2}{r^2}], \quad B_z = 0. \tag{4.10}$$

It is seen that the electric field of an elementary toroidal solenoid moving in the non-dispersive medium strongly resembles the field of an electric quadrupole. As the magnetic field in (4.10) has only the ϕ component, it cannot be reduced to the field of a magnetic quadrupole. Conditionally, it may be called the field of the moving toroidal moment.
The electromagnetic strengths and inductions in the reference frame, where the toroidal dipole is at rest and the medium moves with the velocity $-\vec{v}$, are equal to

$$B'_x = \frac{m_t\gamma}{\epsilon}\beta^2\gamma_n^3(n^2-1)^2\frac{3y'}{r'^5}(1-5\frac{\gamma_n^2 z'^2}{\gamma^2 r'^2}),$$

$$B'_y = -\frac{m_t\gamma}{\epsilon}\beta^2\gamma_n^3(n^2-1)^2\frac{3x'}{r'^5}(1-5\frac{\gamma_n^2 z'^2}{\gamma^2 r'^2}), \quad B'z = 0, \quad \vec{H}' = 0,$$

$$E'_x = \frac{m_t\gamma\beta}{\epsilon}(1-n^2)\gamma_n\frac{3x'}{r'^5}(1-5\frac{\gamma_n^2 z'^2}{\gamma^2 r'^2}), \quad E'_y = \frac{m_t\gamma\beta}{\epsilon}(1-n^2)\gamma_n\frac{3y'}{r'^5}(1-5\frac{\gamma_n^2 z'^2}{\gamma^2 r'^2}),$$

$$E'_z = \frac{m_t\beta}{\epsilon}(1-n^2)\gamma_n^3\frac{3z'}{\gamma r'^5}(1-5\frac{z'^2}{\gamma^2 r'^2}), \quad D'_x = -\beta m_t\gamma(n^2-1)\gamma_n^3\frac{3x'}{r'^5}(1-5\frac{\gamma_n^2 z'^2}{\gamma^2 r'^2}),$$

$$D'_y = -\beta m_t\gamma(n^2-1)\gamma_n^3\frac{3y'}{r'^5}(1-5\frac{\gamma_n^2 z'^2}{\gamma^2 r'^2}), \quad D'_z = -\beta m_t(n^2-1)\frac{\gamma_n^3}{\gamma}\frac{3z'}{r'^5}(3-5\frac{\gamma_n^2 z'^2}{\gamma^2 r'^2}) \quad (4.11).$$

Here $r'^2 = (x'^2 + y'^2) + z'^2\gamma_n^2/\gamma^2$. It is seen that $\vec{H}'$ differs from zero only at the toroidal dipole position (the term with δ function is omitted), while $\vec{B}'$, $\vec{D}'$ and $\vec{E}'$ differ from zero everywhere. In this reference frame there is no relations $\vec{B}' = \mu\vec{H}'$, $\vec{D}' = \epsilon\vec{E}'$ which are valid only in the reference frame where medium is at rest. Instead, Eq. (2.22) should be used.

From the inspection of Eqs. (4.9)-(4.11) we conclude:
i) For a TS being at rest either in vacuum or medium, the EMF differs from zero only inside the TS.
ii) For a TS moving in vacuum with a constant velocity, EMF differs from zero only inside the TS. Withot any calculations this can be proved by applying the Lorentz transformation to the EMF strengths of a TS at rest. Since this transformation is linear and since

EMF strengths vanish for a TS at rest, they vanish for a moving TS as well.
iii) Eqs. (4.9)-(4.11) tell us that EMF of a TS moving in medium differs from zero both inside and outside the TS. At first glance this seems to be incorrect. In fact, let TS initially be at rest in medium. Let's pass to the Lorentz reference frame (LRF) in which TS' velocity is v. In this frame the EMF strengths vanish outside the TS. Both the TS and medium move with the velocity V relative this frame. However, Eqs. (4.9)-(4.11) are valid in the frame relative to which the medium is at rest while a TS moves with the velocity v. Therefore, these reference frames are not equivalent. There is no Lorentz transformation relating them. These important facts were established earlier in the ω representation in Ref. [10].

4.2 The velocity is perpendicular to the torus axis

Let a toroidal solenoid move in medium with the velocity perpendicular to the torus symmetry axis. For definiteness, let the TS move along the x axis. Then, in the LF

$$\rho_{Ch} = -\frac{j_0 v\gamma^2}{c^2 R_0}\frac{z(x-vt)}{\rho_1^2}\delta(R_1-R_0), \quad j_x = -j_0\frac{\gamma^2}{R_0}\frac{z(x-vt)}{\rho_1^2}\delta(R_1-R_0),$$

$$j_y = -j_0\frac{zy}{\rho_1^2}\frac{\delta(R_1-R_0}{R_0}, \quad j_z = j_0\frac{\rho_1-d}{\rho_1}\frac{\delta(R+1-R_0)}{R_0}.$$

Here

$$\rho_1 = \sqrt{(x-vt)^2\gamma^2+y^2}, \quad R_1 = \sqrt{(\rho_1-d)^2+z^2}.$$

It is easy to check that

$$j_x = -\frac{\partial M_y}{\partial z}, \quad j_y = \frac{\partial M_x}{\partial z}, \quad j_z = \frac{1}{\gamma^2}\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}, \quad \rho_{Ch} = -\frac{\beta}{c}\frac{\partial M_y}{\partial z}, \tag{4.12}$$

where

$$M_y = -j_0\gamma^2\frac{x-vt}{\rho_1^2}\Theta(R_0-R_1), \quad M_x = j_0\frac{y}{\rho_1^2}\Theta(R_0-R_1), \quad M_z = 0.$$

Let the minor radius R_0 of a torus tend to zero. Then,

$$\Theta(R_0-R_1) = \pi R_0^2\delta(\rho_1-d)\delta(z)$$

and

$$M_x = -j_0\frac{\pi R_0^2}{d}\frac{\partial}{\partial y}\Theta(d-\rho_1)\delta(z), \quad M_y = j_0\frac{\pi R_0^2}{d}\frac{\partial}{\partial x}\Theta(d-\rho_1)\delta(z).$$

Therefore,

$$\rho_{Ch} = -\frac{\beta j_0\pi R_0^2}{cd}\frac{\partial^2}{\partial x\partial z}\Theta(d-\rho_1)\delta(z), \quad j_x = -\frac{j_0\pi R_0^2}{d}\frac{\partial^2}{\partial x\partial z}\Theta(d-\rho_1)\delta(z),$$

$$j_y = -\frac{j_0\pi R_0^2}{d}\frac{\partial^2}{\partial y\partial z}\Theta(d-\rho_1)\delta(z),$$

$$j_z = \frac{j_0 \pi R_0^2}{d\gamma^2} \frac{\partial^2}{\partial x^2} \Theta(d - \rho_1)\delta(z) + \frac{j_0 \pi R_0^2}{d} \frac{\partial^2}{\partial y^2} \Theta(d - \rho_1)\delta(z).$$

Now we let the major radius d go to zero. Then,

$$\Theta(d - \rho_1) = \frac{\pi d^2}{\gamma}\delta(x - vt)\delta(y), \quad \rho_{Ch} = -\frac{\beta j_0 \pi^2 d R_0^2}{c\gamma} \frac{\partial^2}{\partial x \partial z}\delta(x - vt)\delta(y)\delta(z),$$

$$j_x = -\frac{j_0 \pi^2 d R_0^2}{\gamma} \frac{\partial^2}{\partial x \partial z}\delta(x - vt)\delta(y)\delta(z), \quad j_y = -\frac{j_0 \pi^2 d R_0^2}{\gamma} \frac{\partial^2}{\partial y \partial z}\delta(x - vt)\delta(y)\delta(z),$$

$$j_z = \frac{j_0 \pi^2 d R_0^2}{\gamma^3} \frac{\partial^2}{\partial x^2}\delta(x - vt)\delta(y)\delta(z) + \frac{j_0 \pi^2 d R_0^2}{\gamma} \frac{\partial^2}{\partial y^2}\delta(x - vt)\delta(y)\delta(z).$$

As a result, we arrive at the following electromagnetic potentials:

$$\Phi = -\frac{\beta m_t}{\gamma\epsilon} \frac{\partial^2}{\partial x \partial z}\alpha_1, \quad A_x = -\frac{m_t \mu}{\gamma} \frac{\partial^2}{\partial x \partial z}\alpha_1,$$

$$A_y = -\frac{m_t \mu}{\gamma} \frac{\partial^2}{\partial y \partial z}\alpha_1, \quad A_z = \frac{m_t \mu}{\gamma^3} \frac{\partial^2}{\partial x^2}\alpha_1 + \frac{m_t \mu}{\gamma} \frac{\partial^2}{\partial y^2}\alpha_1, \tag{4.13}$$

where α_1 is given by (3.17). In the manifest form, EMF potentials are given by

$$\Phi = -\frac{3\beta m_t}{\epsilon\gamma\gamma_n^2} \frac{(x - vt)z}{r_m^5}, \quad A_x = -\frac{3\mu m_t}{\gamma\gamma_n^2} \frac{(x - vt)z}{r_m^5},$$

$$A_y = -\frac{3\mu m_t}{\gamma\gamma_n^4} \frac{yz}{r_m^5}, \quad A_z = \frac{m_t \mu}{\gamma}\left[\frac{1}{\gamma^2 r_m^3}\left(1 - 3\frac{x^2}{r_m^2}\right) + \frac{1}{\gamma_n^2 r_m^3}\left(1 - 3\frac{y^2}{{}_m^2\gamma_n^2}\right)\right].$$

Electromagnetic field strengths are

$$E_x = \frac{\beta m_t}{\gamma\epsilon}(1 - n^2)\frac{\partial^3 \alpha_1}{\partial x^2 \partial z}, \quad E_y = \frac{\beta m_t}{\gamma\epsilon}(1 - n^2)\frac{\partial^3 \alpha_1}{\partial x \partial y \partial z},$$

$$E_z = \frac{\beta m_t}{\gamma\epsilon}\left[(n^2 - 1)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \tilde{\Delta}\right]\frac{\partial \alpha_1}{\partial x}, \quad B_x = \frac{\mu m_t}{\gamma}\left[\tilde{\Delta} + \beta^2(n^2 - 1)\frac{\partial^2}{\partial x^2}\right]\frac{\partial \alpha_1}{\partial y},$$

$$B_y = -\frac{\mu m_t}{\gamma}\left[\tilde{\Delta} + \beta^2(n^2 - 1)\frac{\partial^2}{\partial x^2}\right]\frac{\partial \alpha_1}{\partial x}, \quad B_z = 0, \quad \tilde{\Delta} = (1 - \beta_n^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{4.14}$$

It is seen that electromagnetic field strengths equal zero outside the Cherenkov cone, fall like r^{-4} at large distances inside this cone, and are infinite on the Cherenkov cone. Since for $\beta_n < 1$, $\tilde{\Delta}\alpha_1 = -4\pi\delta(x - vt)\delta(y)\delta(z)$, one may drop $\tilde{\Delta}$ operators in (4.14). This confirms the previous result that EMF goes beyond a TS moving in medium.

5 Electromagnetic field of a moving point-like electric dipole

Consider an electric dipole consisting of point electric charges:

$$\rho_d = e[\delta^3(\vec{r} + a\vec{n}) - \delta^3(\vec{r} - a\vec{n})].$$

Here $\vec{r}$ defines the dipole center-of-mass, $2a$ is the distance between charges and vector $\vec{n}$ defines the dipole orientation. Let the dipole move uniformly along the z axis. Then,

$$\rho_d = e\gamma\{\delta(x + an_x)\delta(y + an_y)\delta[(z - vt)\gamma + an_z] - \delta(x - an_x)\delta(y - an_y)\delta[(z - vt)\gamma - an_z]\},$$

$$j_z = v\rho_d.$$

Let the distance between charges tend to zero. Then,

$$\rho_d = 2ea(\vec{n}\vec{\nabla})\delta(x)\delta(y)\delta(z - vt), \quad j_z = v\rho_d.$$

Here

$$(\vec{n}\vec{\nabla}) = \vec{n}_x\nabla_x + \vec{n}_y\nabla_y + \frac{1}{\gamma}\vec{n}_z\nabla_z, \quad \nabla_i = \frac{\partial}{\partial x_i}.$$

The electromagnetic potentials are equal to

$$\Phi = \frac{2ea}{\epsilon}(\vec{n}\vec{\nabla})\alpha, \quad A_z = 2ea\mu\beta(\vec{n}\vec{\nabla})\alpha, \tag{5.1}$$

where α is the same as in (3.7). In the manifest form, the electromagnetic potentials are

$$\Phi = -\frac{2ea}{\sqrt{1 - \beta_n^2}\epsilon r^3}(\vec{n}\vec{r}), \quad A_z = -\frac{2ea\mu}{\sqrt{1 - \beta_n^2}\epsilon r^3}(\vec{n}\vec{r}),$$

$$(\vec{n}\vec{r}) = xn_x + yn_y + n_z(z - vt)\frac{\sqrt{1 - \beta^2}}{1 - \beta_n^2)}, \quad r^2 = x^2 + y^2 + \frac{(z - vt)^2}{1 - \beta_n^2} \tag{5.2}$$

for $\beta_n < 1$ and

$$\Phi = \frac{4ea(\vec{n}\vec{r})}{\epsilon r_1^3}R, \quad A_z = \frac{4ea\mu\beta(\vec{n}\vec{r})}{r_1^3}R \tag{5.3}$$

for $\beta_n > 1$. Here

$$R = \left[\frac{1}{\sqrt{\beta_n^2 - 1}}\Theta(vt - z - \rho\sqrt{\beta_n^2 - 1}) - \frac{r_1^2}{\rho}\delta(vt - z - \rho\sqrt{\beta_n^2 - 1})\right], \quad \text{and}$$

$$r_1^2 = \frac{(z - vt)^2}{\beta_n^2 - 1} - \rho^2$$

The nonvanishing electromagnetic field strengths are

$$E_x = -\frac{2ea}{\epsilon}\frac{\partial}{\partial x}(\vec{n}\vec{\nabla})\alpha, \quad E_y = -\frac{2ea}{\epsilon}\frac{\partial}{\partial y}(\vec{n}\vec{\nabla})\alpha, \quad E_z = -\frac{2ea}{\epsilon}(1 - \beta_n^2)\frac{\partial}{\partial z}(\vec{n}\vec{\nabla})\alpha,$$

$$B_x = 2ea\mu\beta\frac{\partial}{\partial y}(\vec{n}\vec{\nabla})\alpha, \quad B_y = -2ea\mu\beta\frac{\partial}{\partial x}(\vec{n}\vec{\nabla})\alpha. \tag{5.4}$$

It is seen that electromagnetic field strengths vanish outside the Cherenkov cone, inside this cone they fall like r^{-3} at large distances, and they are infinite on the Cherenkov cone.

We limit ourselves to the $\beta_n < 1$ case. The EMF is equal to

$$E_x = \frac{2ea}{\epsilon}\frac{\gamma_n}{r^3}\left[n_x - 3\frac{x}{r^2}(\vec{n}\vec{r})\right], \quad E_y = \frac{2ea}{\epsilon}\frac{\gamma_n}{r^3}\left[n_y - 3\frac{y}{r^2}(\vec{n}\vec{r})\right],$$

$$E_z = \frac{2ea}{\epsilon\gamma}\frac{\gamma_n}{r^3}[n_z - 3\gamma\frac{z-vt}{r^2}(\vec{n}\vec{r})], \quad B_x = -2ea\mu\beta\gamma_n\frac{1}{r_3}[n_y - 3\frac{y}{r^2}(\vec{n}\vec{r})],$$

$$B_y = 2ea\mu\beta\gamma_n\frac{1}{r^3}[n_x - 3\frac{x}{r^2}(\vec{n}\vec{r})], \quad B_z = 0 \qquad (5.5).$$

In the reference frame where the electric dipole is at rest

$$B'_x = \frac{2ea(1-n^2)\beta\gamma\gamma_n}{\epsilon}\frac{1}{r'^3}[n_y - 3\frac{y'}{r'^2}(\vec{n}\vec{r}\,')], \quad B'_y = -\frac{2ea(1-n^2)\beta\gamma\gamma_n}{\epsilon}\frac{1}{r'^3}[n_x - 3\frac{x'}{r'^2}(\vec{n}\vec{r}\,')],$$

$$E'_x = 2ea\frac{\gamma}{\gamma_n\epsilon}\frac{1}{r'^3}[n_x - 3\frac{x'}{r'^2}(\vec{n}\vec{r}\,')], \quad E'_y = 2ea\frac{\gamma}{\gamma_n\epsilon}\frac{1}{r'^3}[n_y - 3\frac{y'}{r'^2}(\vec{n}\vec{r}\,')],$$

$$E'_z = 2ea\frac{\gamma_n}{\gamma\epsilon}\frac{1}{r'^3}[n_z - 3\frac{z'}{r'^2}(\vec{n}\vec{r}\,')], \quad \vec{H}' = 0, \quad D'_x = \frac{2ea\gamma_n}{\gamma\epsilon}\frac{1}{r'^3}[n_x - 3\frac{x'}{r'^2}(\vec{n}\vec{r}\,')],$$

$$D'_y = \frac{2ea\gamma_n}{\gamma\epsilon}\frac{1}{r'^3}[n_y - 3\frac{y'}{r'^2}(\vec{n}\vec{r}\,')], \quad D'_z = 2ea\frac{\gamma_n}{\gamma r'^3}[n_z - 3\frac{z'}{r'^3}(\vec{n}\vec{r}\,')]. \qquad (5.6)$$

We see that $\vec{E}$ resembles the field of electric dipole, while $\vec{H}$, having only two Cartesian components, cannot be interpreted as a field of magnetic dipole.
For the vector $\vec{n}$ oriented along the motion axis, one gets

$$E_x = -6ea\gamma_n^3\frac{x(z-vt)}{\gamma\epsilon r^5}, \quad E_y = -6ea\gamma_n^3\frac{y(z-vt)}{\gamma\epsilon r^5}, \quad E_z = \frac{2ea}{\epsilon}[\frac{\gamma_n}{r^3}(1 - 3\frac{\gamma_n^2(z-vt)^2}{r^2})],$$

$$B_x = 6\mu\beta ea\gamma_n^3\frac{y(z-vt)}{\gamma r^5}, \quad B_y = -6\mu\beta\ ea\gamma_n^3\frac{x(z-vt)}{\gamma r^5}, \qquad (5.7)$$

where $r^2 = x^2 + y^2 + (z-vt)^2\gamma_n^2$ is the same as in (3.7).
For the vector $\vec{n}$ perpendicular to the motion axis (say, $\vec{n}$ is in the x direction), the field strengths are

$$E_x = \frac{2ea}{\epsilon}\frac{\gamma_n}{r^3}[1 - 3\frac{x^2}{r^2}], \quad E_y = -6ea\gamma_n\frac{xy}{\epsilon r^5}, \quad E_z = -6ea\gamma_n\frac{x(z-vt)}{\epsilon r^5},$$

$$B_x = 6ea\mu\gamma_n\beta\frac{xy}{r^5}, \quad B_y = 2ea\mu\beta\frac{\gamma_n}{r^3}[1 - 3\frac{x^2}{r^2}]. \qquad (5.8)$$

6 Electromagnetic field of induced dipole moments

Now we apply the formalism developed by Frank to evaluate the EMF of moving magnetic and electric dipoles.

6.1 Electromagnetic field of moving magnetic dipole

According to Refs. [1-3, 5-11], the moving magnetic dipole $\vec{m}'$ creates the following magnetic $\vec{m}$ and electric $\vec{p}$ dipole moments in the LF:

$$\vec{m} = \vec{m}' - (1 - \sqrt{1-\beta^2})\vec{v}(\vec{v}\vec{m}')/v^2, \quad \vec{p} = (\vec{\beta} \times \vec{m}'), \quad \vec{\beta} = \vec{v}/c. \qquad (6.1)$$

For the $\vec{m}'$ directed along the motion axis, (6.1) passes into

$$m_x = m_y = 0, \quad m_z \equiv m = m'/\gamma, \quad \vec{p} = 0. \qquad (6.2)$$

The EMF of induced dipoles (6.2) being at rest in the instantaneous position of the moving magnetic dipole (this is essentially Frank's prescription) is given by

$$\vec{E}_d = 0, \quad B_x^d = 3m\frac{\gamma x(z-vt)}{r^5}, \quad B_y^d = 3m\frac{\gamma y(z-vt)}{r^5}, \quad B_z^d = -m(\frac{1}{r^3} - 3\frac{\gamma^2(z-vt)^2}{r^5}). \qquad (6.3)$$

Here $r = [x^2+y^2+\gamma^2(z-vt)^2]^{1/2}$. By comparing (6.3) with (3.11), we conclude that the magnetic field of a moving point-like current loop resembles (but not coincides with) that of a magnetic dipole. The nontrivial γ_n dependence in (3.11) tells us that the magnetic field of a moving magnetic dipole cannot be obtained by the simple Frank's prescription (6.1).
Further, Frank's receipt (6.2) gives a zero electric field, while the exact electric field (3.11) differs from zero. Another way to see this is to write out the electric field created by the electric dipole $\vec{p}$ which is at rest in the instantaneous position of a moving magnetic dipole:

$$(E_d)_x = -\frac{1}{r^3}p_x + 3x\frac{xp_x + yp_y + \gamma(z-vt)p_z}{r^5},$$

$$(E_d)_y = -\frac{1}{r^3}p_y + 3y\frac{xp_x + yp_y + \gamma(z-vt)p_z}{r^5},$$

$$(E_d)_z = -\frac{1}{r^3}p_z + 3\gamma z\frac{xp_x + yp_y + \gamma(z-vt)p_z}{r^5}, \qquad (6.4)$$

where $r^2 = x^2 + y^2 + (z-vt)^2\gamma^2$. The exact electric field of a moving point-like current loop has only the ϕ component (see (3.11)). It is easy to check that it is impossible to choose p_x, p_y, p_z in (6.4) in such a way as to vanish simultaneously E_ρ and E_z. This means that the electric field (3.11) produced by a moving magnetic dipole cannot be associated with the field of the induced electric dipole.
For the $\vec{m}'$ perpendicular to the motion axis (for definiteness, let the motion and symmetry axes be along the x and z axes, resp.) Then, Eq.(6.1) gives

$$m_x = m_y = 0, m_z = m = m', \quad p_y = -\beta m.$$

The EMF generated by this dipole moment is

$$(E_d)_x = -3\beta m\gamma\frac{y(x-vt)}{r^5}, \quad (E_d)_y = \frac{\beta m}{r^3} - 3\beta m\frac{y^2}{r^5}, \quad (E_d)_z = -3\beta m\frac{yz}{r^5},$$

$$(B_d)_x = 3\gamma m\frac{z(x-vt)}{r^5}, \quad (B_d)_y = 3\gamma m\frac{yz}{r^5}, \quad (B_d)_z = -\frac{m}{r^3} + 3m\frac{z^2}{r^5}. \qquad (6.5)$$

These expressions slightly resemble the exact ones (3.20), but not reduce to them (again, due to the nontrivial γ_n dependence in (3.20)).
The situation remains essentially the same if instead of $\vec{p}$ given by (6.1), the modified Frank' formula ([3])

$$\vec{p} = n^2(\beta \times m'), \quad n^2 = \epsilon\mu \qquad (6.6)$$

is used.

6.2 Electromagnetic field of moving electric dipole

According to Frank, a moving electric dipole $\vec{p}'$ creates the following magnetic $\vec{m}$ and electric $\vec{p}$ dipole moments in the LF:

$$\vec{p} = \vec{p}' - (1 - \sqrt{1-\beta^2})\vec{v}(\vec{v}\vec{p}')/v^2, \quad \vec{m} = -\vec{\beta} \times \vec{p}'. \tag{6.7}$$

For $\vec{p}'$ aligned along the motion axis z this reduces to

$$p_x = p_y = 0, \quad p_z = p'/\gamma, \quad \vec{m} = 0. \tag{6.8}$$

The EMF of induced dipoles (6.8) being at rest in the instant position of the moving electric dipole is given by

$$E_x = p\frac{x\gamma(z-vt)}{r^5}, \quad E_y = p\frac{y\gamma(z-vt)}{r^5}, \quad E_z = -\frac{p}{r^3} + 3p\frac{\gamma^2(z-vt)^2}{r^5}, \quad \vec{B} = 0. \tag{6.9}$$

By comparing this with (5.7), we conclude that the electric field (6.9) of an induced electric dipole resembles (but not reduces to) the exact electric field (5.7) of a moving electric dipole. On the other hand, the magnetic field vanishes for the induced magnetic moment (6.8) contrary to the exact magnetic field (5.7) of the moving electric dipole. The latter cannot be attributed to the magnetic dipole.
For the electric dipole oriented perpendicularly (say, in the x direction) to the motion direction z, one obtains from (6.7) for the non-vanishing components of induced dipole moments

$$p_x \equiv p = p', \quad m_y = -\beta p. \tag{6.10}$$

The corresponding EMF is

$$E_x = -\frac{p}{r^3} + 3p\frac{x^2}{r^2}, \quad E_y = 3p\frac{xy}{r^5}, \quad E_z = 3p\frac{x\gamma(z-vt)}{r^5},$$

$$B_x = -3\beta p\frac{xy}{r^5}, \quad B_y = \frac{\beta p}{r^3} - 3\beta p\frac{y^2}{r^5}, \quad B_z = -3\beta p\frac{y\gamma(z-vt)}{r^5}. \tag{6.11}$$

By comparing this with (5.8) we conclude that the electric field of an induced dipole moment resembles the exact electric field (5.8) of a moving electric dipole. On the other hand, there are three components of the magnetic field of the induced moment (6.10) and only two exact nonvanishing components in (5.8). Therefore, the exact magnetic field (5.8) of a moving electric dipole cannot be attributed to the induced magnetic dipole (6.10).

7 Discussion and Conclusion

The exact results presented in sects. 3-5 confirm the nontrivial Frank's thought that

> Radiations of magnetic and electric dipoles should have the same angular distributions as they are due to the interference of waves which do not depend on the particle nature.

(Our translation from Ref. [3], p.191). In fact, from comparison of Eqs. (3.11) with (5.7) we conclude that:
i) For $\beta_n < 1$ the magnetic (electric) field of a moving magnetic dipole is of the same functional form as the electric (magnetic) field of a moving electric dipole.
ii) For $\beta_n > 1$, the same Cherenkov singularity is produced by moving electric and magnetic dipole moments: the EMFs produced by them vanish outside the Cherenkov cone and coincide with each other (with the interchange $\vec{E} \Longleftrightarrow \vec{H}$, similarly to the $\beta_n < 1$ case) inside it.
However, the arising electromagnetic fields cannot be obtained with the use of simplified Frank's prescriptions (6.1),(6.6) and (6.7).

It should be mentioned that electromagnetic fields originating from an arbitrary motion of magnetic and electric dipoles were obtained earlier in a number of papers ([30-34]). The nice review of these attempts may be found in Ref. [35]. Electromagnetic fields obtained there were expressed in terms of the so-called retarded times. However, to express the retarded time through the measurable laboratory time is not trivial task at all even for the simplest motion laws. In this paper we succeeded to do this for the point-like magnetic and electric dipoles and toroidal dipole noving uniformly in a nondispersive medium.

We briefly summarize the main results obtained:
1. The exact electromagnetic fields of point-like electric and magnetic dipoles moving in a non-dispersive medium are obtained. In accordance with Frank's prediction, they produce the same electromagnetic fields (with the interchange $\vec{E} \Longleftrightarrow \vec{H}$).
However, the formalism of induced electric and magnetic moments suggested by Frank, does not describe properly the exact electromagnetic fields mentioned above.
2. The exact electromagnetic field of a point-like toroidal solenoid moving in a nondispersive medium is obtained. For the elementary TS' velocity smaller than the light velocity in medium, the electric field of moving TS is similar to the field of an electric quadrupole.

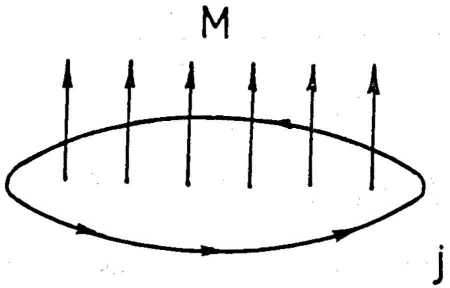

Fig.1: The circular current $\vec{j}$ is equivalent to the magnetization perpendicular to the the current plane.

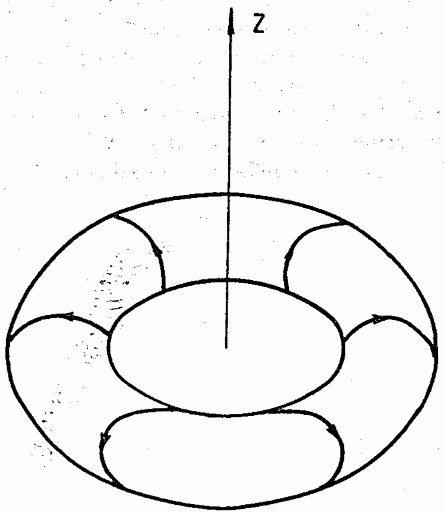

Fig.2: The poloidal current flowing on the torus surface.

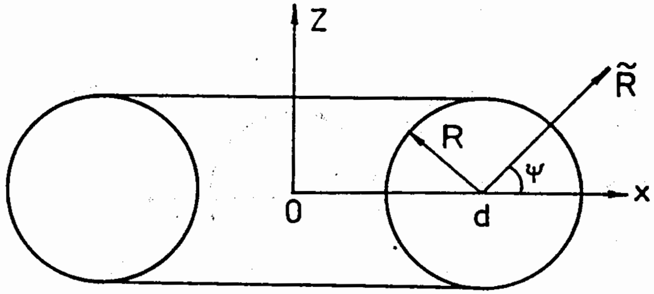

Fig.3: The coordinates $\tilde{R}, \psi$ parametrizing the torus.

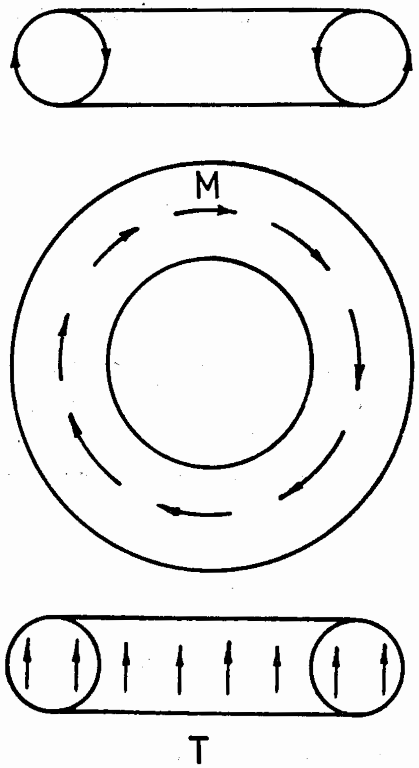

Fig.4: The poloidal current $\vec{j}$ flowing on the torus surface is equivalent to the magnetization $\vec{M}$ confined to the interior of the torus and to the toroidization $\vec{T}$ directed along the torus symmetry axis.

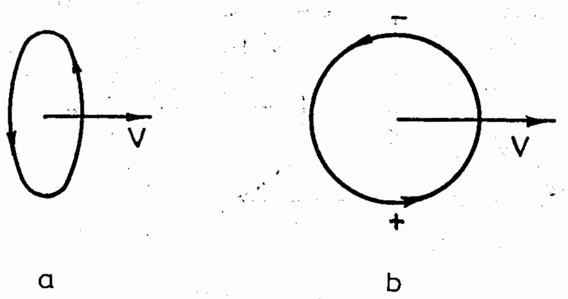

Fig.5: a) There is no induced charge density when the symmetry axis of the current loop is along the velocity;
b) The induced charge density arises when the symmetry axis of the current loop is perpendicular to the velocity.

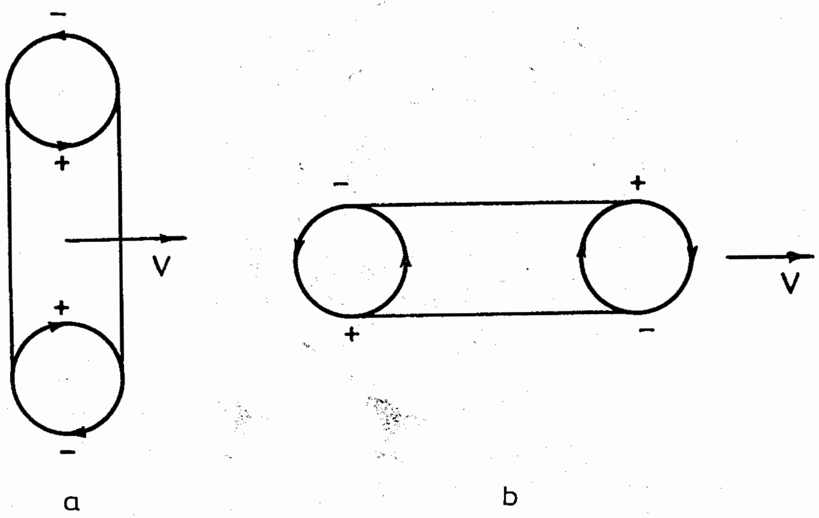

Fig.6: The induced charge densities for the cases when the symmetry axes of a moving toroidal solenoid are along the velocity (a) or perpendicular to it (b).

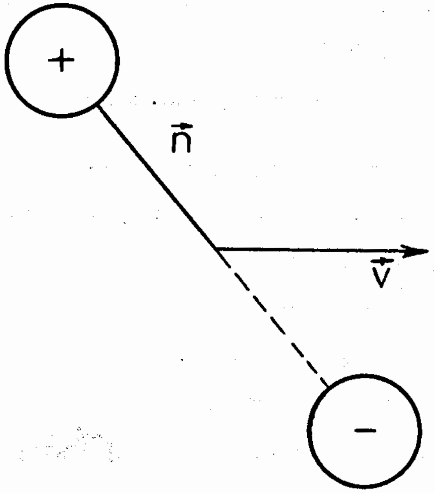

Fig.7: A moving electric dipole with arbitrary orientation relative to its velocity.

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