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HARMONIC SUPERPOTENTIALS AND SYMMETRIES IN GAUGE THEORIES WITH EIGHT SUPERCHARGES

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Гармонические суперпотенциалы и симметрии в калибровочных теориях с восемью суперзарядами

Модели взаимодействия D-мерных гипермультиплетов и суперсимметричных калибровочных мультиплетов с $\mathcal{N} = 8$ суперзарядами ($D \leq 6$) могут быть сформулированы в рамках гармонических суперпространств. Эффективное кулоновское низкоэнергетическое действие для $D = 5$ включает свободный член и член Черна—Саймонса. Мы рассматриваем также неабелево суперполевое $D = 5$ действие Черна—Саймонса. Бигармоническое $D = 3$, $\mathcal{N} = 8$ суперпространство вводится для описания l и r супермультиплетов и зеркальной симметрии. $D = 2$, (4,4) калибровочная теория и взаимодействия гипермультиплетов рассматриваются в тригармоническом суперпространстве. Связи для $D = 1$, $\mathcal{N} = 8$ супермультиплетов решаются с помощью $SU(2) \times Spin(5)$ гармоник. Эффективные калибровочные действия в полных $D \leq 3$, $\mathcal{N} = 8$ суперпространствах содержат ограниченные (гармонические) суперпотенциалы, удовлетворяющие $(6 - D)$ уравнениям Лапласа для калибровочной группы $U(1)$ или соответствующим $(6 - D)p$-мерным уравнениям для групп $[U(1)]^p$. Обобщенные гармонические представления суперпотенциалов связывают эквивалентные суперполевые структуры этих теорий в полном и аналитических суперпространствах. Гармонический подход упрощает доказательства теорем неперенормировки.

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Harmonic Superpotentials and Symmetries in Gauge Theories with Eight Supercharges

Models of interactions of D-dimensional hypermultiplets and supersymmetric gauge multiplets with $\mathcal{N} = 8$ supercharges ($D \leq 6$) can be formulated in the framework of harmonic superspaces. The effective Coulomb low-energy action for $D = 5$ includes the free and Chern—Simons terms. We consider also the non-Abelian superfield $D = 5$ Chern—Simons action. The biharmonic $D = 3$, $\mathcal{N} = 8$ superspace is introduced for a description of l and r supermultiplets and the mirror symmetry. The $D = 2$,(4,4) gauge theory and hypermultiplet interactions are considered in the triharmonic superspace. Constraints for $D = 1$, $\mathcal{N} = 8$ supermultiplets are solved with the help of the $SU(2) \times Spin(5)$ harmonics. Effective gauge actions in the $D \leq 3$, $\mathcal{N} = 8$ superspaces contain constrained (harmonic) superpotentials satisfying the $(6 - D)$ Laplace equations for the gauge group $U(1)$ or corresponding $(6 - D)p$-dimensional equations for the groups $[U(1)]^p$. Generalized harmonic representations of superpotentials connect equivalent superfield structures of these theories in the full and analytic superspaces. The harmonic approach simplifies the proofs of non-renormalization theorems.

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1 Introduction

The harmonic superspace (HS) has firstly been introduced for the off-shell description of matter, gauge and supergravity superfield theories with the manifest $D{=}4, N_4{=}2$ supersymmetry [1, 2]. The $SU(2)/U(1)$ harmonics $u^\pm_i$ and corresponding harmonic derivatives ∂^{++}, ∂^{--} and ∂^0 are used for the consistent solution of the superfield constraints in the $N_4{=}2$ superspace. The basic relations for the harmonics are

$$[\partial^{++},\partial^{--}] = \partial^0 , \quad [\partial^0,\partial^{\pm\pm}] = \pm 2\partial^{\pm\pm} , \tag{1.1}$$
$$\partial^{++} u^+_i = 0, \quad \partial^{++} u^-_i = u^+_i, \quad \partial^0 u^\pm_i = \pm u^\pm_i , \tag{1.2}$$
$$\partial^{--} u^-_i = 0, \quad \partial^{--} u^+_i = u^-_i . \tag{1.3}$$

The HS approach has also been applied to consistently describe hypermultiplets and vector multiplet in $D{=}6, N_6{=}1$ supersymmetry [3, 4]. It is convenient to use the total number of supercharges $\mathcal{N}$ for the classification of all these models in different dimensions D instead of the number of spinor representations for supercharges N_D. Let us review briefly the basic aspects of the $D{=}6, \mathcal{N}{=}8$ harmonic gauge theory. The harmonics $u^\pm_i$ are used to construct the analytic $6D$ coordinates $\zeta{=}(\hat{x}^{\alpha\rho}_A,\ \theta^{\alpha+})$ and the additional spinor coordinate $\theta^{\alpha-}$, where $\alpha,\ \beta,\ \rho\ldots$ are the 4-spinor indices of the $(1,0)$ representation of the $Spin(5,1)$ group and $\theta^{\pm\alpha}{=}u^\pm_i\theta^{i\alpha}$. The harmonized spinor derivatives and harmonic derivatives have the following form in these coordinates:

$$D^+_\alpha = \partial^+_\alpha , \quad D^-_\alpha = -\partial^-_\alpha - i\theta^{\gamma-}\hat{\partial}_{\alpha\gamma} , \tag{1.4}$$
$$D^{++} = \partial^{++} + \frac{i}{2}\theta^{\alpha+}\theta^{\gamma+}\hat{\partial}_{\alpha\gamma} + \theta^{\alpha+}\partial^+_\alpha , \tag{1.5}$$
$$D^{--} = \partial^{--} + \frac{i}{2}\theta^{\alpha-}\theta^{\gamma-}\hat{\partial}_{\alpha\gamma} + \theta^{\alpha-}\partial^-_\alpha , \tag{1.6}$$

where $\hat{\partial}_{\alpha\gamma} = \partial/\partial\hat{x}^{\alpha\gamma}$.

The Grassmann analyticity condition in HS is $D^+_\mu\,\omega{=}0$. Superfield constraints of $D{=}6$ SYM in the ordinary superspace (central basis or CB) are equivalent to the integrability conditions preserving this analyticity. The Yang-Mills prepotential $V^{++}(\zeta,u)$ in the analytic basis (AB) describes the $6D$ vector multiplet ($A_{\alpha\rho}$, λ^α_i, X^{ik}) and possesses the gauge transformation with the analytic matrix parameter $\lambda(\zeta,u)$

$$\delta V^{++} = D^{++}\lambda + [V^{++},\lambda] = \nabla^{++}\lambda . \tag{1.7}$$

The action of the $D{=}6$ SYM theory has the form of integral over the full superspace [3]

$$S(V^{++}) = \frac{1}{g_6^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int d^6x d^8\theta du_1\ldots du_n \frac{\mathrm{Tr}\, V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u^+_1u^+_2)\ldots(u^+_nu^+_1)} \tag{1.8}$$

where g_6 is the coupling constant of dimension $d{=}1$ and $(u^+_1u^+_2)^{-1}$ is the harmonic distribution [2].

The gauge variation of this action

$$\delta S(V^{++}) = \frac{1}{g_6^2}\int d^6x d^8\theta du \mathrm{Tr}\, \nabla^{++}\lambda V^{--} = -\frac{1}{g_6^2}\int d^6x d^8\theta du \mathrm{Tr}\, \lambda D^{--}V^{++} = 0 \tag{1.9}$$

vanishes due to the analyticity of the parameter and prepotential. We have used here the harmonic zero-curvature equation

$$D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}] = 0 \;, \tag{1.10}$$

where V^{--} is the connection for the harmonic derivative D^{--}. Note that reality conditions for the harmonic connections include the special conjugation of harmonics preserving the $U(1)$-charges [1]

$$\widetilde{u_i^{\pm}} = u^{i\pm} \;, \qquad (V^{\pm\pm})^\dagger = -V^{\pm\pm} \;. \tag{1.11}$$

The physical fields of the hypermultiplet f^i and ψ_α and the infinite number of auxiliary fields are components of the analytic $6D$ superfield $q^+(\zeta, u)$. The interaction of the hypermultiplet and gauge field can be written in the analytic superspace

$$S(q^+, V^{++}) = \int d\zeta^{(-4)} du \tilde{q}^+ (D^{++} + V^{++}) q^+ \;, \tag{1.12}$$

where $d\zeta^{(-4)} = d^6x_A (D^-)^4$ is the analytic measure in HS.

Universality of harmonic superspaces is connected with the possibility of constructing $\mathcal{N}$=8 models in D<6 by a dimensional reduction. The HS analysis of the D=4 low-energy effective actions has been considered for the gauge superfields [7] and for the hypermultiplets [8]. The manifestly supersymmetric calculations in HS are in a good agreement with the basic ideas of the Seiberg-Witten theory [6], however, the HS geometry allows us to rewrite the chiral-superspace Coulomb action as the integral in the full superspace

$$i \int d^4x d^4\theta \mathcal{F}(W) + \text{c.c.} = \int d^4x d^8\theta du V^{++} V^{--} [F(W) + \text{c.c.}] \;, \tag{1.13}$$

where $F(W) = -iW^{-2}\mathcal{F}(W)$ is the holomorphic part of the superpotential in this representation. It should be stressed that the Lagrange density in the full superspace is not gauge-invariant in contrast to the chiral density. The superpotential $f(W, \bar{W}) = [F(W) + \text{c.c.}]$ is the most general solution of the constraints

$$D^+_\alpha \bar{D}^+_{\dot\alpha} f(W, \bar{W}) = 0 \;\rightarrow\; \partial_W \bar{\partial}_W f(W, \bar{W}) = 0 \;, \tag{1.14}$$

which follow from the gauge invariance. Representations of the action in the full, analytic and chiral superspaces are important for the HS interpretation of the electric-magnetic duality [9].

The holomorphic action can be reduced to lower dimensions, however, this reduction does not produce the general effective action. The $\mathcal{N}$=8 supersymmetries have some specific features for each dimension based on differences in the structure of Lorentz groups L_D, maximum automorphism groups R_D and the set of central charges Z_D. The main result of this work is a construction of the Coulomb effective actions for the dimensions D=1, 2, 3 and 5 in the full $\mathcal{N}$=8 superspace

$$S_D = \int d^Dx \, d^8\theta \, du \, V^{++} V^{--} f_D(W) \;, \tag{1.15}$$

where $f_D(W)$ is the superpotential and W is the constrained $(6-D)$-component superfield strength for the $U(1)$ gauge prepotential V^{++}. The gauge invariance of this action implies the $(6-D)$-dimensional Laplace equation for the general superpotential

$$\Delta^w_D f_D(W) = 0 \;, \tag{1.16}$$

which generalizes the $2D$-Laplace equation (1.13). The $(6-D)$-harmonic solutions of this equation can be used for a description of non-perturbative solutions in the $\mathcal{N}$=8 gauge theories. We discuss harmonic-integral representations of the $D\leq3$ superpotentials which allow us to construct the equivalent analytic-superspace representations of S_D.

It should be remarked that the function f_D determines σ-model structures and interactions of the $(6-D)$-dimensional scalar field with fermion and vector fields.

Renormalization theorems in this approach are connected with the R_D-invariant solutions of Eq.(1.16)

$$f^R_D(w_D) = g_D^{-2} + k_D w_D^{D-4} \;, \qquad D \neq 4 \;, \tag{1.17}$$

where the invariant superfield w_D can be interpreted as a length in the $(6-D)$-moduli space, and g_D and k_D are coupling constants.

The effective actions of the $[U(1)]^p$ gauge theories are considered also by these methods. The matrix superpotentials of these theories satisfy the $(6-D)p$-dimensional Laplace-type equations. It is interesting that rich harmonic structures of moduli spaces for the $D\leq3, \mathcal{N}$=8 theories arising in connection with the equations for superpotentials generalize naturally the original $SU(2)$-harmonic structure of the $D\geq4, \mathcal{N}$=8 theories.

Sect.2 is devoted to the 5-dimensional HS theories. The effective action of the D=5 Abelian theory contains the free term and the cubic Chern-Simons term. We also construct the non-Abelian superfield Chern-Simons term.

In Sect.3, we consider the biharmonic superspace (BHS) using harmonics of the automorphism group $SU_l(2)\times SU_r(2)$ in the D=3, $\mathcal{N}$=8 models. The l-analytic gauge prepotentials and hypermultiplets have their mirror partners in the r-analytic superspace.

The D=2, (4, 4) models in the triharmonic $SU_c(2)\times SU_l(2)\times SU_r(2)$ superspace (THS) [29, 30, 31] are discussed in Sect.4. We underline the importance of the (4,4) gauge theory and derive the formula for the effective action in the full superspace, which is equivalent to the action in the rl-analytic superspace.

An adequate superfield description of the D=1, $\mathcal{N}$=8 theories requires the use of harmonics for the automorphism group $R_1 = SU_c(2)\times Spin(5)$. We define the corresponding BHS gauge and hypermultiplet models in Sect.5.

Problems of the $\mathcal{N}$=8 gauge theories have earlier been discussed in the framework of the component-field formalism or the formalism with $\mathcal{N}$=4, D=1, 2, 3 superfields (see e.g.[11, 13, 34, 39]). In particular, the $(6-D)$ Laplace equations have been considered in the $\mathcal{N}$=4 superfield formalism of the $\mathcal{N}$=8 gauge theories and in the formalism of the corresponding σ-models. Nevertheless, it should be stressed that the manifestly covariant HS approach provides the most adequate and universal methods to solve the problems of the $\mathcal{N}$=8 theories in all dimensions. A short discussion of these ideas has also been presented in [10].

2 Five-dimensional harmonic gauge theories

Let us consider firstly the harmonic superspace with the D=5, $\mathcal{N}$=8 supersymmetry. The general five-dimensional superspace has the coordinates $z=(x^{\rm m}, \theta^\alpha_i)$, where ${\rm m}$ and α are the 5-vector and 4-spinor indices of the Lorentz group $L_5=SO(4,1)$, respectively, and i is the 2-spinor index of the automorphism group $R_5=SU(2)$. The spinors of L_5 are equivalent to the pair of the $SL(2,C)$ spinors: $\Psi^\alpha=(\psi^a, \bar{\psi}^{\dot a})$.

The invariant symplectic matrices $\Omega_{\alpha\rho}$ and $\Omega^{\alpha\rho}$ can be constructed in terms of the $SL(2,C)$ ε-symbols

$$\Omega_{\alpha\rho}=\begin{pmatrix}\varepsilon_{\alpha\rho} & 0\\ 0 & \varepsilon_{\dot\alpha\dot\rho}\end{pmatrix}\,,\qquad \Omega_{\alpha\rho}\Omega^{\rho\sigma}=\delta^\sigma_\alpha\,. \tag{2.1}$$

These matrices connect spinors with low and upper indices.

The antisymmetric traceless representation of the Γ-matrices contains the $4D$ Weyl matrices σ_m and ε-symbols

$$(\Gamma_m)_{\alpha\beta}=\begin{pmatrix}0 & (\sigma_m)_{\alpha\dot\beta}\\ -(\sigma_m)_{\beta\dot\alpha} & 0\end{pmatrix}\,,\quad (\Gamma_4)_{\alpha\beta}=\begin{pmatrix}i\varepsilon_{\alpha\beta} & 0\\ 0 & -i\varepsilon_{\dot\alpha\dot\beta}\end{pmatrix}\,. \tag{2.2}$$

The corresponding representation of the $5D$ Clifford algebra has the following form:

$$(\Gamma_{\rm m})_{\alpha\beta}(\Gamma_{\rm n})^{\beta\gamma}+(\Gamma_{\rm n})_{\alpha\beta}(\Gamma_{\rm m})^{\beta\gamma}=-2\delta^\gamma_\alpha\eta_{\rm mn}\,, \tag{2.3}$$

where $(\Gamma_{\rm n})^{\beta\gamma}=\Omega^{\beta\rho}\Omega^{\gamma\sigma}(\Gamma_{\rm n})_{\rho\sigma}$ and $\eta_{\rm mn}$ is the metric of the (4,1) space.

The 5-vector projector in the spinor space is

$$(\Pi_5)^{\alpha\gamma}_{\rho\sigma}=\frac{1}{4}(\Gamma^{\rm m})^{\alpha\gamma}(\Gamma_{\rm m})_{\rho\sigma}=\frac{1}{2}(\delta^\alpha_\rho\delta^\gamma_\sigma-\delta^\alpha_\sigma\delta^\gamma_\rho)+\frac{1}{4}\Omega_{\rho\sigma}\Omega^{\alpha\gamma}\,. \tag{2.4}$$

Consider also the relations between the antisymmetric 4-spinor symbol $\mathcal{E}$ and the matrices Ω and Γ

$$\mathcal{E}_{\alpha\rho\mu\nu}=\Omega_{\alpha\rho}\Omega_{\mu\nu}+\Omega_{\alpha\mu}\Omega_{\nu\rho}+\Omega_{\alpha\nu}\Omega_{\rho\mu}=-\frac{1}{2}(\Gamma^{\rm m})_{\alpha\rho}(\Gamma_{\rm m})_{\mu\nu}+\frac{1}{2}\Omega_{\alpha\rho}\Omega_{\mu\nu}\,. \tag{2.5}$$

It is convenient to use the bispinor representation of the $5D$ coordinates and partial derivatives

$$x^{\alpha\rho}=\frac{1}{2}(\Gamma_{\rm m})^{\alpha\rho}x^{\rm m}\,,\quad \partial_{\alpha\rho}=\frac{1}{2}(\Gamma^{\rm m})_{\alpha\rho}\partial_{\rm m}\,. \tag{2.6}$$

The C-conjugation rules for the $Spin(4,1)$ objects are similar to the corresponding rules for (1,0) spinors in the $6D$ space

$$\overline{\theta}{}^\alpha_i\equiv\varepsilon_{ik}C^\alpha_\gamma(\theta^\gamma_k)^*=\theta^\alpha_i\,,\quad (C^2)^\alpha_\gamma=-\delta^\alpha_\gamma\,, \tag{2.7}$$

$$\overline{\Omega}_{\alpha\rho}=-\Omega_{\alpha\rho}\,,\quad \overline{x}^{\alpha\rho}=x^{\alpha\rho}\,,\quad \overline{\partial}_{\alpha\rho}=-\partial_{\alpha\rho}\,. \tag{2.8}$$

The basic relations between the spinor derivatives of the D=5,$\mathcal{N}$=8 superspace have the following form:

$$\{D^k_\alpha\,,\,D^l_\gamma\}=i\varepsilon^{kl}(\partial_{\alpha\gamma}+\frac{1}{2}\Omega_{\alpha\gamma}Z)\,, \tag{2.9}$$

where Z is the real central charge. We shall consider the basic superspace with Z=0 and introduce the central charges via the interaction of gauge superfields satisfying the constraints

$$\{\nabla^k_\alpha\,,\,\nabla^l_\gamma\}=i\varepsilon^{kl}(\nabla_{\alpha\gamma}+\frac{1}{2}\Omega_{\alpha\gamma}W)\,, \tag{2.10}$$

where W is the real superfield.

The spinor $SU(2)/U(1)$ harmonics $u^\pm_i$ can be used to construct the R_5-invariant HS coordinates ζ=($x^{\rm m}_A$, $\theta^{\alpha+}$), $\theta^{\alpha-}$, spinor derivatives $D^\pm_\alpha$ and harmonic derivatives by analogy with Eqs.(1.4-1.6)

$$D^+_\alpha=\partial^+_\alpha\,,\quad D^-_\alpha=-\partial^-_\alpha-i\theta^{\gamma-}\partial_{\alpha\gamma}\,, \tag{2.11}$$

$$D^{++}=\partial^{++}+\frac{i}{2}\theta^{\alpha+}\theta^{\gamma+}\partial_{\alpha\gamma}+\theta^{\alpha+}\partial^+_\alpha\,. \tag{2.12}$$

We shall use the following notation for degrees of the spinor derivatives:

$$D^{(\pm2)}=\frac{1}{4}D^{\alpha\pm}D^\pm_\alpha\,,\quad D^{(\pm2)}_{\alpha\gamma}=(\Pi_5)^{\rho\sigma}_{\alpha\gamma}D^\pm_\rho D^\pm_\sigma\,, \tag{2.13}$$

$$D^{(\pm3)}_\alpha=D^\pm_\alpha D^{(\pm2)}\,,\quad D^{(\pm4)}=2D^{(\pm2)}D^{(\pm2)} \tag{2.14}$$

and the important identities

$$D^{(+2)}D^{(+2)}_{\alpha\gamma}=0\,,\qquad D^{(+2)}_{\alpha\gamma}D^{(+2)}_{\rho\sigma}=-2(\Pi_5)_{\alpha\gamma,\rho\sigma}D^{(+4)}\,, \tag{2.15}$$

$$D^{(+4)}D^{--}D^{--}D^{(+4)}=-2\partial^{\rm m}\partial_{\rm m}D^{(+4)}\,. \tag{2.16}$$

The analytic Abelian prepotential $V^{++}(\zeta,u)$ describes the $5D$ vector supermultiplet. In the WZ-gauge, this harmonic superfield contains the real scalar field Φ, the Maxwell field $A_{\rm m}$, the isodoublet of spinors λ^α_i and the auxiliary isotriplet X^{ik}

$$\begin{aligned}V^{++}_{WZ}=\;&i\Theta^{(+2)}\Phi(x_A)+\Theta^{(+2)\alpha\rho}A_{\alpha\rho}(x_A)\\&+\Theta^{(+2)}\theta^{\alpha+}u^-_i\lambda^i_\alpha(x_A)+i[\Theta^{(+2)}]^2u^-_ku^-_jX^{kj}(x_A)\,,\end{aligned} \tag{2.17}$$

where

$$\Theta^{(+2)}=\frac{1}{4}\theta^{\alpha+}\theta^+_\alpha\,,\qquad \Theta^{(+2)\alpha\rho}=(\Pi_5)^{\alpha\rho}_{\mu\nu}\theta^{\mu+}\theta^{\nu+}\,. \tag{2.18}$$

The real superfield strength of this theory can be written in terms of the harmonic connection $V^{--}(V^{++})$ (see Eqs. (1.10) and (2.29))

$$W=-2iD^{(+2)}V^{--}\,. \tag{2.19}$$

This superfield satisfies the following constraints:

$$\nabla^{++}W=D^{++}W+[V^{++},W]=0\,, \tag{2.20}$$

$$D^{(+2)}_{\alpha\rho}W=0\,. \tag{2.21}$$

The Abelian superfield W does not depend on harmonics.

The $5D$ SYM action has the universal form (1.8) in the full harmonic superspace. The SYM equations have the vacuum Abelian solution v^{++}=$i\Theta^{(+2)}Z$ where Z is the linear combination of the Cartan generators of the gauge group (see the analogous D=4 solution in ref.[8]). This vacuum solution spontaneously breaks the gauge symmetry, but it conserves the D=5 supersymmetry with the central charge and produces BPS masses of the Z-charged fields.

Chiral superspaces are not Lorentz-covariant in the case D=5, so one can use the full and analytic superspaces only. It is readily to construct the most general low-energy effective $U(1)$-gauge action in the full $\mathcal{N}$=8 harmonic superspace

$$S_5=\int d^5xd^8\theta du\;V^{++}V^{--}[g_5^{-2}+k_5W]\,, \tag{2.22}$$

where g_5 is the coupling constant of dimension 1/2, and k is the dimensionless constant of the $5D$ Chern-Simons interaction. The linear superpotential $f_5=g_5^{-2}+k_5W$ is a solution of the constraints

$$D^{\pm\pm}f_5=0\,,\qquad D^{(+2)}_{\alpha\beta}f_5=0\,, \tag{2.23}$$

which arise from the gauge invariance of S_5.

Note that the R_5 invariance of the effective action can be broken by the Fayet-Iliopoulos term in the analytic superspace

$$S_{FI}=\int d\zeta^{(-4)}du\,\xi^{ik}u^+_iu^+_kV^{++}\,, \tag{2.24}$$

which implies also the spontaneous breaking of supersymmetry.

The gauge-invariant Chern-Simons term for the group $[U(1)]^p$ contains the following cubic interactions of the Abelian superfields V^{++}_B

$$\int d^5xd^8\theta du\,k_{BCD}V^{++}_BV^{--}_CW_D\,, \tag{2.25}$$

where k_{BCD} are coupling constants and $B,C,D=1\ldots p$.

It is not difficult to construct the non-Abelian $5D$ Chern-Simons term S^5_{CS} starting from the following formula of its variation

$$\begin{aligned}\delta S^5_{CS}&=k_5\int d^5xd^8\theta du\,\mathrm{Tr}\,\delta V^{++}[V^{--},\,D^{(+2)}V^{--}]\\&=k_5\int d\zeta^{(-4)}du\,\mathrm{Tr}\,\delta V^{++}D^{(+4)}[V^{--},D^{(+2)}V^{--}]\,,\end{aligned} \tag{2.26}$$

which guarantees the gauge invariance taking into account Eqs. (1.7,1.10,2.20) and (2.21)

$$\begin{aligned}\delta_\lambda S^5_{CS}&=k_5\int d\zeta^{(-4)}du\,\mathrm{Tr}\,\lambda D^{(+4)}[D^{(+2)}V^{--},\nabla^{++}V^{--}]\\&=k_5\int d\zeta^{(-4)}du\,\mathrm{Tr}\,\lambda D^{(+4)}[D^{(+2)}V^{--},D^{--}V^{++}]=0\,.\end{aligned} \tag{2.27}$$

The non-polynomial formula for S^5_{CS} can be written as an integral over the auxiliary variable s

$$S^5_{CS}=k_5\int_0^1 ds\int d^5xd^8\theta du\,\mathrm{Tr}\,V^{++}[V^{--}(sV^{++}),\,D^{(+2)}V^{--}(sV^{++})]\,, \tag{2.28}$$

where the perturbative solution for V^{--} [5] is used

$$V^{--}(sV^{++})=\sum_{n=1}^{\infty}(-s)^n\int du_1\ldots du_n\frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u^+u^+_1)\ldots(u^+_nu^+)}\,. \tag{2.29}$$

The next-to-leading order effective Abelian $5D$ action can be written in terms of the manifestly gauge invariant function H(W).

3 Three-dimensional biharmonic superspace

Three-dimensional supersymmetric gauge theories have been intensively studied in the framework of new nonperturbative methods [13, 14, 15]. Superfield description of the simplest $D=3$, $\mathcal{N}=2,4$ theories and various applications have earlier been discussed in refs. [16]-[20]. Three-dimensional harmonic superspaces were considered in refs.[21, 22]. The most interesting features of the $D=3$ theories are connected with the Chern-Simons terms for gauge fields and also with the mirror symmetry between vector multiplets and hypermultiplets.

The $D=3$, $\mathcal{N}=8$ gauge theory can be constructed in the superspace with the automorphism group $R_3=SU_l(2)\times SU_r(2)$. Coordinates of the corresponding general superspace are $z=(x^{\alpha\beta},\ \theta^\alpha_{ia})$. We use here the two-component indices $\alpha,\ \beta\ldots$ for the space-time group $SL(2,R)$, $i,\ k\ldots$ for the group $SU_l(2)$ and $a,\ b\ldots$ for $SU_r(2)$, respectively.

The relations between basic spinor derivatives are

$$\{D^{ka}_\alpha,D^{lb}_\beta\}=i\varepsilon^{kl}\varepsilon^{ab}\partial_{\alpha\beta}+i\varepsilon^{kl}\varepsilon_{\alpha\beta}Z^{ab}\,, \tag{3.1}$$

where $\partial_{\alpha\beta}=\partial/\partial x^{\alpha\beta}$ and Z^{ab} are the central charges which commute with all generators exept for the generators of $SU_r(2)$. These central charges can be interpreted as covariantly constant Abelian gauge superfields by analogy with [8].

The superfield constraints of the $\mathcal{N}=8$ SYM theory in the central basis can be written as follows:

$$\{\nabla^{ka}_\alpha,\nabla^{lb}_\beta\}=i\varepsilon^{kl}\varepsilon^{ab}\nabla_{\alpha\beta}+i\varepsilon_{\alpha\beta}\varepsilon^{kl}W^{ab}\,, \tag{3.2}$$

where ∇_M are covariant derivatives with superfield connections and W^{ab} is the constrained superfield of the SYM theory (l-vector supermultiplet)

$$\nabla^{ka}_\alpha W^{bc}+\nabla^{kb}_\alpha W^{ca}+\nabla^{kc}_\alpha W^{ab}=0\,. \tag{3.3}$$

Note that gauge transformations in CB have the standard form

$$\delta\nabla^{ka}_\alpha=[\tau(z),\nabla^{ka}_\alpha]\,,\qquad \delta W^{ab}=[\tau(z),W^{ab}]\,. \tag{3.4}$$

It is evident that one can consider the mirror r-versions of superfield constraints for the vector multiplet and hypermultiplets. We shall define the biharmonic superspace which has simple properties with respect to the exchange $l\leftrightarrow r$. The mirror symmetry connects l-vector multiplets with r-hypermultiplets and vice versa.

Let us consider the l-harmonics $u^\pm_i\equiv u^{(\pm1,0)}_i$ of the group $SU_l(2)$ and the analogous r-harmonics $v^{(0,\pm1)}_a$ of the group $SU_r(2)$. The notation of charges in BHS is (q_1,q_2), but one can use also the notation with the one charge for the l-harmonic superspace, for instance, $D^{\pm\pm}_l$. The spinor and harmonic derivatives have the following form in the l-analytic coordinates $\zeta_l=(x^{\alpha\beta}_l,\theta^{\alpha+}_a)$ and $\theta^{\alpha-}_a$:

$$D^{b+}_\alpha=u^+_iD^{ib}_\alpha=\partial^{b+}_\alpha\,,\quad D^{b-}_\alpha=u^-_iD^{ib}_\alpha=-\partial^{b-}_\alpha+i\theta^{\beta b-}\partial^l_{\alpha\beta}\,, \tag{3.5}$$

$$D^{++}_l=\partial^{++}_l-\frac{i}{2}\theta^{\alpha+}_a\theta^{\beta a+}\partial^l_{\alpha\beta}+\theta^{\alpha+}_a\partial^{a+}_\alpha\,. \tag{3.6}$$

The following relations will be used in this section:

$$\{D^{a+}_\alpha,\,D^{b-}_\beta\}=-i\varepsilon^{ab}\partial^l_{\alpha\beta},\quad [D^{--},\,D^{a+}_\alpha]=D^{a-}_\alpha\,, \tag{3.7}$$

$$D^{(+2)}_{\alpha\beta}=\frac{1}{2}D^{+a}_\alpha D^+_{a\beta}\,,\qquad D^{ab(+2)}=\frac{1}{2}D^{\alpha a+}D^{b+}_\alpha\,, \tag{3.8}$$

$$D^{(+2)}_{\alpha\beta}D^{ab(+2)}=0\,,\qquad D^{(+2)}_{ab}D^{(+2)}_{cd}=(\varepsilon_{ac}\varepsilon_{bd}+\varepsilon_{bc}\varepsilon_{ad})(D^+)^4\,. \tag{3.9}$$

The l-harmonic superspace is adequate to the solution of the constraints (3.2)

$$u_i^+ u_k^+ \{\nabla_\alpha^{ia}, \nabla_\beta^{kb}\} \equiv \{\nabla_\alpha^{a+}, \nabla_\beta^{b+}\} = 0 , \qquad (3.10)$$
$$\nabla_\alpha^{a+} = g^{-1}(z,u) D_\alpha^{a+} g(z,u) , \qquad (3.11)$$

where $g(z,u)$ is the bridge matrix [1]. The l-analytic prepotential of the SYM theory is

$$V_l^{++} \equiv V_l^{(2,0)} = (D^{++}g)g^{-1} , \qquad D_\alpha^{a+} V_l^{++} = 0 , \qquad (3.12)$$
$$\delta g = \lambda g - g\tau . \qquad (3.13)$$

The components of this superfield can be determined in the WZ gauge

$$(V_l^{++})_{WZ} = \theta^{\alpha a+}\theta_\alpha^{b+}\Phi_{ab}(x_l) + \theta^{\alpha a+}\theta_a^{\beta+}A_{\alpha\beta}(x_l)$$
$$+\theta^{\alpha a+}\theta_\alpha^{b+}\theta_b^{\beta+}u_k^-\lambda_{a\beta}^k(x_l) + i(\theta^+)^4 u_k^- u_j^- X^{kj}(x_l) . \qquad (3.14)$$

The superfield strength of the D=3,$\mathcal{N}$=8 gauge theory in the analytic basis contains the corresponding harmonic connection $V_l^{--}(V_l^{++})$

$$W^{ab} = -iD^{ab(+2)}V_l^{--} , \qquad (3.15)$$
$$\delta W^{ab} = [\lambda,\ W^{ab}] , \qquad D^{++}W^{ab} + [V_l^{++}, W^{ab}] = 0 . \qquad (3.16)$$

It satisfies the following constraints:

$$D_\alpha^{a+}W^{bc} + D_\alpha^{b+}W^{ca} + D_\alpha^{c+}W^{ab} = 0 , \qquad (3.17)$$
$$D_{\alpha\beta}^{(+2)}W^{bc} = 0 , \qquad (3.18)$$

which are equivalent to the CB-constraints (3.3).

The superfield W^{ab} does not depend on harmonics in the gauge group $U(1)$. The vacuum Abelian solution of the SYM theory

$$v^{++} = i\theta^{\alpha a+}\theta_\alpha^{b+} Z_{ab} \qquad (3.19)$$

is covariant with respect to the supersymmetry with central charges Z_{ab} by analogy with the case D=4 [8].

The l-analytic hypermultiplet $q^+ \equiv q^{(1,0)}$ has the standard minimal interaction with $V_l^{++} \equiv V^{(2,0)}$ (see (1.12)). By analogy with refs.[3, 8], one can construct the free HS propagator for this superfield in the covariantly constant background (3.19)

$$i\langle q^+(1)|\bar{q}^+(2)\rangle = -\frac{1}{\Box_1^Z}(D_1^+)^4(D_2^+)^4 e^{(v_2-v_1)}\delta^{11}(z_1-z_2)\frac{1}{(u_1^+u_2^+)^3} , \qquad (3.20)$$

where $\Box^Z = \partial^{\alpha\beta}\partial_{\alpha\beta} + Z^{ab}Z_{ab}$ and $D^{++}v = v^{++}$. The manifestly supersymmetric perturbation theory is the important advantage of the HS approach.

One can consider also the alternative version of l-hypermultiplet ω_l and l-linear multiplet $L^{(2,0)}$, $D_l^{(2,0)}L^{(2,0)}$=0.

The low-energy $U(1)$ effective action can be expressed in terms of the superpotential $f(W^{ab})$ which does not depend on $u^\pm$

$$S_3 = \int d^3x d^8\theta du\ V_l^{++}V_l^{--}f_3(W^{ab}) . \qquad (3.21)$$

The gauge invariance produces the following constraint :

$$\delta S_3 = -2\int d^3x d^8\theta du\lambda D^{--}V_l^{++}f_3(W^{ab})$$
$$\sim \int d^3x(D^-)^4 du\lambda\partial^{\alpha\beta}V_l^{++}D_{\alpha\beta}^{(+2)}f_3(W^{ab}) = 0 , \qquad (3.22)$$

where the analyticity of V_l^{++} and relations (1.10) and $d^8\theta$=$(D^-)^4(D^+)^4$ are used.

This constraint on the superpotential is equivalent to the $3D$ Laplace equation

$$D_{\alpha\beta}^{(+2)}f_3(W^{ab}) = 0 \ \rightarrow\ \frac{\partial}{\partial W^{ab}}\frac{\partial}{\partial W_{ab}}f_3(W^{ab}) = 0 . \qquad (3.23)$$

The general superpotential breaks the $SU_r(2)$ invariance. The R_3-invariant superpotential has the following form:

$$f_3^R(w_3) = g_3^{-2} + k_3 w_3^{-1} , \qquad w_3 = \sqrt{W^{ab}W_{ab}} , \qquad (3.24)$$

where g_3 is the coupling constant of dimension $d= -1/2$, and k_3 is the dimensionless constant of the $\mathcal{N}$=8 $WZNW$-type interaction of the vector multiplet. This superpotential is singular at the point Z_{ab}=0 of the moduli space. The field model is well defined in the shifted variables $\hat{W}_{ab}$=$W_{ab} - Z_{ab}$ for nonvanishing central charges. It should be remarked that the superfield interactions of the $3D$-vector multiplets with dimensionless constants (Chern-Simons terms) have earlier been constructed for the case $\mathcal{N}$=4 [18] and $\mathcal{N}$=6 [20, 21].

The general solution of Eq.(3.23) can be written in the harmonic-twistor representation

$$f_3(W^{ab}) = \int dv F_3[W^{(0,2)}, v_\alpha^{(0,\pm1)}] , \qquad W^{(0,2)} = v_a^{(0,1)}v_b^{(0,1)}W^{ab} , \qquad (3.25)$$

where F_3 is an arbitrary function with q=$(0,0)$, and $W^{(0,2)}$ is the r-harmonic projection of the basic superfield. The proof is based on the following r-harmonic representation of the Laplace operator:

$$\varepsilon_{ac}\varepsilon_{bd}\frac{\partial}{\partial W_{ab}}\frac{\partial}{\partial W_{cd}} \sim \frac{\partial}{\partial W^{(0,2)}}\frac{\partial}{\partial W^{(0,-2)}} - \left(\frac{\partial}{\partial W^{(0,0)}}\right)^2 , \qquad (3.26)$$
$$\varepsilon_{ac} = v_a^{(0,1)}v_c^{(0,-1)} - v_c^{(0,1)}v_a^{(0,-1)} . \qquad (3.27)$$

This solution is a covariant form of the well-known integral representation of the $3D$-harmonic functions [22].

Let us introduce the following definitions and relations for the spinor derivatives in the biharmonic superspace:

$$D_\alpha^{(\pm1,\pm1)} = u_i^{(\pm1,0)}v_a^{(0,\pm1)}D_\alpha^{ia} , \qquad D^{(\pm2,\pm2)} = D^{(\pm1,\pm1)\alpha}D_\alpha^{(\pm1,\pm1)} , \qquad (3.28)$$
$$[D_l^{(\pm2,0)}, D_\alpha^{(\mp1,\pm1)}] = D_\alpha^{(\pm1,\pm1)} , \qquad [D_r^{(0,\pm2)}, D_\alpha^{(\pm1,\mp1)}] = D_\alpha^{(\pm1,\pm1)} , \qquad (3.29)$$
$$[D_l^{(\pm2,0)}, D_\alpha^{(\pm1,\pm1)}] = [D_r^{(0,\pm2)}, D_\alpha^{(\pm1,\pm1)}] = 0 , \qquad (3.30)$$
$$D^{(\pm4,0)} = D^{(\pm2,2)}D^{(\pm2,-2)} , \qquad D^{(0,\pm4)} = D^{(2,\pm2)}D^{(-2,\pm2)} . \qquad (3.31)$$

The constraints (3.17) are equivalent to the following conditions in BHS:

$$D_\alpha^{(\pm1,1)}W^{(0,2)} = 0 , \qquad D_r^{(0,2)}W^{(0,2)} = 0 . \qquad (3.32)$$

By analogy with the D=4 analytic linear multiplet L^{++} ,($D^{++}L^{++}$=0) we shall call this 3D representation *the r-linear multiplet*. These restrictions on $W^{(0,2)}$ are evident in the BHS representation

$$W^{(0,2)} = -iD^{(2,2)}V_l^{(-2,0)} = -i\int du D^{(-2,2)}V_l^{(2,0)} \, . \tag{3.33}$$

Consider the r-harmonic decomposition of the full spinor measure

$$d^8\theta = D^{(0,-4)}D^{(0,4)} \, . \tag{3.34}$$

Using this decomposition and Eqs.(3.21,3.25) and (3.33) we can construct an equivalent form of the effective action in the r-analytic superspace

$$S_3 = \int d^3x D^{(0,-4)} dv [W^{(0,2)}]^2 F_3[W^{(0,2)}, \; v^{(0,\pm1)}] \, . \tag{3.35}$$

It should be underlined that this action with the gauge-invariant analytic Lagrangian can be generalized to the case of non-Abelian SYM theory.

Let us consider now the set of l-analytic prepotentials $V_{lB}^{(2,0)}$ in the $[U(1)]^p$ gauge theory and the corresponding r-analytic superfields $W_B^{(0,2)}(V_{lB}^{(2,0)})$. The effective action of this theory in the r-analytic superspace is

$$S_3^p = \int d^3x D^{(0,-4)} dv \sum_{B,C=1}^{p} W_B^{(0,2)} W_C^{(0,2)} F_{BC}^3[W_A^{(0,2)}, \; v^{(0,\pm1)}] \, , \tag{3.36}$$

where F_{BC} are real q=(0,0) functions of the superfields $W_1^{(0,2)}, \ldots W_p^{(0,2)}$ and v-harmonics. The corresponding effective action in the full superspace contains the matrix superpotential of the $[U(1)]^p$ gauge theory

$$S_3^p = \sum_{B,C=1}^{p} \int d^3x \, d^8\theta \, du \, dv \; V_{lB}^{(2,0)} V_{lC}^{(-2,0)} f_{BC}^3(W_1^{ab}, \ldots W_p^{ab}) \, , \tag{3.37}$$

$$f_{BC}^3(W_1^{ab}, \ldots W_p^{ab}) = \int dv F_{BC}^3[W_A^{(0,2)}, \; v^{(0,\pm)}] \, . \tag{3.38}$$

It is evident in the v-integral representation that this superpotential satisfies the following conditions:

$$\frac{\partial}{\partial W_M^{cd}} \frac{\partial}{\partial W_{cdN}} f_{BC}^3 = 0 \, , \tag{3.39}$$

$$D_{\alpha\beta}^{(+2)} f_{BC}^3 = 0 \, . \tag{3.40}$$

These conditions guarantee the gauge invariance of S_3^p in the full superspace.

The r-forms of the hypermultiplet constraints have been discussed in ref.[22]. Consider the superfield constraints for these hypermultiplets in the framework of BHS

$$D_\alpha^{(\pm1,1)} q_r^{a(0,1)} = 0 \, , \qquad q_r^{a(0,1)} = (q_r^{(0,1)}, \bar{q}_r^{(0,1)}) \, , \tag{3.41}$$

$$D_\alpha^{(\pm1,1)} \omega_r = 0 \, , \qquad D_l^{(\pm2,0)}(\omega_r, q_r^{a(0,1)}) = 0 \, , \tag{3.42}$$

where $D_l^{(\pm2,0)}$ are the l-harmonic derivatives.

These hypermultiplets are dual to each other and also to the r-linear analytic multiplet

$$q_r^{a(0,1)} = v^{a(0,1)}\omega_r + v^{a(0,-1)}L^{(0,2)} \, . \tag{3.43}$$

The duality relation between the ω_r and r-linear multiplet is described by the action

$$\int d^3x D^{(0,-4)} dv \{\omega_r [D_r^{(0,2)} L^{(0,2)}] + F^{(0,4)}[L^{(0,2)}, \; v^{(0,\pm1)}]\} \, , \tag{3.44}$$

where $F^{(0,4)}$ is an arbitrary r-analytic function.

It is clear that the l-analytic hypermultiplets $q_l^{(1,0)}$ and ω_l are dual to the alternative r-version of the vector multiplet which can be described by the r-analytic prepotential $V_r^{(0,2)}$.

4 Two-dimensional (4,4) harmonic superspaces

The two-dimensional (4,4) and (4,0) σ-models have been discussed in the field-component formalism and in the framework of the ordinary or harmonic superspaces [24]-[33]. The (4,4) gauge theory has been considered in the component formalism and in the (2,2) superspace [34, 35, 36]. We shall study the geometry of this theory in the manifestly covariant harmonic formalism which is convenient for the superfield quantum calculations.

The maximum automorphism group of the (4,4) superspace is $SO_l(4)\times SO_r(4)$; however, we shall mainly use the group R_2=$SU_c(2)\times SU_l(2)\times SU_r(2)$. Let us choose the left and right coordinates in the (4,4) superspace

$$z_l = (y, \; \theta^{i\alpha}) \, , \qquad z_r = (\bar{y}, \; \bar{\theta}^{ia}) \, , \tag{4.1}$$

where $y = (1/\sqrt{2})(t+x)$ and $\bar{y} = (1/\sqrt{2})(t-x)$ are the light-cone 2D coordinates; and the following types of 2-spinor indices are used: $i, k, \ldots$ for $SU_c(2)$; $\alpha, \beta \ldots$ for $SU_l(2)$ and $a, b \ldots$ for $SU_r(2)$, respectively. The $SO(1,1)$ weights of coordinates are (1, 1/2) for z_l and (−1, −1/2) for z_r. The algebra of spinor derivatives in this superspace

$$\{D_{k\alpha}, D_{l\beta}\} = i\varepsilon_{kl}\varepsilon_{\alpha\beta}\partial_y \, , \tag{4.2}$$

$$\{\bar{D}_{ka}, \bar{D}_{lb}\} = i\varepsilon_{kl}\varepsilon_{ab}\bar{\partial}_y \, , \tag{4.3}$$

$$\{D_{k\alpha}, \bar{D}_{lb}\} = i\varepsilon_{kl}Z_{\alpha b} \tag{4.4}$$

contains the central charges $Z_{\alpha b}$.

The CB-geometry of the (4,4) SYM theory is described by the superfield constraints

$$\{\nabla_{k\alpha}, \nabla_{l\beta}\} = i\varepsilon_{kl}\varepsilon_{\alpha\beta}\nabla_y \, , \tag{4.5}$$

$$\{\bar{\nabla}_{ka}, \bar{\nabla}_{lb}\} = i\varepsilon_{kl}\varepsilon_{ab}\bar{\nabla}_y \, , \tag{4.6}$$

$$\{\nabla_{k\alpha}, \bar{\nabla}_{lb}\} = i\varepsilon_{kl}W_{\alpha b} \, , \tag{4.7}$$

where ∇_M=$D_M + A_M$ is the covariant derivative for the corresponding coordinate. The gauge-covariant superfield $W_{\alpha b}$ satisfies the constraints of the (4,4) vector multiplet which are equivalent to the constraints of the so-called twisted multiplet [24, 28].

The authors of refs.[30, 31, 32] have discussed three types of harmonics: $u_i^{\pm}$=$u_i^{(\pm1,0,0)}$ for $SU_c(2)/U_c(1)$; $l_\alpha^{(0,\pm1,0)}$ for $SU_l(2)/U_l(1)$; and $r_a^{(0,0,\pm1)}$ for $SU_r(2)/U_r(1)$. We use the notation with 3 charges in the triharmonic superspace (THS) and the standard notation in the c-harmonic superspace. The basic geometric structures of the gauge theory are mainly connected with the c-harmonics $u^\pm_i$ and the corresponding analytic coordinates $\zeta_c=(y_c,\ \theta^{\alpha+})$, $\theta^{\alpha-}$ and $\bar\zeta_c=(\bar y_c,\ \bar\theta^{a+})$, $\bar\theta^{a-}$.

The c-harmonized spinor derivatives and harmonic derivatives have the following form in the superspace with vanishing central charges:

$$D^+_\alpha = \partial^+_\alpha\ , \qquad D^-_\beta = -\partial^-_\alpha - i\theta^-_\alpha\partial^c_y\ , \tag{4.8}$$

$$\bar D^+_a = \partial^+_a\ , \qquad \bar D^-_a = -\bar\partial^-_a - i\bar\theta^-_a\bar\partial^c_y\ , \tag{4.9}$$

$$D^{++}_c = \partial^{++}_c + \frac{i}{2}\theta^{+\alpha}\theta^+_\alpha\partial^c_y + \frac{i}{2}\bar\theta^{a+}\bar\theta^+_a\bar\partial^c_y + \theta^{\alpha+}\partial^+_\alpha + \bar\theta^{a+}\partial^+_a\ . \tag{4.10}$$

The basic combinations of the spinor derivatives are

$$(D^\pm)^2 = D^{\pm\alpha}D^\pm_\alpha\ , \quad (\bar D^\pm)^2 = \bar D^{a\pm}\bar D^\pm_a\ , \quad (D^\pm)^4 = (D^\pm)^2(\bar D^\pm)^2\ . \tag{4.11}$$

The c-harmonic projections of the constraints (4.5-4.7) are equivalent to the integrability conditions of the c-analyticity by analogy with the $D\geq3$, $\mathcal{N}=8$ theories

$$\{\nabla^+_\alpha, \nabla^+_\beta\} = \{\nabla^+_\alpha, \bar\nabla^+_b\} = \{\bar\nabla^+_a, \bar\nabla^+_b\} = 0\ , \tag{4.12}$$

where $\nabla^+_\alpha = u^+_i\nabla^i_\alpha$ and $\bar\nabla^+_a = u^+_i\bar\nabla^i_a$.

The prepotential of the (4,4) gauge theory in this version of HS is the c-analytic harmonic connection $V^{++}_c(\zeta_c\ ,\bar\zeta_c,\ u) \equiv V^{(2,0,0)}_c$ which determines the second harmonic connection $V^{--}_c \equiv V^{(-2,0,0)}_c$. The WZ gauge for this prepotential has the following form:

$$(V^{++}_c)_{WZ} = \theta^{\alpha+}\bar\theta^{b+}\Phi_{\alpha b}(y_c\,,\bar y_c) + (\theta^+)^2\bar A(y_c\,,\bar y_c) + (\bar\theta^+)^2 A(y_c\,,\bar y_c)$$
$$+(\bar\theta^+)^2\theta^{\alpha+}u^-_i\lambda^i_\alpha(y_c\,,\bar y_c) + (\theta^+)^2\bar\theta^{+a}u^-_i\bar\lambda^i_a(y_c\,,\bar y_c) + i(\theta^+)^2(\bar\theta^+)^2u^-_k u^-_j X^{kj}(y_c\,,\bar y_c) \tag{4.13}$$

where the components of the $2D$ vector multiplet are defined.

The gauge-covariant superfield strength can be constructed by analogy with $D=3$

$$W_{\alpha b} \equiv (\sigma^m)_{\alpha b}W_m = -iD^+_\alpha\bar D^+_b V^{--}_c \tag{4.14}$$

where $(\sigma^m)_{\alpha b}$ are the Weyl matrices for $SU_l(2)\times SU_r(2)$ and W_m is the 4-vector representation of this superfield.

In the Abelian case, the constraints for this superfield are

$$D^+_\alpha W_{\beta b} = \frac{1}{2}\varepsilon_{\alpha\beta}D^{+\rho}W_{\rho b}\ , \qquad \bar D^+_a W_{\alpha b} = \frac{1}{2}\varepsilon_{ab}\bar D^{+c}W_{\alpha c}\ , \tag{4.15}$$

$$D^{\pm\pm}_c W_{\alpha b} = 0\ , \qquad (D^+)^2 W_{\alpha a} = (\bar D^+)^2 W_{\alpha a} = 0\ . \tag{4.16}$$

The universal harmonic construction of the $U(1)$ effective action with 8 supercharges has the following form in the case $D=2$:

$$S_2 = \int d^2x d^8\theta du V^{++}_c V^{--}_c f_2(W_m)\ , \qquad (D^+)^2 f_2(W_m) = (\bar D^+)^2 f_2(W_m) = 0\ , \tag{4.17}$$

where $d^2x = dtdx \equiv dyd\bar y$.

The general (4,4) superpotential satisfies the $4D$ Laplace equation

$$\Delta^w_4 f_2(W_m) = 0\ , \qquad \Delta^w_4 = \Big(\frac{\partial}{\partial W_m}\Big)^2\ . \tag{4.18}$$

The $4D$ Laplace equation in the (4,4) σ-models has been discussed, for instance, in refs.[27, 33].

The R_2-invariant (4,4) superpotential is

$$f^R_2(w_2) = g^{-2}_2 + k_2 w^{-2}_2\ , \quad w_2 = \sqrt{W^{\alpha a}W_{\alpha a}}\ . \tag{4.19}$$

An analogous function has been considered in the derivation of the R_2-invariant (2,2) Kähler potential of the D=2, (4,4) gauge theory [35]. Note that the Kähler potential of the (2,2) formalism is gauge-invariant by definition, and the $4D$ Laplace equation arises in this approach from the restrictions of the (4,4) supersymmetry; while in our (4,4) formulation the analogous condition on the (4,4) superpotential (4.18) follows from the gauge invariance. The manifestly (4,4) covariant formalism of the harmonic gauge theory simplifies the proof of the non-renormalization theorem.

The c-analytic (4,4) hypermultiplets q^+_c and ω_c have the minimal interactions with V^{++}_c. The corresponding HS Feynmann rules can be formulated by analogy with ref. [2]. The (4,4) HS methods can be useful in the analysis of the vector-hypermultiplet Matrix models with (8,8) supersymmetry.

The THS projections of the $2D$ spinor derivatives are

$$D^{(\pm1,\pm1,0)} = u^{(\pm1,0,0)}_i l^{(0,\pm1,0)}_\alpha D^{\alpha i}\ , \qquad \bar D^{(\pm1,0,\pm1)} = u^{(\pm1,0,0)}_i r^{(0,0,\pm1)}_a \bar D^{ai}\ . \tag{4.20}$$

The rl-version of the c-vector multiplet (4.14) has the following form:

$$W^{(0,1,1)} = -iD^{(1,1,0)}\bar D^{(1,0,1)}V^{(-2,0,0)}_c = -i\int du D^{(-1,1,0)}\bar D^{(-1,0,1)}V^{(2,0,0)}_c\ . \tag{4.21}$$

This superfield satisfies the conditions of the rl-analyticity

$$D^{(\pm1,1,0)}W^{(0,1,1)} = 0\ , \qquad \bar D^{(\pm1,0,1)}W^{(0,1,1)} = 0 \tag{4.22}$$

and the harmonic conditions

$$D^{(\pm2,0,0)}_c W^{(0,1,1)} = D^{(0,2,0)}_l W^{(0,1,1)} = D^{(0,0,2)}_r W^{(0,1,1)} = 0 \tag{4.23}$$

which are analogous to the constraints on the $q^{(1,1)}$ superfield of ref.[30] (this notation does not indicate the $U_c(1)$ charge). Note that the vector multiplet (4.21) contains the $2D$ vector field instead of the auxiliary scalar component in the superfield $q^{(1,1)}$.

Let us consider the rl-analytic coordinates

$$\zeta_l = (y_l, \theta^{(\pm1,1,0)})\ , \qquad \theta^{(\pm1,\pm1,0)} = u^{(\pm1,0,0)}_i l^{(0,\pm1,0)}_\alpha\theta^{i\alpha}\ , \tag{4.24}$$

$$y_l = y + \frac{i}{2}[\theta^{(1,-1,0)}\theta^{(-1,1,0)} - \theta^{(-1,-1,0)}\theta^{(1,1,0)}]\ , \tag{4.25}$$

$$\bar\zeta_r = (\bar y_r, \bar\theta^{(\pm1,0,1)})\ , \qquad \bar\theta^{(\pm1,0,\pm1)} = u^{(\pm1,0,0)}_i r^{(0,0,\pm1)}_a\bar\theta^{ia}\ , \tag{4.26}$$

$$y_r = y + \frac{i}{2}[\bar\theta^{(1,0,-1)}\bar\theta^{(-1,0,1)} - \bar\theta^{(-1,0,-1)}\bar\theta^{(1,0,1)}]\ . \tag{4.27}$$

The spinor and harmonic derivatives have the following form in these coordinates:

$$D^{(\pm1,1,0)}=\pm\partial^{(\pm1,1,0)}\,,\qquad D^{(\pm1,-1,0)}=\mp\partial^{(\pm1,-1,0)}+i\theta^{(\pm1,-1,0)}\partial^l_y\,, \tag{4.28}$$
$$\bar{D}^{(\pm1,0,1)}=\pm\bar{\partial}^{(\pm1,0,1)}\,,\qquad \bar{D}^{(\pm1,0,-1)}=\mp\bar{\partial}^{(\pm1,0,-1)}+i\bar{\theta}^{(\pm1,-1,0)}\bar{\partial}^r_y\,, \tag{4.29}$$
$$D^{(0,2,0)}_l=\partial^{(0,2,0)}_l+i\theta^{(1,1,0)}\theta^{(-1,1,0)}\partial^l_y+\theta^{(1,1,0)}\partial^{(-1,1,0)}+\theta^{(-1,1,0)}\partial^{(1,1,0)}\,, \tag{4.30}$$
$$D^{(0,0,2)}_r=\partial^{(0,0,2)}_r+i\bar{\theta}^{(1,0,1)}\bar{\theta}^{(-1,0,1)}\bar{\partial}^r_y+\bar{\theta}^{(1,0,1)}\bar{\partial}^{(-1,0,1)}+\bar{\theta}^{(-1,0,1)}\bar{\partial}^{(1,0,1)}\,. \tag{4.31}$$

The c-analytic coordinates in the THS notation are

$$\zeta_c=(y_c,\theta^{(1,\pm1,0)})\,,\qquad y_c=y+\frac{i}{2}[\theta^{(-1,1,0)}\theta^{(1,-1,0)}+\theta^{(1,1,0)}\theta^{(-1,-1,0)}]\,, \tag{4.32}$$
$$\bar{\zeta}_c=(\bar{y}_c,\bar{\theta}^{(1,0,\pm1)})\,,\qquad \bar{y}_c=\bar{y}+\frac{i}{2}[\bar{\theta}^{(-1,0,1)}\bar{\theta}^{(1,0,-1)}+\bar{\theta}^{(1,0,1)}\bar{\theta}^{(-1,0,-1)}]\,. \tag{4.33}$$

It is important that all coordinates $\zeta_c,\bar{\zeta}_c,\zeta_l$ and $\bar{\zeta}_r$ are separately real with respect to the corresponding conjugation.

The solution of the $4D$ Laplace equation (4.18) has the simple harmonic representation

$$f_2(W_{\alpha a})=\int dl dr F_2[W^{(0,1,1)},\ l,\ r]\,, \tag{4.34}$$

where F_2 is the real function and $W^{(0,1,1)}=l^{(0,1,0)}_\alpha r^{(0,0,1)}_a W^{\alpha a}$ (4.21). The proof is based on the THS decomposition of the $4D$ Laplace operator

$$\frac{\partial}{\partial W^{\alpha b}}\frac{\partial}{\partial W_{\alpha b}}\sim\frac{\partial}{\partial W^{(0,1,1)}}\frac{\partial}{\partial W^{(0,-1,-1)}}-\frac{\partial}{\partial W^{(0,1,-1)}}\frac{\partial}{\partial W^{(0,-1,1)}}\,. \tag{4.35}$$

Note that the formal change of the density in (4.34)

$$F_2[W^{(0,1,1)},\ l,\ r]\ \rightarrow\ F'[W^{(0,1,1)},W^{(0,1,-1)},\ l,\ r] \tag{4.36}$$

does not produce more general superpotentials. This can be easily shown for the polynomial solutions of Eq.(4.18).

Consider the THS decomposition of the Grassmann measure

$$d^8\theta=D^{(0,-2,-2)}D^{(0,2,2)}\,, \tag{4.37}$$
$$D^{(0,\pm2,\pm2)}=D^{(1,\pm1,0)}D^{(-1,\pm1,0)}\bar{D}^{(1,0,\pm1)}\bar{D}^{(-1,0,\pm1)}\,. \tag{4.38}$$

Using this decomposition and Eqs.(4.17,4.21) one can obtain the following equivalent representation of the effective (4,4) action in the rl-analytic superspace:

$$S_2=\int dl dr d^2x D^{(0,-2,-2)}[W^{(0,1,1)}]^2F_2[W^{(0,1,1)},\ l,\ r]\,. \tag{4.39}$$

One can construct the effective (4,4) action for the gauge group $[U(1)]^p$ in the rl-analytic and full superspaces by analogy with the case $D=3$ (3.36,3.37).

An analogous action of the $q^{(1,1)}$ multiplet and dual superfields $\omega^{(\pm1,\mp1)}$ has been considered in refs.[30, 31, 32]. The relation between the c-analytic gauge superfield and rl-analytic hypermultiplets is a specific manifestation of the $2D$ mirror symmetry [34].

The triharmonic superspace is convenient for the classification of the (4,4) supermultiplets. Let us consider, for instance, the cr-analytic superfield $Q^{(1,0,1)}_{cr}(\zeta_c,\bar{\zeta}_r,u,r)$ satisfying the subsidiary harmonic conditions

$$D^{(2,0,0)}_c Q^{(1,0,1)}_{cr}=0\,,\qquad D^{(0,0,2)}_r Q^{(1,0,1)}_{cr}=0\,, \tag{4.40}$$

where the analytic coordinates (4.32) and (4.27) are used. The cl-analytic superfield $Q^{(1,1,0)}_{cl}(\bar{\zeta}_c,\zeta_l,u,l)$ can be defined analogously.

5 One-dimensional harmonic superspaces

The one-dimensional σ-models have been considered in the component formalism and also in the framework of the superspaces with $\mathcal{N}$=1,2 and 4 [36, 37, 38]. Recently, the $\mathcal{N}$=4 superspace has been used also for the proof of the non-renormalization theorem in the $\mathcal{N}$=8 gauge theory [39].

We shall consider the D=1, $\mathcal{N}$=8 superspace which is based on the maximum automorphism group $R_1=SU_c(2)\times Spin(5)$ and has coordinates $z=(t,\ \theta^\alpha_i)$ ($i,k,l\ldots$ are the 2-spinor indices and $\alpha,\beta,\rho\ldots$ are the 4-spinor indices of the group $Spin(5)=USp(4)$). The algebra of spinor derivatives is

$$\{D^k_\alpha,D^l_\rho\}=i\varepsilon^{kl}\Omega_{\alpha\rho}\partial_t+i\varepsilon^{kl}Z_{\alpha\rho}\,, \tag{5.1}$$

where $Z_{\alpha\rho}$ are central charges.

Conjugation rules in the group $Spin(5)$ differ from the corresponding rules in $Spin(4,1)$ (2.8)

$$\overline{\theta^\alpha_i}=\theta^i_\alpha\,,\quad \overline{\Omega_{\alpha\rho}}=-\Omega^{\alpha\rho}\,,\quad \overline{Z_{\alpha\rho}}=Z^{\alpha\rho}\,. \tag{5.2}$$

The CB-geometric superfield constraints of the $\mathcal{N}$=8 SYM theory are

$$\{\nabla^k_\alpha,\nabla^l_\rho\}=i\varepsilon^{kl}\Omega_{\alpha\rho}(\partial_t+A_t)+i\varepsilon^{kl}W_{\alpha\rho}\,, \tag{5.3}$$

where a traceless bispinor superfield representation of the $1D$ vector multiplet $W_{\alpha\rho}(z)$ is defined.

The harmonics $u^\pm_i$ can be used for a construction of the D=1 c-analytic coordinates $\zeta_c=(t_c,\ \theta^{+\alpha})$

$$t_c=t+\frac{i}{2}\theta^\alpha_k\theta_{l\alpha}u^{k+}u^{l-}\,,\qquad \theta^{+\alpha}=u^+_k\theta^{k\alpha}\,. \tag{5.4}$$

The algebra of the c-harmonized $1D$ spinor derivatives resembles the corresponding algebra of the $5D$ derivatives (2.13-2.15) with $Spin(5)$ indices instead of the $Spin(4,1)$ indices. In the case of vanishing central charges we have

$$D^+_\alpha=\partial^+_\alpha\,,\qquad D^-_\alpha=-\partial^-_\alpha+i\theta^-_\alpha\partial^c_t\,, \tag{5.5}$$
$$D^{++}_c=\partial^{++}_c-\frac{i}{2}\theta^{\alpha+}\theta^+_\alpha\partial^c_t+\theta^{\alpha+}\partial^+_\alpha\,. \tag{5.6}$$

The constraints (5.3) correspond to the integrability conditions of the c-analyticity

$$\{\nabla^+_\alpha,\nabla^+_\gamma\}=0\,,\qquad \nabla^+_\alpha=u^+_i\nabla^i_\alpha\,. \tag{5.7}$$

The c-analytic prepotential $V^{++}_c(\zeta_c,u)$ describes the $1D$ vector multiplet and contains also the pure gauge one-dimensional field A

$$\begin{aligned}(V^{++}_c)_{WZ}&=\Theta^{(+2)}A(t_c)+\Theta^{(+2)\alpha\rho}\Phi_{\alpha\rho}(t_c)\\&+\Theta^{(+2)}\theta^{\alpha+}u^-_k\lambda^k_\alpha(t_c)+i[\Theta^{(+2)}]^2u^-_ku^-_jX^{kj}(t_c)\,,\end{aligned} \tag{5.8}$$

where the notation (2.18) is used. Of course, one can use the subsidiary gauge condition $A(t_c)$=0.

The AB-superfield strength of the corresponding harmonic gauge theory is the 5-vector W_m with respect to $Spin(5)$

$$W_{\alpha\rho} \equiv \frac{1}{2}(\Gamma^m)_{\alpha\rho} W_m = -iD^{(+2)}_{\alpha\rho} V^{--}\,, \qquad \Omega^{\alpha\rho} W_{\alpha\rho} = 0\,, \tag{5.9}$$

where the Γ matrices of $Spin(5)$ are introduced.

The constraints for this superfield have the following form:

$$D^+_\alpha W_{\beta\gamma} = \frac{2}{5}\Omega_{\alpha\beta} D^{+\sigma} W_{\sigma\gamma} - \frac{2}{5}\Omega_{\alpha\gamma} D^{+\sigma} W_{\sigma\beta} + \frac{1}{5}\Omega_{\beta\gamma} D^{+\sigma} W_{\sigma\alpha}\,, \tag{5.10}$$

$$D^{(+2)} W_{\alpha\rho} = 0\,, \qquad \nabla^{++} W_{\alpha\rho} = 0\,. \tag{5.11}$$

In the Abelian gauge group, the superfield $W_{\alpha\rho}$ does not depend on harmonics, and four components of this superfield $W_{13}, W_{14}, W_{23}, W_{24}$ satisfy the conditions of different twisted chiralities, e.g.

$$D^\pm_1 W_{13} = D^\pm_3 W_{13} = 0\,, \qquad D^\pm_1 W_{14} = D^\pm_4 W_{14} = 0\,. \tag{5.12}$$

The D=1 low-energy effective action has the following universal form:

$$S_1 = \int dt d^8\theta du\; V^{++}_c V^{--}_c\; f_1(W_m)\,. \tag{5.13}$$

The gauge invariance of S_1 is equivalent to the $5D$ Laplace equation for the superpotential

$$D^{(+2)} f_1(W_m) = 0 \;\rightarrow\; \Delta^w_5 f_1(W_m) = 0\,. \tag{5.14}$$

The R_1-invariant D=1 superpotential is

$$f^R_1(w_1) = g_1^{-2} + k_1 w_1^{-3}\,, \qquad w_1 = (W^{\rho\sigma} W_{\rho\sigma})^{1/2}\,. \tag{5.15}$$

Note that the same function determines the Kähler potential of the D=1 gauge theory in the $\mathcal{N}$=4 superfield formalism [39].

The c-analytic hypermultiplets $q^+(\zeta_c, u)$ and $\omega_c(\zeta_c, u)$ can be introduced by analogy with HS of higher dimensions. These superfields have the R_1-invariant minimal interactions with the prepotential V^{++}_c.

Let us introduce now the biharmonic $1D$-superspace using the $SU_c(2)$ harmonics $u^{(\pm1,0,0)}_i = u^\pm_i$ and harmonics $v^{(0,\pm1,0)}_\alpha$, $v^{(0,0,\pm1)}_\alpha$ of the group $USp(4)$ [40]. The basic relations for the v-harmonics are

$$\Omega^{\alpha\rho} v^{(0,a,0)}_\alpha v^{(0,-b,0)}_\rho = \delta^{ab}\,, \tag{5.16}$$

$$\Omega^{\alpha\rho} v^{(0,0,a)}_\alpha v^{(0,0,-b)}_\rho = \delta^{ab}\,,\,, \tag{5.17}$$

$$\Omega^{\alpha\rho} v^{(0,a,0)}_\alpha v^{(0,0,b)}_\rho = 0\,. \tag{5.18}$$

where $a, b = \pm1$ and δ_{ab} is the Kronecker symbol. These harmonics determine the 8-dimensional coset space H_8=$USp(4)/U(1)\times U(1)$.

The harmonic derivatives $D^{(0,\pm2,0)}_v$, $D^{(0,0,\pm2)}_v$ and $D^{(0,\pm1,\pm1)}_v$ are defined in ref.[40]

$$D^{(0,\pm2,0)}_v v^{(0,\mp1,0)}_\alpha = v^{(0,\pm1,0)}_\alpha\,, \quad D^{(0,0,\pm2)}_v v^{(0,0,\mp1)}_\alpha = v^{(0,0,\pm1)}_\alpha\,, \tag{5.19}$$

$$D^{(0,\pm2,0)}_v v^{(0,\pm1,0)}_\alpha = D^{(0,\pm2,0)}_v v^{(0,0,\pm1)}_\alpha = D^{(0,0,\pm2)}_v v^{(0,\pm1,0)}_\alpha = D^{(0,0,\pm2)}_v v^{(0,0,\pm1)}_\alpha = 0\,, \tag{5.20}$$

$$D^{(0,\pm1,\pm1)}_v v^{(0,0,\mp1)}_\alpha = v^{(0,\pm1,0)}_\alpha\,, \quad D^{(0,\pm1,\pm1)}_v v^{(0,0,\pm1)}_\alpha = D^{(0,\pm1,\pm1)}_v v^{(0,\pm1,0)}_\alpha = 0\,. \tag{5.21}$$

The algebra of harmonic derivatives on H_8 contains also the $U(1)$-charges D^0_{v2} and D^0_{v3}. The harmonic derivatives on the coset $SU_c(2)/U_c(1)$ are $D^{(\pm2,0,0)}_c$ and D^0_c.

The biharmonized spinor derivatives are

$$D^{(\pm1,\pm1,0)} = u^{(\pm1,0,0)}_i v^{(0,\pm1,0)}_\alpha D^{i\alpha}\,, \qquad D^{(\pm1,0,\pm1)} = u^{(\pm1,0,0)}_i v^{(0,0,\pm1)}_\alpha D^{i\alpha}\,. \tag{5.22}$$

The basic superfield of the biharmonic approach is the v-projection of the bispinor superfield (5.9)

$$W^{(0,1,1)} = -iD^{(1,1,0)} D^{(1,0,1)} V^{(-2,0,0)}_c = -i\int du\, D^{(-1,1,0)} D^{(-1,0,1)} V^{(2,0,0)}_c\,, \tag{5.23}$$

where the c-harmonic connections are used. This superfield is v-analytic

$$D^{(\pm1,1,0)} W^{(0,1,1)} = 0, \qquad D^{(\pm1,0,1)} W^{(0,1,1)} = 0 \tag{5.24}$$

and also satisfies the harmonic constraints

$$D^{(\pm2,0,0)}_c W^{(0,1,1)} = 0, \quad \mathcal{D}^A_v W^{(0,1,1)} = 0\,, \tag{5.25}$$

where $\mathcal{D}^A_v$ is the triplet of harmonic derivatives conserving the v-analyticity (5.24)

$$\mathcal{D}^A_v = (D^{(0,1,1)}_v, D^{(0,2,0)}_v, D^{(0,0,2)}_v)\,, \tag{5.26}$$

$$[\mathcal{D}_A, D^{(\pm1,1,0)}] = [\mathcal{D}_A, D^{(\pm1,0,1)}] = 0\,. \tag{5.27}$$

The v-analytic coordinates ζ_v=$(t_v\,, \theta^{(\pm1,1,0)}, \theta^{(\pm1,0,1)})$ can be defined by analogy with (5.4)

$$t_v = t + \tfrac{i}{2}[\theta^{(-1,-1,0)}\theta^{(1,1,0)} - \theta^{(1,-1,0)}\theta^{(-1,1,0)} - \theta^{(1,0,-1)}\theta^{(-1,0,1)} + \theta^{(-1,0,-1)}\theta^{(1,0,1)}]\,, \tag{5.28}$$

$$\theta^{(\pm1,\pm1,0)} = u^{(\pm1,0,0)}_i v^{(0,\pm1,0)}_\alpha \theta^{i\alpha}\,, \qquad \theta^{(\pm1,0,\pm1)} = u^{(\pm1,0,0)}_i v^{(0,0,\pm1)}_\alpha \theta^{i\alpha}\,. \tag{5.29}$$

The BHS representation of the general $1D$ superpotential (5.14) is

$$f_1(W_{\alpha\rho}) = \int dv F_1[W^{(0,1,1)},\; v\alpha]\,, \tag{5.30}$$

where the real function F_1 determines the general solution of the $5D$ Laplace equation. Partial solutions can contain, for instance, functions of two variables W_{13} and W_{14}.

Using this relation and Eq.(5.23) we can obtain the v-analytic representation of the $1D$ effective action (5.13)

$$S_1 = \int dv dt_v D^{(0,-2,-2)} [W^{(0,1,1)}(\zeta_v)]^2 F_1[W^{(0,1,1)}(\zeta_v),\; v\alpha]\,, \tag{5.31}$$

where the following Grassmann measure is used:

$$D^{(0,-2,-2)} = D^{(1,-1,0)} D^{(-1,-1,0)} D^{(1,0,-1)} D^{(-1,0,-1)}\,. \tag{5.32}$$

The effective action for an arbitrary gauge group can be constructed in the v-analytic superspace. The matrix superpotential for the gauge group $[U(1)]^p$ in the full superspace satisfies the following conditions:

$$\frac{\partial}{\partial W^{\gamma\sigma}_M} \frac{\partial}{\partial W_{\gamma\sigma_N}} f^1_{BC}(W^{\alpha\rho}_1, \ldots W^{\alpha\rho}_p) = 0\,, \qquad D^{(+2)} f^1_{BC} = 0\,. \tag{5.33}$$

It is not difficult to define the triplet of v-analytic superfields which is dual to the superfield $W^{(0,1,1)}$

$$\omega_v = (\omega_v^{(0,1,-1)}, \omega_v^{(0,-1,1)}, \omega_v^{(0,0,0)}) \,, \tag{5.34}$$

$$D^{(\pm1,1,0)}\omega_v = D^{(\pm1,0,1)}\omega_v = D_c^{(\pm2,0,0)}\omega_v = 0 \,. \tag{5.35}$$

These superfields have an infinite number of auxiliary components.

The interpolating term for the duality relation has the following form:

$$\int dv dt D^{(0,-2,-2)}[W^{(0,1,1)}D_v^{(0,1,1)}\omega_v^{(0,0,0)} + W^{(0,1,1)}D_v^{(0,2,0)}\omega_v^{(0,-1,1)} + W^{(0,1,1)}D_v^{(0,0,2)}\omega_v^{(0,1,-1)}] \,. \tag{5.36}$$

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