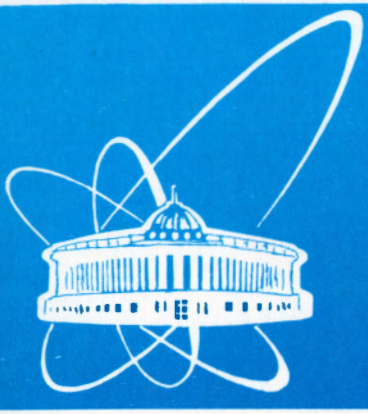

ОБЪЕДИНЕННЫЙ
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A MULTILOOP IMPROVEMENT OF NON-SINGLET QCD EVOLUTION EQUATIONS

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Многопетлевое улучшение несинглетных эволюционных уравнений КХД

Развит подход для вычисления вкладов «во всех петлях», происходящих от ренормалонных цепочек, в несинглетные ядра эволюции. Получены замкнутые выражения для сумм таких вкладов в ядра $P(z)$ эволюционного уравнения ДГЛАП и в ядро $V(x,y)$ для эволюционного уравнения ЕР-БЛ. Эти вклады доминируют при больших значениях b_0, первого коэффициента β-функции. Вычисления проведены в ковариантной ξ-калибровке в MS-подобной схеме. Найдено обобщение приближения «наивной неабелизации» путем специального выбора калибровки $\xi = -3$. Для улучшенного таким образом уравнения ЕР-БЛ получены решения, близкие по форме к однопетлевым. Обсуждаются pro и contra подхода, а также его следствия в описании жестких процессов.

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A Multiloop Improvement of Non-Singlet QCD Evolution Equations

An approach is elaborated for calculation of «all loop» contributions to the non-singlet evolution kernels from the diagrams with renormalon chain insertions. Closed expressions are obtained for sums of contributions to kernels $P(z)$ for the DGLAP equation and $V(x,y)$ for the «nonforward» ER-BL equation from these diagrams that dominate for a large value of b_0, the first β-function coefficient. Calculations are performed in the covariant ξ-gauge in a MS-like scheme. It is established that a special choice of the gauge parameter $\xi = -3$ generalizes the standard «naive nonabelianization» approximation. The solutions are obtained to the ER-BL evolution equation (taken at the «all loop» improved kernel), which are in form similar to one-loop solutions. A consequence for QCD descriptions of hard processes and the benefits and incompleteness of the approach are briefly discussed.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1 Introduction

Evolution kernels are main ingredients of the well-known evolution equations for the parton distribution of DIS processes [1] and for parton wave functions [2] in hard exclusive reactions. These equations describe the dependence of parton distribution functions and parton wave functions on the renormalization parameter μ^2. The calculations performed beyond the one-loop approximation for the forward DGLAP evolution kernel $P(z)$ [3, 4], and what is more, for the nonforward Efremov-Radyushkin–Brodsky-Lepage (ER-BL) kernel $V(x,y)$ [5, 6] were challenged and complicated technical tasks. 15 years later, the 3-loop results for these kernels are not known yet, except for the first few elements of anomalous dimension in DIS, obtained numerically in [7]. In this situation, it seems useful to try other ways to gain knowledge about high-order corrections to these kernels and to the solutions to the corresponding equations.

Here I discuss the results of the diagrammatic analysis and multiloop calculations of the DGLAP kernel $P(z)$ and ER-BL kernel $V(x,y)$ in a certain class of the "all-order" approximation of perturbative QCD (pQCD). The corresponding diagrams include the chains of one-loop self-energy parts (renormalon chains) into the one-loop diagrams (see Fig. 1). The regular method of calculation and resummation of the indicated classes of diagrams for these kernels based upon their simple forest structure has been suggested in [8]. There was established that the resulting series possesses a nonzero convergent radius, therefore the infrared renormalons are absent in the kernels. The results of that summation for both the kinds of kernels (DGLAP and ER-BL) obtained earlier in the framework of a scalar model in six dimensions with the Lagrangian $L_{int} = g\sum_i^{N_f}(\psi_i^*\psi_i\varphi)_{(6)}$ with N_f of the scalar "quark" flavours (ψ_i) and "gluon" (φ) are analyzed here for non-singlet QCD kernels. For the readers convenience some important results of the paper [8] would be recalled.

The insertion of the chain into the "gluon" line ("chain-1" in [8]) of the diagram in Fig.1 a,b and resummation over all bubbles transforms the one-loop kernel $aP_0(z) = a\bar{z} \equiv a(1-z)$ into the "improved" kernel $P^{(1)}(z;A)$

$$aP_0(z) = a\bar{z} \stackrel{chain^{-1}}{\longrightarrow} P^{(1)}(z;A) = a\bar{z}\left[(z)^{-A}(1-A)\frac{\gamma_\varphi(0)}{\gamma_\varphi(A)}\right];\ A = aN_f\gamma_\varphi(0),\ a = \frac{g^2}{(4\pi)^3}. \quad (1)$$

Here, $\gamma_{\psi(\varphi)}(\varepsilon)$ are one-loop coefficients of the anomalous dimensions of quark (gluon at $N_f = 1$) fields in D-dimension ($D = 6 - 2\varepsilon$) discussed in [8]; for the scalar model $\gamma_\psi(\varepsilon) = \gamma_\varphi(\varepsilon) = B(2-\varepsilon, 2-\varepsilon)C(\varepsilon)$, and $C(\varepsilon)$ is a scheme-dependent factor ($C(0) = 1$) corresponding to a certain choice of an $\overline{\rm MS}$ –like scheme. The argument A of the function $\gamma_\varphi(A)$ in (1) is a standard anomalous dimension (AD) of a "gluon" field. On the other hand, the result (1) corresponds to resummation of a class of series like $a\,(A\ln[1/z])^n$-series (see Table 1 in [8]) into the kernel which dominate at large N_f.

The resummation of this "chain-1" subseries into an analytic function in A should not be taken by surprise. Really, the considered problem can be connected with the calculation of large N_f asymptotics of ADs' in order of $1/N_f$. An approach was suggested by A. Vasil'ev and colleagues at the beginning of the 80's [10] to calculate the renormalization-group functions in this limit, they used the conformal properties of the theory at the critical point $g = g_c$ corresponding to the non-trivial zero g_c of the D-dimensional β-function. This approach was extended by J. Gracey for the calculation of

ADs' of composite operators of DIS in QCD in any order n of PT, [11]. I used another approach which is close to [12]; contrary to the large N_f asymptotic method, it does not appeal to the value of parameters $N_f T_R$, $C_A/2$ or C_F, associated with different kinds of loops in QCD. To illustrate this feature, let us consider the insertions of chains of one-loop self-energy parts into the "quark" line of diagram Fig.1a ("chain-2" in [8]). Contributions of these diagrams calculated in the framework of the above scalar model do not contain the parameter N_f, nevertheless, they can be summarized into the kernel $P^{(2)}(z;B)$ [8]

$$aP_0(z) = a\bar{z} \stackrel{chain^{-2}}{\longrightarrow} P^{(2)}(z;B) = a\bar{z}\left(1 + B\frac{d}{dB}\right)\left[(\bar{z})^{-B}\frac{\gamma_\psi(0)}{\gamma_\psi(B)}\right]; B = a\gamma_\psi(0), \bar{z} \equiv 1 - z, \quad (2)$$

according to the same approach. This corresponds to summation of various series like $a\,(B\ln[1/\bar{z}])^n$-series into the kernel. The operator $(1 + B\ d/dB)$ appearing in front of formula (2) expresses an inherent combinatoric factor to these diagrams. Following that line, the "improved" QCD kernel $P^{(1)}(z;A)$ was obtained in [13] for the general case of a mixed chain (quark and gluon bubble chain) in ξ– gauge.

Here, we present the QCD results similar to Eq.(1), in the covariant ξ– gauge for the DGLAP non-singlet kernel $P(z;A)$. Analytic properties of the function $P(z;A)$ in variable A are analyzed. The assumption of the "Naive Nonabelianization" (NNA) approximation [14] for the kernel calculation [15] is discussed and its generalization based on $\xi = -3$ gauge is suggested. The numerical importance of the resummation in this case is demonstrated. The ER-BL evolution kernel $V(x,y)$ is obtained in the same multiloop approximation as the DGLAP kernel, by using exact relations between the P and V kernels [6, 8] for a class of "triangular diagrams". The considered class of diagrams represents the leading b_0-contributions to both the kinds of kernels. Partial solutions for the ER-BL equation, $\Phi_n(x,A)$, are derived. The multiloop "improved" kernels $P(z;A)$, $V(x,y;A)$ and solutions $\Phi_n(x,A)$ are compared with the exact results in 3(2)-loop approximation.

2 Triangular diagrams for the DGLAP evolution kernel

Here, the results of the bubble chain resummation for QCD diagrams in Fig.1 for the DGLAP kernel are discussed. These classes of diagrams generate, in particularly, contributions $\sim a_s\,(A\ln[1/z])^n$ in any order n of pQCD. Based on the resummation method of Ref. [8] in the QCD version, one can derive the kernels $P^{(1a,b,c)}$ (corresponding to the diagrams in Fig.1 a,b,c) in the covariant ξ–gauge[1], whose explicit expressions are presented in [13]. They contribute to the total kernel $P^{(1)}(z;A,\xi)$ that has the expected "plus form"

$$P^{(1)}(z;A,\xi) = a_s C_F 2\cdot\left[\bar{z}z^{-A}(1-A)^2 + \frac{2z^{1-A}}{1-z}\right]_+ \frac{\gamma_g(0,\xi)}{\gamma_g(A,\xi)}, \quad (3)$$

$$a_s P_0(z) = a_s C_F 2\cdot\left[\bar{z} + \frac{2z}{1-z}\right]_+, \quad (4)$$

[1]The gauge parameter ξ is defined via the gluon propagator in the lowest order $iD_{\mu\nu}(k^2) = \frac{-i\delta^{ab}}{k^2+i\epsilon}\left(g_{\mu\nu} + (\xi-1)\frac{k_\mu k_\mu}{k^2}\right)$

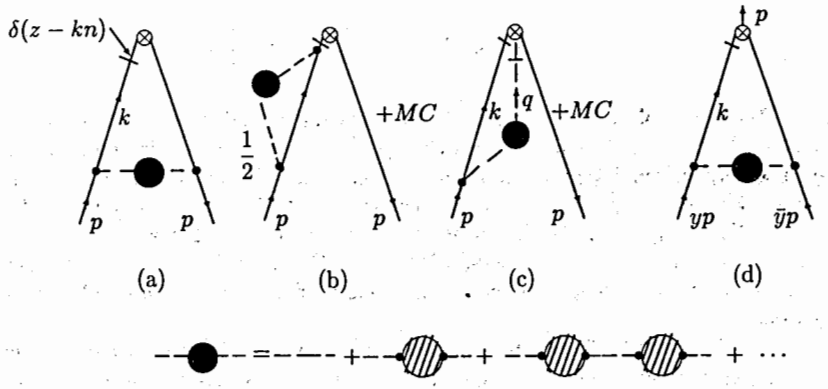

Figure 1: The diagrams in figs. 1a – 1c are "triangular" diagrams for the QCD DGLAP kernel; dashed lines for gluons, solid lines for quarks; black circles denote the sum of all kinds of the one-loop insertions (dashed circles), both quark and gluon (ghost) or mixed chains; the slash on the line denotes the delta function $\delta(z-kn)$ (k is the momentum on the line) which is traced to the representation of the composite operator $\otimes$, see [6] for details; MC denotes the mirror–conjugate diagram; 1d is an example of a diagram for the nonforward ER-BL kernel.

where $a_s = \frac{\alpha_s}{4\pi}$, $C_F = (N_c^2-1)/2N_c$, $C_A = N_c$, $T_R = \frac{1}{2}$ are the Casimirs of SU(N_c) group, and $A = -a_s\gamma_g(0,\xi)$. For comparison with (3), the one-loop result $a_s P_0(z)$ is written also down, the latter can be obtained as the limit $P^{(1)}(z;A\to 0,\xi)$. The $\gamma_g(0,\xi)$ is the one-loop coefficient of the standard AD of the gluon field, while the function $\gamma_g(\varepsilon,\xi)$ is the coefficient of the anomalous dimension in D-dimension, here $D = 4-2\varepsilon$. In other words, it is the coefficient $Z_1(\varepsilon)$ of a simple pole in the expansion of the gluon field renormalization constant Z that includes both its finite part and all the powers of the ε-expansion. So, one can conclude that the "all-order" result in (3) is completely determined by the single quark (or/and gluon) bubble diagram. The function $\gamma_g(\varepsilon)$ thus defined is an analytic function in the variable ε by construction, see [8]. Equation (3) is valid for any kind of insertions, *i.e.*, $\gamma_g = \gamma_g^{(q)}$ for the quark loop, $\gamma_g = \gamma_g^{(g)}$ for the gluon (ghost) loop, or for their sum

$$\gamma_g(A,\xi) = \gamma_g^{(q)}(A) + \gamma_g^{(g)}(A,\xi); \quad (5)$$

when both the kinds of insertions are taken into account. Note that in (3) the $\delta(1-z)$ - terms appearing in the partial contributions (see [13]) are exactly accumulated in the form of the $[\ldots]_+$ prescription, and the ξ - terms successfully cancel. This is due to the evident current conservation for the case of quark bubble insertions; including the gluon bubbles into consideration merely modifies the effective AD $\gamma_g^{(q)}(A) \to \gamma_g(A,\xi)$, conserving the structure of result (3), see [8, 13]. Substituting the well-known expressions of $\gamma_g(\varepsilon)$ from the quark or gluon (ghost) loops (see, *e.g.*, [17])

$$\gamma_g^{(q)}(\varepsilon) = -8N_f T_R B(D/2, D/2)C(\varepsilon), \quad (6)$$

$$\gamma_g^{(g)}(\varepsilon,\xi) = \frac{C_A}{2}B(D/2-1, D/2-1)\left(\left(\frac{3D-2}{D-1}\right)+\right.$$

$$(1-\xi)(D-3)+\left(\frac{1-\xi}{2}\right)^2\varepsilon\Bigg)C(\varepsilon), \qquad (7)$$

into formula (3) one can obtain $P^{(1)}(z;A,\xi)$ for both the quark and gluon loop insertions simultaneously. Here, the coefficient $C(\varepsilon)=\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)$ implies a certain choice of the $\overline{\rm MS}$ scheme where every loop integral is multiplied by the scheme factor $\Gamma(D/2-1)(\mu^2/4\pi)^\varepsilon$ ($\overline{\rm MS}_1$ scheme). The renormalization scheme dependence of $P^{(1)}(z;A)$ is accumulated by the factor $C(\varepsilon)$. For another popular definition of a minimal scheme, when a scheme factor is chosen as $\exp(c\cdot\varepsilon)$, $c=-\gamma_E+\ldots$ instead of $\Gamma(D/2-1)$ ($\overline{\rm MS}_2$ scheme), the coefficient $C(\varepsilon)$ does not contain any scheme "traces" in final expressions for the renormalization-group functions.

Of course, the final result (3) will be gauge-dependent in virtue of the evident gauge dependence of the gluon loop contribution $\gamma_g^{(g)}(\varepsilon,\xi)$. A new expansion parameter $A(\xi)$ in this case,

$$A(\xi)=-a_s\gamma_g(0,\xi)=-a_s\left(\gamma_g^{(g)}(0,\xi)+\gamma_g^{(q)}(0)\right)=-a_s\left[\left(\frac{5}{3}+\frac{(1-\xi)}{2}\right)C_A-\frac{4}{3}N_fT_R\right], \qquad (8)$$

is the contribution to the one-loop renormalization of the gluon field. The positions of zeros of the function $\gamma_g(A,\xi)$ in A, which represent the poles of $P(z;A,\xi)$, also depend on ξ. The kernel $P^{(1)}(z;A,\xi)$ becomes gauge-invariant if we restrict themselfves only to the quark-loop insertions, *i.e.*, $\gamma_g\to\gamma_g^{(q)}$; $A\to A^{(q)}=-a_s\gamma_g^{(q)}(0)=a_s\frac{4}{3}T_RN_f$, and $P^{(1)}(z;A,\xi)$ is reduced to $P_q^{(1)}(z;A^{(q)})$, as it is presented in [8]. It is instructive to outline analytic properties of $P_q^{(1)}(z;A^{(q)})$ in $A^{(q)}$ based on Eq. (3) and on the explicit form for $\gamma_g^{(q)}$ in (6): (i) the range of convergence of the PT series corresponds to the left zero of the function $\gamma_g^{(q)}(A)$ and is equal to $A_0=5/2$, which corresponds to $\alpha_s^0=15\pi/N_f$, so, this range looks very broad [2], $\alpha_s<5\pi$ at $N_f=3$; (ii) the resummation into $P_q^{(1)}(z;A)$ is substantial, two zeros of the function $P_q^{(1)}(z;A)$ in A appear within the range of convergence (in $\overline{\rm MS}_1$ scheme). Of couse, the moments of this reduced kernel $P_q^{(1)}(z;A^{(q)})$ agree with the generating function for the anomalous dimensions, obtained earlier in [11].

3 A modified NNA version for kernel calculations

The expansion of $P_q^{(1)}(z;A)$ in A provides the leading $a_s\,(a_sN_f\ln[1/z])^n$ dependence of the kernels with a large number N_f in any order n of PT [8]. But these contributions do not numerically dominate for real numbers of flavours $N_f=4,\ 5,\ 6$. That can be verified by comparing the total numerical results for 2- and 3-loop ADs' of composite operators (ADCO) presented in [7] with their N_f-leading terms, see Table 1. There the contributions to coefficients of different Casimirs in the ADCO are presented. To obtain a satisfactory agreement at least with the two-loop results, one should take into account the contribution from **subleading** N_f-terms. As a first step, let us consider the contribution

[2]Here we consider the evolution kernel $P(z,A)$ itself. We do not consider that the factorization scale μ^2 of hard processes would be chosen large enough, $\mu^2\geq m_\rho^2$, where the ρ–meson mass m_ρ represents the characteristic hadronic scale. For this reason, the used coupling $\alpha_s(\mu^2)$ could not be too large.

from the completed renormalization of the gluon line, which should generate **a part of subleading** terms. Below, we examine an exceptional choice of the gauge parameter $\xi=-3$. For this gauge the coefficient of one-loop gluon AD $\gamma_g(0,-3)$ coincides with b_0, the one-loop coefficient of the β-function [3] and $A(-3)=-a_sb_0$. Therefore this gauge can be used for reformulating the so-called [14] NNA proposition to kernel calculations. Note, just this value of ξ has been used in [19] to estimate the total gluon contribution only from the gluon bubble in order a_s^2 to the process of e^+e^- annihilation. Other interesting applications of this gauge to approximate the exact loop results have been considered in [20, 21]..

To obtain the NNA result in a usual way, one should substitute the coefficient b_0 for $\gamma_g^{(q)}(0)$ in the expression for $A^{(q)}$ by hand (see, *e.g.*, [15]). Note, the use of such an NNA procedure does not improve $P_q^{(1)}(z;A)$ and leads to poor results even for the two-loop level, *i.e.*, for the $a_s^2\,P_1(z)$ term of the expansion, see [22]. The NNA trick expresses common hope that the main logarithmic contribution can follow from the renormalization of the coupling constant g_s. This renormalization appears as a sum of contributions from all the sources of renormalization of g_s at the vertexes of triangular diagrams. Let us consider the gluonic, vertex, and quark line renormalizations successively in the case of the $\xi=-3$ gauge. The one-loop gluon renormalization in this gauge imitates the contributions from all other sources and the coefficient b_0 **appears naturally** via of $\gamma_g(0,-3)$. At the same time, in the one-loop vertex renormalization constant Z_{1F},

$$1-Z_{1F}\sim a_s\left[C_F\xi+\frac{C_A}{4}(3+\xi)\right],$$

the nonabelian part vanishes at $\xi=-3$, while the corresponding abelian part, $a_sC_F\xi$, is compensated by the renormalization of the quark line of a triangular diagram, $-a_sC_F\xi$, due to the Ward identity[4]. So, due to the cancellations, only the gluon contribution survives in g_s renormalization and provides the expected b_0-term, $a_sb_0\ln[z]$. These properties of cancellation can be illustrated by the well-known diagram by diagram results for two-loop $P_1(z)$ presented in Feynman gauge in [4, 6] (for $V_1(x,y)$ in [6, 9]). Indeed, the terms, connected with the quark field/vertex renormalization are proportional to $\ln[1-z]$ in these diagrams and really cancel in the gauge invariant sum of all contributions. In contrast to that, the $\ln[z]$-terms collect the coefficient a_sb_0. Though we should not take into account the self-energy chain ("chain-2" in the Intr.) and "rainbow" graph insertions into the quark line unless the vertex of the triangular diagram, dressed in the same manner, is included into consideration, we see that their contributions should be cancelled in the first *log*-parts for the discussed gauge. For these reasons we can guess the gauge $\xi=-3$ "exceptional" for the one-loop chain dressing.

To analyze the resulting effect of "all-loop" resummation for the case $\xi=-3$ in (3), let us choose the common factor $\gamma_g(0,-3)/\gamma_g(A(-3),-3)$ in formula (9) (below the notation $a=a_sb_0=-A(-3)$ is introduced),

$$P^{(1)}(z;-a,-3)=a_sC_F2\cdot\left[\bar{z}z^a(1+a)^2+\frac{2z^{1+a}}{1-z}\right]_+\frac{\gamma_g(0,-3)}{\gamma_g(-a,-3)}, \qquad (9)$$

[3]Here, for the $\beta(a_s)$-function we adopt $\beta(a_s)=-b_0a_s^2+\ldots$, $b_0=\frac{11}{3}C_A-\frac{4}{3}N_fT_R$

[4]This reason was noted also in [21]

$$\frac{\gamma_g(0,-3)}{\gamma_g(-a,-3)} = \frac{\Gamma(2+2a)(3+2a)}{(\Gamma(1+a))^2\, C(-a)} \frac{b_0}{(4a^2+a(3b_0+2)+3b_0)}, \qquad (10)$$

for a crude measure of the modification of the kernel in comparison with the one-loop result $a_s P_0(z)$. The factor (as well as the whole kernel $P(z;-a,-3)$) has no singularity in a for $a > 0$. Considering the curve of this factor in the argument a in Fig.2, one can conclude:

(i) the factor $\gamma_g(0,-3)/\gamma_g(-a,-3)$ significantly grows with argument a in the range of the standard PT validity. Really, this factor reaches 1.42 for the $\overline{\rm MS}_1$ scheme (1.26 for the $\overline{\rm MS}_2$ scheme), if we take the naive boundary of validity of the standard PT, $a_0 = 0.5$, $\alpha_s^0 = \frac{2\pi}{b_0} \approx 0.7$ that corresponds to the value of α_s on the hadronic scale; thus, the resummation is numerically important in this range, see Fig.2.

(ii) scheme dependence looks not too strong for acceptable values of parameter a.

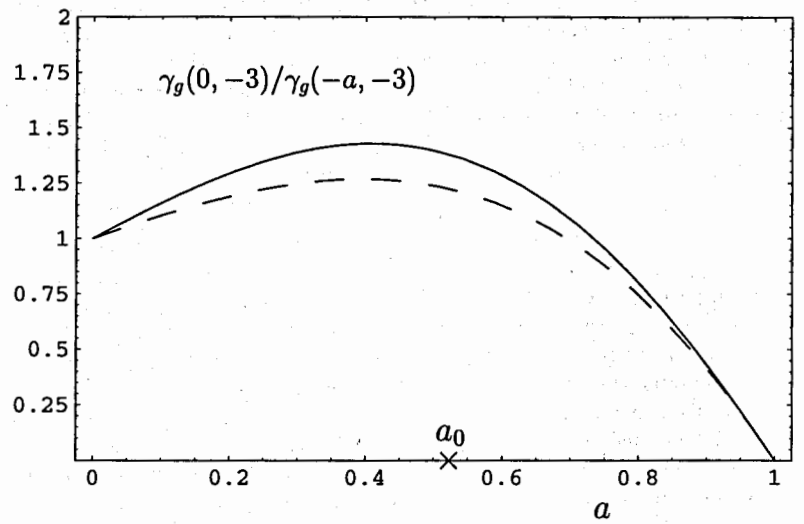

Figure 2: The curves of the factor $\gamma_g(0,-3)/\gamma_g(-a,-3)$ in a; the solid line corresponds to $\overline{\rm MS}_1$ scheme; the dashed line corresponds to $\overline{\rm MS}_2$ scheme

Note that Eqs.(3, 9) could not provide the valid asymptotic behavior of the kernels for $z \to 0$. A similar z-behavior is determined by the double-logarithmic corrections which are most singular at zero, like $a_s \left(a_s \ln^2[z]\right)^n$ [18]. These contributions appear due to renormalization of the composite operator in the diagrams by ladder graphs, *etc.* , rather than by the triangular ones. But, Eq.(9) can provide a main z-behavior for not too small z due to simple-logarithmic corrections. To obtain the low boundary of this z-region, let us compare effects from simple and double logarithmic contributions taking into account the main singular terms up to 3 loops;

$$\begin{aligned} |_{z\downarrow 0} P(z) = \quad & a_s 2C_F \\ + \quad & a_s^2 2C_F \left[\ldots + b_0 \ln[z] + (2C_A - 3C_F)\ln^2[z]\right] \\ + \quad & a_s^3 2C_F \left[\ldots + \left(b_0 \frac{11b_0 - 2}{3} \ln[z] + \frac{b_0^2}{2}\ln^2[z]\right) + \frac{C_F^2 - 6[C_F - C_A/2]^2}{3}\ln^4[z]\right] \\ + \quad & \ldots, \end{aligned} \qquad (11)$$

The first terms in the squar brackets in (11) follow directly from the expansion of Eq.(9) in a; the second term in the second line is the double-log from the exact two-loop calculations; and the last term in the third line was predicted by J.Blumlein&A.Vogt in [18]. From (11) rough estimate follows to the boundary of validity of Eq.(9), $z \simeq 0.1 - 0.05$ at moderate $\alpha_s \simeq 0.3 - 0.1$. The most singular $\ln^4[z]$-term in (11) becomes important for $z \leq 10^{-3}$.

Table 1. The results of $\Gamma_{(1,2)}(n)$ calculations ($\Gamma(n) = \int_0^1 dx x^n P(z)$) performed in different ways, exact numerical results from [7] and approximation obtained from $P(z, A, \xi)$ with $\xi = -3$; both numerical and analytic **exact results** are marked by the bold print:

	$\Gamma_{(1)}(n)$		$\Gamma_{(2)}(n)$		
	$C_F C_A$	$N_f \cdot C_F$	$C_A^2 C_F$	$N_f \cdot C_F C_A$	$N_f^2 \cdot C_F$
n=2					
Exact	**13.9**	**−2.3704**	**86.1 + 21.3 ζ(3)**	**−12.9 − 21.3 ζ(3)**	**−0.9218**
$\xi = -3$	11.3		−42.0	12.9	
n=4:					
Exact	**23.9**	**−4.9152**	**140.0 + 19.2 ζ(3)**	**−18.1 − 41.9 ζ(3)**	**−1.5814**
$\xi = -3$	23.5		−76.0	23.	
n=6					
Exact	**29.7**	**−6.4719**	**173 + 19.01 ζ(3)**	**−20.4 − 54.0 ζ(3)**	**−1.9279**
$\xi = -3$	31.1		−95.6	28.5	
n=8					
Exact	**33.9**	**−7.6094**	**196.9 + 18.98 ζ(3)**	**−21.9 − 62.7 ζ(3)**	**−2.1619**
$\xi = -3$	36.3		−109.0	32.3	
n=10					
Exact	**37.27**	**−8.5095**	**216.0 + 18.96 ζ(3)**	**−23.2 − 69.6 ζ(3)**	**−2.3366**
$\xi = -3$	41.00		−119.28	35.24	
n=12					
Exact	**40.02**	**−9.2555**	**?**	**?**	**−2.4753**
$\xi = -3$	44.64		−127.61	37.58	

It seems naturally to combine the improved by the simple-logs kernel $P^{(1)}(z;-a,-3)$ with the first double-logs contribution from the exact two-loop calculations into a modernized kernel [5] $\tilde{P}(z)$,

$$\tilde{P}(z) = P^{(1)}(z;-a,-3) + a_s^2 C_F \left[(P_0(z) C_A - (1+z) C_F) \ln^2[z] - 4(C_F - C_A/2) P_0(-z) F(z) \right]_+ ,$$

which works up to $z \sim 10^{-3}$.

At the end let us consider the integral characteristics of the kernel $P^{(1)}(z;-a,-3)$ to compare with the exact results. The expansion of this kernel in a generates partial kernels $a_s^2 P_{(1)}(z)$, $a_s^3 P_{(2)}(z), \ldots$ which in turn produce ADCO $a_s^2 \cdot \Gamma_{(1)}(n)$, $a_s^3 \cdot \Gamma_{(2)}(n)$, ... according to the relation $\Gamma(n) = \int_0^1 dz z^n P(z)$. Let us compare these elements of ADCO and a few numerical exact results from [7] collected in Table 1:

(i) evidently, the leading N_f-contributions are reproduced exactly for any $\Gamma_{(j)}(n)$;

(ii) we consider there the subleading contributions to the coefficient $\Gamma_{(1)}(n)$ generated by gluon loops and associated with the Casimirs $C_F C_A/2$, the C_F^2-term is missed, but its contribution is numerically insignificant. It is seen that in this order the $C_F C_A$-terms are rather close to exact values (the accuracy is about 10% for $n > 2$) and our approximation works rather well;

(iii) in the next order, the contributions to $\Gamma_{(2)}(n)$ associated with the coefficients $N_f \cdot C_F C_A$ and $C_A^2 C_F$ arise, while the terms with the Casimirs C_F^3, $N_f \cdot C_F^2$, $C_F^2 C_A$ are missed. In the third order, contrary to the previous item, all the generated terms are opposite in sign to the exact values, and the "$\xi = -3$ approximation" doesn't work at all. So, we need the next step to improve the agreement with 3-loop results – to obtain the subleading N_f-terms by an exact calculation.

4 The nonforward ER-BL evolution equation and its solution

Here we present the results of the bubble resummation for the ER-BL kernel $V(x,y)$. The latter can be derived in the same manner as it was done for the DGLAP kernel $P(z)$, see Appendix A in [8]. On the other hand, $V(x,y)$ can be obtained as a "by-product" of the previous results for $P(z)$, *i.e.*, we use again [8, 13] the exact relations between the V and P kernels established in any order of PT [6] for triangular diagrams. These relations were obtained by comparing counterparts for the same triangular diagrams considered in "forward", Fig.1a, and "nonforward", Fig.1d, kinematics.

Collecting the contributions from triangular diagrams, see [13], one arrives at the final expression for $V^{(1)}$ in the "main bubbles" approximation

$$V^{(1)}(x,y;A,\xi) = a_s C_F 2 \left[\theta(y>x) \left(\frac{x}{y}\right)^{1-A} \left(1 - A + \frac{1}{y-x}\right) \right]_+ \frac{\gamma_g(0,\xi)}{\gamma_g(A,\xi)} + (x \to \bar{x}, y \to \bar{y}) \qquad (12)$$

that has a "plus form" again due to the vector current conservation. The contribution $V^{(1)}$ in (12) should dominate for $N_f \gg 1$ in the kernel V. Besides, the function

[5] Here the double-*log*'s part is rewritten from [4]; $F(z) = \frac{1}{2}\ln^2[z] - 2\ln[z]\ln[1+z] - 2Li_2(-z) - Li_2(1)$

$V^{(1)}(x,y;A,\xi)$ possesses an important symmetry of its arguments x and y. Indeed, the function $\mathcal{V}(x,y;A,\xi) = V^{(1)}(x,y;A,\xi) \cdot (\bar{y}y)^{1-A}$ is symmetric under the change $x \leftrightarrow y$, $\mathcal{V}(x,y) = \mathcal{V}(y,x)$. This symmetry allows us to obtain the eigenfunctions $\psi_n(x)$ of the "reduced" evolution equation [9]

$$\int_0^1 V^{(1)}(x,y;A)\psi_n(y;A)dy = \Gamma(n;A)\psi_n(x;A), \qquad (13)$$

$$\psi_n(y;A) = (\bar{y}y)^{d_\psi(A)-\frac{1}{2}} \frac{C_n^{\left(d_\psi(A)\right)}(y-\bar{y})}{N(n,A)}; \; d_\psi(A) = (D_A - 1)/2; \; D_A = 4 - 2A; \qquad (14)$$

$$N(n,A) = 2^{1-4d_\psi(A)}\pi\Gamma(n+2d_\psi(A))/\left(n!\,(n+d_\psi(A))\left(\Gamma(d_\psi(A))^2\right)\right),$$

and $d_\psi(A)$ is the effective dimension of the quark field when the AD $A(\xi)$ is taken into account; $C_n^{(\alpha)}(z)$ are the Gegenbauer polynomials of an order of α; $N(n,A)$ is the norm of $C_n^{(\alpha)}(y-\bar{y})$, [25]. The partial solutions $\Phi(x;a_s,l)$ of the original ER-BL-equation (where $l \equiv \ln(\mu^2/\mu_0^2)$)

$$\mu^2 \frac{d}{d\mu^2}\Phi(x;a_s,l) = \int_0^1 V^{(1)}(x,y;A)\;\Phi(y;a_s,l)dy \qquad (15)$$

are proportional to these eigenfunctions $\psi_n(x;A)$ for a special case of the stopped evolution $a_s = a_s^*$, $\beta(a_s^*) = 0$, see, *e.g.*, [23, 8]. The result (14) for the eigenfunctions at $\xi = -3$, has been confirmed in [16] by "a partial resummation of conformal anomalies" and in a suggestion of a large value of b_0. Let us examine $\psi_n(x;-a)$ in (14) as an approximation to the exact two-loop solution derived in a closed form in [23]. Expanding, *e.g.*, $\psi_0(y;-a)$ in parameter a we can express $\psi_0^{appr}(x)$ versus the exact solution $\psi_0^{exact}(x)$

$$\psi_0(x;-a) \to \psi_0^{appr}(x) = 6x\bar{x}\left\{1 + a_s b_0\left(\ln(x\bar{x}) + \frac{5}{3}\right)\right\}, \qquad (16)$$

$$\psi_0^{exact}(x) = 6x\bar{x}\left\{1 + a_s b_0\left(\ln(x\bar{x}) + \frac{5}{3}\right) + a_s C_F\left(\ln^2\left(\frac{\bar{x}}{x}\right) + 2 - \frac{\pi^2}{2}\right)\right\}. \qquad (17)$$

The term $\psi_0^{appr}(x)$ coincides with the "conformal symmetry-predicted" (CSP) part in (17), (proportional to b_0), this part dominates in $\psi_0^{exact}(x)$ in the mid-region of the parameter x, $0.3 < x < 0.7$. The other part in (17) is generated by the "additional conformal symmetry breaking term" [23]; it contributes in the opposite phase to the first one and it is large and enhanced near the end points. For the latter reason, $\psi_n^{appr}(x)$ become useless at $n \geq 2$ even for the mid-region x description, see [23].

In the general case $\beta(a_s) \neq 0$ let us start with an ansatz for the partial solution of Eq.(15), $\Phi_n(x;a_s,l) \sim \chi_n(a_s,l) \cdot \psi_n(x;A)$, with the boundary condition $\chi_n(a_s,0) = 1$; $\Phi_n(x;a_s,0) \sim \psi_n(x;A)$. For this ansatz, Eq.(15) reduces to

$$\left(\mu^2\partial_{\mu^2} + \beta(a_s)\partial_{a_s}\right)\ln\left(\Phi_n(x;a_s,l)\right) = \Gamma(n;A). \qquad (18)$$

In the case $n = 0$, the AD of the vector current $\Gamma(0;A) = 0$, and the solution of the homogeneous equation in (18) provides the "asymptotic wave function"

$$\Phi_0(x;a_s,l) = \psi_0(x;\bar{A}) = \frac{1}{N(0,\bar{A})}((1-x)x)^{(1-\bar{A})} \qquad (19)$$

where $\bar{A} = -\bar{a}_s(\mu^2)\gamma(0,\xi)$, $\bar{a}_s(\mu^2)$ is the running coupling corresponding to a β-function $\beta(a_s)$. Similar solutions have been discussed in [15] in the framework of the standard NNA approximation. Solving simultaneously Eq. (18) and the renormalization-group equation for the coupling constant $\bar{a}_s$, we arrive at the partial solution $\Phi_n(x;\bar{a}_s,l)$ in the form

$$\Phi_n(x,\bar{a}_s) \sim \chi_n(\mu^2)\cdot\psi_n(x;\bar{A});\ \text{where}\ \chi_n(\mu^2) = \exp\left\{-\int_{a_s(\mu_0^2)}^{a_s(\mu^2)}\frac{\Gamma(n,A)}{\beta(a)}da\right\}. \qquad (20)$$

An adequate choice of β-function in (20) must correspond to the same modified NNA approximation that was applied for $\Gamma(n,A)$ calculation, but it is absent jet. The β-function in a large N_f expansion, that is equivalent to quark bubbles resummation, has been computed in [26].

5 Conclusion

Here, I present closed expressions in the "all order" approximation for the DGLAP kernel $P(z)$ and ER-BL kernel $V(x,y)$ resulting from resummation of a certain class of QCD diagrams with the renormalon chain insertions. The contributions from these diagrams, $P^{(1)}(z;A)$ and $V^{(1)}(z;A)$, give the leading N_f dependence of the kernels for a large number of flavours $N_f \gg 1$. These multiloop "improved" kernels are generating functions to obtain contributions to partial kernels like $a_s^{(n+1)}P_{(n)}(z)$ in any order n of the perturbation expansion. Here $A \sim a_s$ is the new expansion parameter that coincides (in magnitude) with the anomalous dimension of the gluon field. On the other hand, the method of calculation suggested in [8] does not depend on the nature of self-energy insertions and does not appeal to the value of parameters N_fT_R, $C_A/2$ or C_F associated with different loops. This allows us to obtain contributions from chains with different kinds of self-energy insertions, both quark and gluon (ghost) loops, see [13]. The price for this generalization is the gauge dependence of final results for $P^{(1)}(z;A(\xi),\xi)$ and $V^{(1)}(z;A(\xi),\xi)$ on the gauge parameter ξ.

The result for the DGLAP nonsinglet kernel $P^{(1)}(z;A(\xi),\xi)$ is presented in (3) in the covariant ξ-gauge, it looks similar in form to the simple one-loop kernel. The analytic properties of this kernel in the variable a_s are discussed for an exceptional gauge parameter $\xi = -3$. This choice of the gauge allows one to generalize the naive nonabelianization suggestion and provides the leading b_0-behavior of the kernel for large $b_0 \gg 1$. For this gauge $P^{(1)}(z;A(-3),-3)$ in (9) works up to $z \simeq 0.1 - 0.05$ at moderate $\alpha_s = 0.3 - 0.1$, and reproduces two-loop anomalous dimensions $a_s^2\Gamma_{(1)}(n)$ with a good accuracy, while the standard "naive nonabelianization" proposition fails at this level. But on the next three loop level the "$\xi = -3$ approximation" is insufficient, see quantities $\Gamma_{(2)}(n)$ in Table 1.

The contribution $V^{(1)}(x,y;A(\xi),\xi)$ to the nonforward ER-BL kernel (12) is obtained for the same classes of diagrams as a "byproduct" of the previous technique [13, 6]. The partial solutions (13), (20) to the multiloop improved ER-BL equation are derived, that are similar in form to the one-loop solutions. The form of these solutions appearing at $\xi = -3$ was confirmed independently in [16]. The lowest harmonic $\psi_0(x;\bar{A}(-3))$ roughly imitates the x-behavior in the mid-region of the exact two-loop solution ([23]).

The obtained results are certainly useful for an independent check of complicated computer calculations in higher orders of perturbation theory, similar to [7]; they are

useful for the analysis of evolution "at small x"; they may be a starting point for further multiloop approximation procedures.

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